

## ESTIMATES FOR MULTIPLICATIVE FUNCTIONS, I.

**Karl-Heinz Indlekofer**<sup>1</sup> (Paderborn, Germany)

**Erdener Kaya** (Cologne, Germany)

Communicated by Imre Kátaı

(Received April 18, 2024; accepted May 6, 2024)

**Abstract.** In this paper we give upper estimates and asymptotic results for some classes of multiplicative functions. For the proofs we mainly use tools from the convolution arithmetic of number theoretical functions (see [4]).

### 1. Introduction

In this paper we consider multiplicative functions  $f : \mathbb{N} \rightarrow \mathbb{C}$  satisfying

- (i)  $|f(p)| \leq B$  for all primes  $p$ ,  $B \geq 1$ ,
- (ii)  $\sum_{p,k \geq 2} \frac{|f(p^k)|}{p^k} < \infty$ .

Our aim is to give upper estimates and asymptotic results for

$$(1.1) \quad M(f, x) := \sum_{n \leq x} f(n) \quad \text{as } x \rightarrow \infty.$$

For multiplicative functions  $f$  satisfying (i) and (ii) we define the *exponentially multiplicative* function  $f_0$  by ( $p$  prime)

$$f_0(p^k) = \frac{(f(p))^k}{k!} \quad (k \in \mathbb{N}).$$

---

*Key words and phrases:* Convolution of arithmetical functions, Multiplicative functions, asymptotic results on arithmetic functions.

*2010 Mathematics Subject Classification:* 11A25, 11K65, 11N37.

<sup>1</sup> Supported by a grant of Deutsche Forschungsgemeinschaft Project Number 289 386 6657.

Then

$$f = h * f_0,$$

where  $h$  is multiplicative and ( $p$  prime)

$$(1.2) \quad h(p^k) = \sum_{\substack{l,m \\ l+m=k}} \frac{(-1)^l}{l!} (f(p))^l f(p^m) \quad (k \in \mathbb{N}).$$

Especially  $h(p) = 0$ . Obviously

$$\begin{aligned} \sum_{p,k \geq 2} \frac{|h(p)^k|}{p^k} &\leq \sum_{\substack{p,l,m \\ l+m \geq 2}} \frac{1}{l!} \frac{|f(p)|^l |f(p^m)|}{p^{l+m}} = \\ &= \sum_p \left( \left( e^{\frac{|f(p)|}{p}} - 1 \right) \frac{|f(p)|}{p} + e^{\frac{|f(p)|}{p}} - 1 - \frac{|f(p)|}{p} + \right. \\ &\quad \left. + e^{\frac{|f(p)|}{p}} \sum_{m \geq 2} \frac{|f(p^m)|}{p^m} \right) = \\ &= \sum_p O\left(\frac{|f(p)|^2}{p^2}\right) + O(1) \sum_{m \geq 2} \frac{|f(p^m)|}{p^m} < \\ &< \infty \end{aligned}$$

and

$$(1.3) \quad \sum_{n=1}^{\infty} \frac{|h(n)|}{n} < \infty.$$

Now we define  $\mathcal{M}_1$  by

$$\mathcal{M}_1 := \{f : \mathbb{N} \rightarrow \mathbb{C} : f \text{ multiplicative; (i), (ii), (iii) hold}\},$$

where

$$(iii) \quad \sum_{\substack{p^k \leq x \\ k \geq 2}} |f(p^k)| = O\left(\frac{x}{\log x}\right).$$

Further, let

$$(iv) \quad \sum_{\substack{p^k \leq x \\ k \geq 2}} \frac{|f(p^k)|}{p^k} \log p^k < \infty,$$

$$(v) \quad \sum_{\substack{p^k \leq x \\ k \geq 2}} \frac{|f(p^k)|}{p^k} (\log p^k)^2 < \infty$$

and

$$(vi) \exp\left(\sum_{p \leq y} \frac{|f(p)| - 1}{p}\right) \ll \exp\left(\sum_{p \leq x} \frac{|f(p)| - 1}{p}\right) \quad \text{if } 1 \leq y \leq x.$$

Then we define the spaces

$$\mathcal{M}_3 \subset \mathcal{M}_2 \subset \mathcal{M}_1 \quad \text{and} \quad \mathcal{M}_4$$

by

$$\mathcal{M}_2 := \{f \in \mathcal{M}_1, (iv) \text{ holds}\}$$

$$\mathcal{M}_3 := \{f \in \mathcal{M}_1, (v) \text{ holds}\}$$

$$\mathcal{M}_4 := \{f : \mathbb{N} \rightarrow \mathbb{C} : f \text{ multiplicative; (i), (ii), (vi) hold}\}$$

It is easy to see that (iv) implies (iii). Further, it is well known that, if (i) and (ii) hold,

$$(1.4) \quad \sum_{n \leq x} |f_0(n)| \ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right)$$

and, if (vi) holds,

$$(1.5) \quad \sum_{n \leq x^\varepsilon} \frac{|f_0(n)|}{n} \ll \varepsilon \sum_{n \leq x} \frac{|f_0(n)|}{n}.$$

**Remark 1.1.** Let

$$\sum_{p \leq x} \frac{|f(p)|}{p} \log p \sim \alpha \log x, \quad x \rightarrow \infty,$$

hold with a constant  $\alpha > 1$ . Then one can show (see Lemma 2.2. of [7]) that (vi) is valid.

## 2. Classical results

If  $f \in \mathcal{M}_4$  then, by (1.3),

$$(2.1) \quad \begin{aligned} \left| \sum_{n \leq x} f(n) \right| &\leq \sum_{n \leq x} |h(n)| \sum_{m \leq \frac{x}{n}} |f_0(m)| \ll \\ &\ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right). \end{aligned}$$

If  $f \in \mathcal{M}_1$  then, by (1.2),

$$\begin{aligned} \sum_{p^k \leq x} |h(p^k)| \log p^k &\leq \sum_{\substack{p^{v+\mu} \leq x \\ v+\mu \geq 2}} \frac{|f(p)|^v}{v!} |f(p^\mu)| \log x \leq \\ &\leq \sum_{p^\mu \leq x} |f(p^\mu)| e^B \log x = O(x) \end{aligned}$$

and (see [7, (3.6)] and A. G. Postnikov [10, p. 201])

$$\begin{aligned} (2.2) \quad \sum_{n \leq x} |h(n)| \log n &\leq \sum_{n \leq x} |h(n)| \sum_{\substack{p^k \leq \frac{x}{n} \\ k \geq 2}} |h(p^k)| \log p^k \ll \\ &\ll x \sum_{n \leq x} \frac{|h(n)|}{n} \ll x. \end{aligned}$$

By partial summation we obtain

$$(2.3) \quad \sum_{n \leq x} |h(n)| = O\left(\frac{x}{\log x}\right).$$

Now, it is easy to show that, if  $f \in \mathcal{M}_1$  the estimate (2.1) holds. For this we put, as above,  $f = h * f_0$  and obtain

$$\begin{aligned} |M(f)| &\leq \sum_{m \leq x^{1/2}} |f_0(m)| \sum_{n \leq \frac{x}{m}} |h(n)| + \sum_{n \leq x^{1/2}} |h(n)| \sum_{m \leq \frac{x}{n}} |f_0(m)| =: \\ &=: \Sigma_1 + \Sigma_2. \end{aligned}$$

Then

$$\begin{aligned} \Sigma_1 &\ll \frac{x}{\log x} \sum_{m \leq x^{1/2}} \frac{|f_0(m)|}{m} \ll \\ &\ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right) \end{aligned}$$

and

$$\Sigma_2 \ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right) \sum_{n \leq x^{\frac{1}{2}}} \frac{|h(n)|}{n},$$

which proves (2.1). Collecting the results we arrive at

**Proposition 2.1.** *Let  $f \in \mathcal{M}_1 \cup \mathcal{M}_4$ . Then the estimate (2.1) holds.*

### 3. Estimates by convolution arithmetic

Using identities in the convolution arithmetic Indlekofer (see [4] and [7]) could prove estimates for  $M(f, x)$  in the case  $f \in \mathcal{M}_1 \cup \mathcal{M}_4$ .

For this define, if the arithmetical function  $f$  is given, the arithmetical function  $L_0 f$  by  $L_0 f(n) = f(n) \log n$  ( $n \in \mathbb{N}$ ). With the function  $M$  ( $M(u) = M(f, u)$  for  $u \geq 1$ ), the identity function  $\mathbf{1}$  and  $L$ , defined by  $L(u) = \log u$  ( $u \geq 1$ ), we use the convolutions

$$\begin{aligned} (L * f)(u) &= \sum_{n \leq u} \left(\log \frac{u}{n}\right) f(n), \\ (M * f)(u) &= \sum_{n \leq u} M\left(\frac{u}{n}\right) f(n), \\ (\mathbf{1} * f)(u) &= \sum_{n \leq u} f(n) \end{aligned}$$

and define  $\Lambda_f$  by

$$(L_0 f)(n) = (f * \Lambda_f)(n).$$

Then (see [4, p. 135])  $M = \mathbf{1} * f = \mathbf{1} * (h * f_0)$  and

$$(3.1) \quad L^2 M = M * (\Lambda_{f_0} * \Lambda_{f_0} + L_0 \Lambda_{f_0}) + (R_1 + R_2) * \Lambda_{f_0} + L(R_1 + R_2),$$

where

$$(3.2) \quad \begin{aligned} R_1 &= L * f, \\ R_2 &= \mathbf{1} * L_0 h * f_0. \end{aligned}$$

Now let  $\mathcal{M}'_1$  be the set of  $f \in \mathcal{M}_1$  such that

$$(3.3) \quad \sum_{n \leq x^\varepsilon} \frac{|f_0(n)|}{n} \leq \delta(\varepsilon) \sum_{n \leq x} \frac{|f_0(n)|}{n},$$

where  $\delta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Then we prove

**Theorem 3.1.** *Let  $f \in \mathcal{M}'_1 \cup \mathcal{M}_4$ . Then*

$$(3.4) \quad |M(x)| \leq 2B^2 \int_1^x \frac{|M(u)|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right)\right)$$

as  $x \rightarrow \infty$ .

**Proof.** We follow the lines of proof in [7, §3]. Then

$$|\Lambda_{f_0} * \Lambda_{f_0} + L_0 \Lambda_{f_0}| \leq B^2(\Lambda * \Lambda + L_0 \Lambda),$$

and we obtain by Lemma 2.4 of [7] the first summand of the right side of (3.4). For  $R_1(x)$  we conclude as in (3.4) of [7]

$$(3.5) \quad |R_1(x)| = O\left(\frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right)\right).$$

In the case

$$R_2(x) = \sum_{n \leq x} h(n) \log n \sum_{m \leq \frac{x}{n}} f_0(m)$$

we show, if  $f \in \mathcal{M}_4$ , by (1.3)

$$(3.6) \quad |R_2(x)| \ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right) \sum_{n \leq x} \frac{|h(n)| \log n}{n} = \\ = o\left(x \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right)\right).$$

If  $f \in \mathcal{M}'_1$ , then

$$R_2(x) = \sum_{m \leq x^\varepsilon} f_0(m) \sum_{n \leq \frac{x}{m}} h(n) \log n + \sum_{n \leq x^{1-\varepsilon}} h(n) \log n \sum_{m \leq \frac{x}{n}} f_0(m) := \\ := \Sigma_1 + \Sigma_2.$$

By (2.2)

$$|\Sigma_1| \ll x \sum_{m \leq x^\varepsilon} \frac{|f_0(m)|}{m} \leq \delta(\varepsilon) x \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right)$$

and

$$|\Sigma_2| \ll \frac{1}{\varepsilon} \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right) \sum_{n \leq x^{1-\varepsilon}} \frac{|h(n)| \log n}{n}$$

which implies by (1.3)

$$(3.7) \quad |R_2(x)| = o\left(x \exp\left(\sum_{p \leq x} \frac{|f(p)|}{p}\right)\right).$$

Consider

$$(R_1 * \Lambda_{f_0})(x) = (L * (f * \Lambda_{f_0}))(x) = \sum_{n \leq x} \left(\log \frac{x}{n}\right) (f * \Lambda_{f_0})(n).$$

Observing

$$\sum_{n \leq y} |f(n)| \sum_{p \leq \frac{y}{n}} |f(p)| \log p \ll y \sum_{n \leq y} \frac{|f(n)|}{n} \ll y \exp \left( \sum_{p \leq y} \frac{|f(p)|}{p} \right)$$

we conclude (cf. [7, (3.4)])

$$(3.8) \quad |(R_1 * \Lambda_{f_0})(x)| \ll x \exp \left( \sum_{p \leq x} \frac{|f(p)|}{p} \right).$$

Since  $L_0 f_0 = f_0 * \Lambda_{f_0}$  we have

$$R_2 * \Lambda_{f_0} = (\mathbf{1} * L_0 h * f_0) * \Lambda_{f_0} = \mathbf{1} * L_0 h * L_0 f_0$$

and

$$(3.9) \quad \begin{aligned} |(R_2 * \Lambda_{f_0})(x)| &\leq \log x (\mathbf{1} * |L_0 h| * |f_0|)(x) = \\ &= o \left( x \log x \exp \left( \sum_{p \leq x} \frac{|f(p)|}{p} \right) \right). \end{aligned}$$

This ends the proof of Theorem 3.1. ■

If  $f \in \mathcal{M}_3$  we can show

**Corollary 3.1.** *Let  $f \in \mathcal{M}_3$ . Then*

$$|M(x)| \leq 2B^2 \int_1^x \frac{|M(u)|}{u^2} du + O \left( \frac{x}{\log^2 x} \exp \left( \sum_{p \leq x} \frac{|f(p)|}{p} \right) \right).$$

**Proof.** For  $f \in \mathcal{M}_3$  define the multiplicative function  $\tilde{h}$  by

$$\tilde{h}(p^k) = \begin{cases} 0 & \text{if } k = 1 \\ 4|h(p^k)| \log p^k & \text{if } k \geq 2. \end{cases}$$

Then

$$\sum_{p,k \geq 2} \frac{\tilde{h}(p^k) \log p^k}{p^k} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\tilde{h}(n)}{n} < \infty,$$

$$\sum_{n \leq x} \tilde{h}(n) \ll \frac{x}{\log x}.$$

By the elementary inequality  $a + b \leq ab$  for  $a, b \geq 2$  we conclude

$$4 \log n \leq \prod_{p^k || n} 4 \log p^k$$

which implies

$$|h(n)| \log n \leq \frac{1}{4} \tilde{h}(n).$$

Thus

$$\begin{aligned} |R_2(x)| &\leq \sum_{n \leq x^{1/2}} |f_0(n)| \sum_{m \leq \frac{x}{n}} \tilde{h}(m) + \sum_{m \leq x^{1/2}} \tilde{h}(m) \sum_{n \leq \frac{x}{m}} |f_0(n)| = \\ &= O\left(\frac{x}{\log x} \exp\left(\sum_{n \leq x} \frac{|f(p)|}{p}\right)\right), \end{aligned}$$

which leads as above to the assertion of Corollary 3.1. ■

#### 4. Estimates using generating functions

Let  $f = h * f_0 \in \mathcal{M}'_1 \cup \mathcal{M}_4$ . Put

$$K_0 = \mathbf{1} * \Lambda_{f_0} * f.$$

Then (see (3.2))

$$\begin{aligned} LM &= \mathbf{1} * L_0 f + R_1 = \\ &= \mathbf{1} * \Lambda_f * f + R_1 = \\ (4.1) \quad &= \mathbf{1} * (\Lambda_h + \Lambda_{f_0}) * f + R_1 = \\ &= K_0 + R_1 + R_2. \end{aligned}$$

Now

$$\begin{aligned} \int_1^x \frac{|M(u)|}{u^2} du &= \int_1^{x^\varepsilon} \frac{|M(u)|}{u^2} du + \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2} du = \\ &= I_1 + I_2. \end{aligned}$$



Then

$$I_1 \leq \sum_{n \leq x^\varepsilon} \frac{|f(n)|}{n} \ll \sum_{n \leq x^\varepsilon} \frac{|f_0(n)|}{n} \leq \delta(\varepsilon) \exp \left( \sum_{p \leq x} \frac{|f(p)|}{p} \right)$$

if  $f \in \mathcal{M}'_1$  and

$$I_1 \ll \varepsilon \exp \left( \sum_{p \leq x} \frac{|f(p)|}{p} \right)$$

if  $f \in \mathcal{M}_4$ . Concerning  $I_2$  we observe (cf. (3.6), (3.7))

$$M(u) = \frac{K_0(u)}{\log u} + o \left( \frac{u}{\log u} \exp \left( \sum_{p \leq u} \frac{|f(p)|}{p} \right) \right)$$

if  $f \in \mathcal{M}'_1 \cup \mathcal{M}_4$ . Thus

$$\int_1^x \frac{|M(u)|}{u^2} du = \int_{x^\varepsilon}^x \frac{|K_0(u)|}{u^2 \log u} du + R(x) \exp \left( \sum_{p \leq x} \frac{|f(p)|}{p} \right),$$

where

$$R(x) \ll \max(\varepsilon, \delta(\varepsilon)) + o \left( \log \frac{1}{\varepsilon} \right).$$

Now

$$\begin{aligned} \int_{x^\varepsilon}^x \frac{|K_0(u)|}{u^2 \log u} du &\leq \frac{1}{\varepsilon} \frac{1}{\log x} \int_{x^\varepsilon}^x \frac{|K_0(u)|}{u^2} du \leq \\ &\leq \frac{1}{\varepsilon \log x} \left( \int_2^x \frac{|K_0(u)|^2}{u^3} du \right)^{1/2} \left( \int_{x^\varepsilon}^x \frac{du}{u} \right)^{1/2} \leq \\ &\leq \frac{1}{\varepsilon} \left( \frac{1}{\log x} \int_2^x \frac{|K_0(u)|^2}{u^3} du \right)^{1/2}. \end{aligned}$$

Put  $\sigma = 1 + \frac{1}{\log x}$ . Then

$$\int_2^x \frac{|K_0(u)|^2}{u^3} du \leq e^2 \int_2^x \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}} du.$$

Since  $(s = \sigma + it)$

$$\int_0^{\infty} K_0(e^u) e^{-us} e^{iut} du = \frac{F'_0(s)}{F_0(s)} F'(s),$$

where  $F(s) = H(s)F_0(s)$  and

$$(4.2) \quad F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \quad F_0(s) = \sum_{n=1}^{\infty} \frac{f_0(n)}{n^s}$$

we conclude, by Parseval's equality,

$$(4.3) \quad \int_1^{\infty} \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}} du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{F'_0(s)}{F_0(s)} \right|^2 |F(s)|^2 \frac{dt}{|s|^2}.$$

## 5. Main result

We assume that  $f, g \in \mathcal{M}'_1 \cup \mathcal{M}_4$  where  $g \geq 0$  and  $|f| \leq g$ . Put  $g = h_1 * g_0$  where  $g_0$  is exponentially multiplicative with  $g_0(p) = g(p)$  ( $p$  prime) and let  $G(s) = H_1(s)G_0(s)$ . Consider

$$(5.1) \quad \sum_p \frac{g(p) - \operatorname{Re} f(p) p^{-it}}{p}, \quad t \in \mathbb{R}.$$

In this paper we deal with the case that (5.1) diverges for every  $t \in \mathbb{R}$ . Then  $|F(s)| = o(G(\sigma))$  uniformly for all  $t \leq K$  as  $\sigma \rightarrow 1$ . By (4.3) we have

$$(5.2) \quad \int_{-\infty}^{\infty} \left| \frac{F'_0(s)}{F_0(s)} \right|^2 \frac{|F(s)|^2}{|s|^2} dt \leq o(G^2(\sigma)) \int_{|t| \leq K} \left| \frac{F'_0(s)}{F_0(s)} \right| \frac{dt}{|s|^2} + G^2(\sigma) \int_{|t| \geq K} \left| \frac{F'_0(s)}{F_0(s)} \right| \frac{dt}{|s|^2}.$$

The integrals in (5.2) can be estimated by

$$\sum_{\substack{k \in \mathbb{Z} \\ |k| \leq K}} \frac{1}{k^2 + 1} \int_{|t-k| \leq \frac{1}{2}} \left| \frac{F'_0(s)}{F_0(s)} \right|^2 dt$$

and

$$\sum_{\substack{k \in \mathbb{Z} \\ |k| \geq K}} \frac{1}{k^2 + 1} \int_{|t-k| \leq \frac{1}{2}} \left| \frac{F'_0(s)}{F_0(s)} \right|^2 dt,$$

respectively. Further, we have

$$\frac{F'_0(s)}{F(s)} = - \sum_p \frac{f(p) \log p}{p^s}$$

and

$$- \sum_p \frac{\log p}{p^s} = - \frac{\zeta'(s)}{\zeta(s)} + O(1)$$

for  $\sigma > 1$ . Since  $|f(p)| \leq Bb_n$  in the proof of Theorem 4.10 in G. Tenenbaum [11] we conclude

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{F'_0}{F_0} \left( 1 + \frac{1}{\log x} + ik + it \right) \right|^2 dt \leq 3B^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\zeta'}{\zeta} \left( 1 + \frac{1}{\log x} + ik + it \right) \right|^2 dt \ll \log x.$$

Thus the right-hand side of (5.2) can be estimated by

$$(5.3) \quad o(\log x(G(\sigma)^2) + O(\log x(G(\sigma)^2)) \sum_{|k| \geq K} \frac{1}{k^2 + 1}.$$

Since

$$G(\sigma) \ll \exp \left( \sum_{p \leq x} \frac{g(p)}{p} \right)$$

we finish the proof of

**Theorem 5.1.** *Let  $f, g \in \mathcal{M}_1 \cup \mathcal{M}_4$  such that  $|f| \leq g$ . If (5.1) diverges for all  $t \in \mathbb{R}$  then*

$$\sum_{n \leq x} f(n) = o \left( \frac{x}{\log x} \exp \left( \sum_{p \leq x} \frac{g(p)}{p} \right) \right).$$

**Remark 5.1.** (i) If  $g = \mathbf{1}$  and  $|f(n)| \leq 1$  for all  $n \in \mathbb{N}$  then Theorem 5.1. contains the corresponding results of G. Halász [2], K.-H. Indlekofer [3], [4], [5]. Especially, if  $\mu$  denotes the Möbius function we obtain the Prime Number Theorem in the form

$$\sum_{n \leq x} \mu(n) = o(x) \text{ as } x \rightarrow \infty.$$

(ii) If there is  $\mathbb{P}_1 \subset \mathbb{P}$  such that, for some  $\alpha > 0$ ,

$$(5.4) \quad \sum_{\substack{p \leq x \\ p \in \mathbb{P}_1}} \frac{g(p) \log p}{p} \sim \alpha \log x \quad \text{as } x \rightarrow \infty$$

then

$$\sum_{n \leq x^\varepsilon} \frac{g_0(n)}{n} \ll \varepsilon^\alpha \exp \left( \sum_{p \leq x} \frac{g(p)}{p} \right)$$

and Theorem 5.1 contains the corresponding results of P.D.T.A. Elliott [1], K.-H. Indlekofer, I. Kátai, R. Wagner [6], E. Kaya [9].

**Remark 5.2.** Let  $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$  and  $g_0 = g_1 * g_2$ , where

$$g_i(p) = \begin{cases} g(p) & \text{if } p \in \mathbb{P}_i \\ 0 & \text{if } p \notin \mathbb{P}_i \end{cases} \quad (i = 1, 2).$$

If (5.4) holds for some  $\alpha > 0$ , then

$$\sum_{n \leq x} g_0(n) \geq \sum_{n \leq x^{1/2}} g_2(n) \sum_{m \leq \frac{x}{n}} g_1(m).$$

A Result of E. Wirsing [12] gives for  $\alpha$  typical inner sum the asymptotic estimate

$$\{1 + o(1)\} \frac{e^{-\gamma\alpha}}{\Gamma(\alpha)} \frac{x}{n \log \frac{x}{n}} \prod_{p \leq \frac{x}{n}} \left( 1 + \frac{g_1(p)}{p} \right).$$

Thus the double sum exceeds a constant multiple of

$$\frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{g_1(p)}{p} \right) \sum_{n \leq x^{1/2}} \frac{g_2(n)}{n}.$$

Then

$$\sum_{n \leq x} g(n) \gg \sum_{n \leq x} g_0(n) \gg \frac{x}{\log x} \exp \left( \sum_{p \leq x} \frac{g(n)}{n} \right).$$

**Remark 5.3.** In [8] we deal with the case that (5.1) converges for some  $t = t_0 \in \mathbb{R}$ .

## References

- [1] **Elliott, P.D.T.A.**, Multiplicative function mean values: Asymptotic estimates, *Functiones et Approximatio*, **56(2)** (2017), 217–238.
- [2] **Halász, G.**, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta. Mat. Sci. Hung.*, **219** (1986), 365–403.
- [3] **Indlekofer, K.-H.**, On the prime number theorem, *Annales Univ. Sci. Budapest., Sect. Comp.*, **27** (2007), 167–185.
- [4] **Indlekofer, K.-H.**, Identities in the convolution arithmetic of number theoretical functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **28** (2008), 203–325.
- [5] **Indlekofer, K.-H.**, On a quantitative form of Wirsing’s mean-value theorem for multiplicative functions, *Publ. Math. Debrecen*, **75** (2009), 143–157.
- [6] **Indlekofer, K.-H., I. Kátai and R. Wagner**, A comparative result for multiplicative functions, *Liet. Matem. Rink.*, **41(2)** (2001), 183–201.
- [7] **Indlekofer, K.-H.**, A new proof of a result of Wirsing for multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **55** (2023), 151–174.
- [8] **Indlekofer, K.-H. and E. Kaya**, Estimates for multiplicative functions, II. (In preparation).
- [9] **Kaya, E.**, On some results of Indlekofer for multiplicative functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **54** (2023), 199–209.
- [10] **Postnikov, A.G.**, *Introduction to Analytic Number Theory*, Translation of mathematical monographs, Vol. 68., American Math. Soc., 1988.
- [11] **Tenenbaum, G.**, *Introduction to Analytic and Probabilistic Number Theory*, Graduate Studies in Mathematics, Vol. 163., American Math. Soc., 2015.
- [12] **Wirsing, E.**, Das asymptotische Verhalten von Summen über multiplikative Funktionen II, *Acta Math. Acad. Sci. Hung.*, **18** (1967), 411–467.

**K.-H. Indlekofer**

Faculty of Computer Science  
Electrical Engineering and Mathematics  
University of Paderborn  
Warburger Straße 100  
D-33098 Paderborn, Germany  
k-heinz@math.uni-paderborn.de

**E. Kaya**

kayaerdener@mersin.edu.tr

