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THE BOUNDEDNESS OF THE HARDY–LITTLEWOOD MAXIMAL OPERATOR ON ORLICZ–LORENTZ–KARAMATA SPACES

Zhiwei Hao (Xiangtan, People's Republic of China)

Guangheng Xie and Dejian Zhou

(Changsha, People's Republic of China)

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Abstract. In this paper, we prove that the Hardy–Littlewood maximal operator is bounded on the Orlicz–Lorentz–Karamata space $L_{\Phi,q,b}(\mathbb{R}^n)$. More precisely, we give a sufficient condition for the boundedness of the Hardy–Littlewood maximal operator on $L_{\Phi,q,b}(\mathbb{R}^n)$ when a Young function $\Phi \in \nabla_2$, $1 \leq q \leq \infty$ and b is a slowly varying function.

1. Introduction

The aim and the idea of Karamata's paper [13], which defined and gave the basic properties of the new classes of functions, called slowly varying function. Karamata proved some fundamental theorems such as the Representation Theorem, the Uniform Convergence Theorem and the Characterization Theorem (see [14]). These results are the basis for the theory and numerous applications. In 2000, Edmunds et al. [4] introduced a new class of function spaces, that is, Lorentz–Karamata spaces. we briefly recall the definition of the Lorentz–Karamata space as follows (see Section 2 for any unexplained terminology): let

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$0 < p < \infty$, $0 < q \leq \infty$ and b be a slowly varying function. The Lorentz–Karamata space $L_{p,q,b}(\mathbb{R}^n)$, consists of the set of all measurable functions f on $\mathbb{R}^n$ with $\|f\|_{p,q,b} < \infty$, where

$$\|f\|_{p,q,b} = \begin{cases} \left(\int\limits_0^\infty \left(t^{\frac{1}{p}} \gamma_b(t) f^*(t)\right)^q \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} \gamma_b(t) f^*(t) & \text{if } q = \infty. \end{cases}$$

By taking different p, q and b, these spaces generalize the classical Lebesgue space, Lorentz spaces, Zygmund spaces, Lorentz–Zygmund spaces and the generalized Lorentz–Zygmund space. This class not only offers a more general and unified insight for these families of spaces, but also provides a framework in which it is easier to appreciate the central issues of different results, see [3, 5, 6, 7, 11, 12, 15, 17] and the references therein.

As a generalization of the Lorentz–Karamata space, Hao et al. [9] introduced the definition of Orlicz–Lorentz–Karamata spaces $L_{\Phi,q,b}$, where Φ is an Orlicz function (A function Φ is said to be an Orlicz function, if it is non-decreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t > 0$ and $\Phi(t) \to \infty$ when $t \to \infty$), $0 < q \leq \infty$ and b is a slowly varying function. Note that if $b \equiv 1$, the space $L_{\Phi,q,b}$ gives to the Orlicz–Lorentz space $L_{\Phi,q}$ studied in [8]; if $q = \infty$, the space $L_{\Phi,q,b}$ becomes the weak Orlicz–Karamata space introduced in [19]; if $q = \infty$ and $b \equiv 1$, the space $L_{\Phi,q,b}$ goes back to the weak Orlicz space showed in [16]. For more values of Φ, q and b, see Section 2.

The Orlicz–Lorentz–Karamata space is much more wider than the above spaces. The development of different space theory enriches the theory of harmonic analysis and the Hardy–Littlewood maximal operator has obtained a mount of investigation. For example, Liu and Wang [16] studied the boundedness of the Hardy–Littlewood maximal operator and other operators on weak Orlicz spaces. Very recently, Hatano et al. [10] investigated the boundedness of the Hardy–Littlewood maximal operator on Orlicz–Lorentz spaces, which extended the result of [16]. As we all known, the Hardy–Littlewood maximal operator has many elegant properties [2] and often plays a key role in many quantitative estimations. It can control various operators appeared in harmonic analysis, and therefore, its boundedness is of great importance. Motivated by this, we study the Hardy–Littlewood maximal operator on Orlicz–Lorentz–Karamata spaces in this article. More precisely, this paper is to show that the Hardy–Littlewood maximal operator is bounded on the Orlicz–Lorentz–Karamata space $L_{\Phi,q,b}(\mathbb{R}^n)$ for a Young function $\Phi \in \nabla_2$, $1 \leq q \leq \infty$ and b is a slowly varying function. It is worthwhile to mention that our result improves the boundedness of the Hardy–Littlewood maximal operator from Liu and Wang [16] when $b \equiv 1$ and $q = \infty$ and Hatano et al. [10] when $b \equiv 1$, respectively.

At the end of this section, we make some conventions. Throughout this paper, the symbol $f \lesssim g$ means that there exists a positive constant C such that $f \le Cg$. If $f \lesssim g \lesssim f$, then we write $f \approx g$ and say that f is equivalent to g. The constant C_p depends only on p and may be different from line to line.

2. Preliminaries

In this section, we introduce some notations and lemmas that will be used in next section.

2.1. The Hardy–Littlewood maximal operator

The *Hardy–Littlewood maximal operator* M is defined by setting, for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$,

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int\limits_Q |f(y)| dy, \tag{2.1}$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$, whose edges are parallel to the coordinate axes of $\mathbb{R}^n$, that contain x.

The well-known theorem of Hardy, Littlewood and Wiener states that if $f \in L_p(\mathbb{R}^n)$, then

$$\|Mf\|_p \le C_{p,n} \|f\|_p \quad 1 < p \le \infty, \tag{2.2}$$

where the constant $C_{p,n}$ depends only on p and n, see [18].

2.2. Young functions

For a function $\Phi : [0, \infty] \to [0, \infty]$, let

$$a(\Phi) = \sup \left\{ t \ge 0 : \Phi(t) = 0 \right\} \quad \text{and} \quad b(\Phi) = \inf \left\{ t \ge 0 : \Phi(t) = \infty \right\}.$$

An increasing function $\Phi : [0, \infty] \to [0, \infty]$ is called a Young function, if it satisfies the following properties:

(i) $0 \le a(\Phi) < \infty$, $0 < b(\Phi) \le \infty$;

(ii) $\lim_{t \to +0} \Phi(t) = \Phi(0) = 0$;

(iii) Φ is convex on $[0, b(\Phi))$;

(iv) if $b(\Phi) = \infty$, then $\lim_{t \to \infty} \Phi(t) = \Phi(\infty) = \infty$;

(v) if $b(\Phi) < \infty$, then $\lim_{t \to b(\Phi) - 0} \Phi(t) = \Phi(b(\Phi))$.

A Young function $\Phi : [0,\infty] \to [0,\infty]$ is said to satisfy the Δ_2-condition or the doubling condition, denoted by $\Phi \in \Delta_2$, if there exists a constant $\alpha \geq 1$ such that

$$\Phi(2r) \leq \alpha\Phi(r), \quad \forall\, r > 0.$$

A Young function $\Phi : [0,\infty] \to [0,\infty]$ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a constant $\alpha > 1$, called the ∇_2-constant, such that

$$\Phi(r) \leq \frac{1}{2\alpha}\Phi(\alpha r), \quad \forall\, r > 0.$$

Obviously, if $\Phi \in \nabla_2$, then there exists a constant $p \in (1,\infty)$ such that the function $t \to t^{-1/p}\Phi^{-1}(t)$ is equivalent to a non-increasing function.

Lemma 2.1. ([10]) *Let Φ be a Young function. $\Phi \in \nabla_2$ if and only if there exists a constant $\alpha > 1$ such that*

$$\Phi^{-1}(2\alpha u) \leq \alpha\Phi^{-1}(u), \quad \forall\, u \geq 0.$$

In this case, α can be taken as the ∇_2-constant of Φ.

2.3. Slowly varying functions

A Lebesgue measurable function $b : [1,\infty) \to (0,\infty)$ is said to be a slowly varying function, if for any given $\epsilon > 0$, the function $t^{\epsilon}b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon}b(t)$ is equivalent to a non-increasing function on $[1,\infty)$.

Let b be a slowly varying function on $[1,\infty)$. For convenience, we define

$$\gamma_b(t) = b(\max\{t, 1/t\}), \quad t \in (0,\infty).$$

The useful properties on slowly varying function are given below.

Lemma 2.2. ([3]) *Let b be a slowly varying function. Then the following conclusions hold:*

(i) *For any given $\epsilon > 0$, the function $t^{\epsilon}\gamma_b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon}\gamma_b(t)$ is equivalent to a non-increasing function on $(0,\infty)$.*

(ii) *For any $r > 0$,*

$$\gamma_b(rt) \approx \gamma_b(t), \quad t > 0.$$

2.4. Orlicz–Lorentz–Karamata spaces

Now we present the definition of Orlicz–Lorentz–Karamata spaces. Denote by $L^0(\mathbb{R}^n)$ the space of all measurable functions.

Definition 2.1. Let Φ be an Orlicz function, b be a slowly varying function and $0 < q \leq \infty$. The Orlicz–Lorentz–Karamata space $L_{\Phi,q,b}(\mathbb{R}^n)$, consists of the set of all functions $f \in L^0(\mathbb{R}^n)$ with $\|f\|_{\Phi,q,b} < \infty$, where

$$\|f\|_{\Phi,q,b} = \begin{cases} \left(\int\limits_0^\infty \left(\frac{1}{\Phi^{-1}(1/t)} \gamma_b(t) f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{t>0} \frac{1}{\Phi^{-1}(1/t)} \gamma_b(t) f^*(t) & \text{if } q = \infty. \end{cases}$$

Here $f^*(t) = \inf \{ s > 0 : |\{ x \in \mathbb{R}^n : |f(x)| > s \}| \leq t \}$ $(\inf \emptyset = \infty)$ is the non-incre-asing rearrangement function of f on $(0, \infty)$.

Note that if $\Phi(t) = t^p$ for $0 < p < \infty$, the space $L_{\Phi,q,b}$ gives to the Lorentz–Karamata space $L_{p,q,b}$; if $\Phi(t) = t^p$ for $0 < p < \infty$ and $b \equiv 1$, the space $L_{\Phi,q,b}$ is the classical Lorentz space $L_{p,q}$. Also, if $\Phi(t) = t^q$ for $0 < q < \infty$ and $b \equiv 1$, then the space $L_{\Phi,q,b}$ is the usual Lebesgue space L_q.

3. Main results

We shall provide the boundedness of the Hardy–Littlewood maximal operator on the Orlicz–Lorentz–Karamata space.

Theorem 3.1. *Let Φ be a Young function with $\Phi \in \nabla_2$, $1 \leq q \leq \infty$ and b be a slowly varying function. If $f \in L_{\Phi,q,b}(\mathbb{R}^n)$, then*

$$\|Mf\|_{\Phi,q,b} \lesssim \|f\|_{\Phi,q,b}.$$

Before proving Theorem 3.1, we recall the definition of the generalized Lorentz space.

Definition 3.1. Let $0 < q \leq \infty$ and $\phi : (0, \infty) \to (0, \infty)$ be a measurable function. We define the generalized Lorentz space $\Lambda_{\phi,q}(\mathbb{R}^n)$ by the set of all functions $f \in L^0(\mathbb{R}^n)$ with the finite quasi-norm

$$\|f\|_{\Lambda_{\phi,q}} = \begin{cases} \left(\int\limits_0^\infty \left(\phi(t) f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty, \\ \operatorname*{ess\,sup}_{t>0} \phi(t) f^*(t) & \text{if } q = \infty. \end{cases}$$

Remark 3.1. Let $0 < q \leq \infty$. If

$$\phi(t) = \frac{1}{\Phi^{-1}(1/t)} \gamma_b(t), \tag{3.1}$$

then $\Lambda_{\phi,q}(\mathbb{R}^n)$ is reduced to the Orlicz–Lorentz–Karamata space $L_{\Phi,q,b}(\mathbb{R}^n)$.

Proposition 3.2. ([1, Corollary 1.9]) *Let $1 \le q < \infty$ and $\phi : (0,\infty) \to (0,\infty)$ be a measurable function. Then M is bounded on $\Lambda_{\phi,q}(\mathbb{R}^n)$ if and only if for every $r > 0$,*

$$\int_r^\infty \Big(\frac{\phi(t)}{t}\Big)^q \frac{dt}{t} \lesssim \frac{1}{r^q} \int_0^r \phi(t)^q \frac{dt}{t}. \tag{3.2}$$

Now we begin to prove Theorem 3.1.

Proof. Firstly, we prove the case of $1 \le q < \infty$. It suffices to prove that ϕ, given by (3.1), satisfies (3.2) of Proposition 3.2. Fix $r > 0$, we get

$$\int_r^\infty \Big(\frac{1}{t\Phi^{-1}(1/t)}\gamma_b(t)\Big)^q \frac{dt}{t} = \int_1^\infty \Big(\frac{1}{rt\Phi^{-1}(1/rt)}\gamma_b(rt)\Big)^q \frac{dt}{t}$$

and

$$\int_0^r \Big(\frac{1}{\Phi^{-1}(1/t)}\gamma_b(t)\Big)^q \frac{dt}{t} = \int_0^1 \Big(\frac{1}{\Phi^{-1}(1/rt)}\gamma_b(rt)\Big)^q \frac{dt}{t}$$

by making change of variables. Let α be the ∇_2-constant. By using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\int_r^\infty \Big(\frac{1}{t\Phi^{-1}(1/t)}\gamma_b(t)\Big)^q \frac{dt}{t} \le \\
\le\ & \sum_{j=1}^\infty \int_{(2\alpha)^{j-1}}^{(2\alpha)^j} \Big(\frac{1}{rt\Phi^{-1}(1/r(2\alpha)^j)}\gamma_b(rt)\Big)^q \frac{dt}{t} \lesssim \\
\lesssim\ & \sum_{j=1}^\infty \int_{(2\alpha)^{j-1}}^{(2\alpha)^j} \Big(\frac{1}{r(2\alpha)^{j-1}\Phi^{-1}(1/r(2\alpha)^j)}\gamma_b\big(r(2\alpha)^{j-1}\big)\Big)^q \frac{dt}{t} \le \\
\le\ & \sum_{j=1}^\infty \Big(\frac{1}{r(2\alpha)^{j-1}\Phi^{-1}(1/r)}\Big)^q \Big(\log(2\alpha)^j - \log(2\alpha)^{j-1}\Big)\alpha^{qj}\gamma_b\big(r(2\alpha)^{j-1}\big)^q \approx \\
\approx\ & \log(2\alpha)\Big(\frac{1}{r\Phi^{-1}(1/r)}\gamma_b(r)\Big)^q \sum_{j=1}^\infty \frac{1}{(2\alpha)^{q(j-1)}} \cdot \alpha^{qj} \approx \\
\approx\ & \frac{1}{r^q}\Big(\frac{1}{\Phi^{-1}(1/r)}\Big)^q \gamma_b(r)^q
\end{aligned}$$

and

$$\int_0^r \Big(\frac{1}{\Phi^{-1}(1/t)}\gamma_b(t)\Big)^q \frac{dt}{t} \geq$$

$$\geq \sum_{j=1}^{\infty} \int_{\frac{1}{(2\alpha)^j}}^{\frac{1}{(2\alpha)^{j-1}}} \Big(\frac{1}{\Phi^{-1}((2\alpha)^j/r)}\gamma_b(rt)\Big)^q \frac{dt}{t} \geq$$

$$\geq \sum_{j=1}^{\infty} \frac{1}{\alpha^{qj}} \Big(\frac{1}{\Phi^{-1}(1/r)}\Big)^q \cdot r \int_{\frac{1}{(2\alpha)^j}}^{\frac{1}{(2\alpha)^{j-1}}} \Big((rt)^{-\frac{1}{q}}\gamma_b(rt)\Big)^q dt \gtrsim$$

$$\gtrsim \Big(\frac{1}{\Phi^{-1}(1/r)}\Big)^q \cdot \sum_{j=1}^{\infty} \Big(\frac{1}{(2\alpha)^{j-1}} - \frac{1}{(2\alpha)^j}\Big) \cdot (2\alpha)^{j-1} \cdot \frac{1}{\alpha^{qj}} \cdot \gamma_b\Big(\frac{r}{(2\alpha)^{j-1}}\Big)^q \approx$$

$$\approx \Big(\frac{1}{\Phi^{-1}(1/r)}\Big)^q \gamma_b(r)^q \sum_{j=1}^{\infty} \frac{1}{\alpha^{qj}} \approx$$

$$\approx \Big(\frac{1}{\Phi^{-1}(1/r)}\Big)^q \gamma_b(r)^q.$$

It follows from the above inequalities that ϕ satisfies (3.2).

Secondly, we prove the case of $q = \infty$. According to Theorem 3.4 in [10], we know that if the inequality

$$\text{(3.3)} \qquad \operatorname*{ess\,sup}_{t>0} \frac{\phi(t)}{t} \int_0^t \frac{ds}{\operatorname*{ess\,sup}_{0<\tau<s} \phi(\tau)} < \infty$$

holds, then one can see that M is bounded on $\Lambda_{\phi,\infty}(\mathbb{R}^n)$. Let ϕ be as in (3.1). Now we verify that ϕ satisfies (3.3). Fix $t > 0$. Since $\Phi \in \nabla_2$, then there exists a constant $p \in (1, \infty)$ such that $t^{-1/p}\Phi^{-1}(t)$ is equivalent to a non-increasing function on $(0, \infty)$. Hence, $\frac{\Phi^{-1}(1/t)}{(1/t)^{1/p}}$ is equivalent to a non-decreasing function on $(0, \infty)$. We estimate

$$\frac{\gamma_b(t)}{t\Phi^{-1}(1/t)} \int_0^t \frac{1}{\sup_{0<\tau<s} \frac{1}{\Phi^{-1}(1/\tau)}\gamma_b(\tau)} ds =$$

$$= \frac{\gamma_b(t)}{t\Phi^{-1}(1/t)} \int_0^t \frac{1}{\sup_{0<\tau<s} \frac{1}{\Phi^{-1}(1/\tau)} \cdot \frac{1}{\tau} \cdot \tau\gamma_b(\tau)} ds \leq$$

$$
\begin{aligned}
&\le \frac{\gamma_b(t)}{t\Phi^{-1}(1/t)} \int_0^t \frac{\Phi^{-1}(1/s)\cdot s}{s\cdot\gamma_b(s)} ds = \\
&= \frac{\gamma_b(t)}{t\Phi^{-1}(1/t)} \int_0^t \frac{\Phi^{-1}(1/s)}{(1/s)^{1/p}} \frac{1}{s^{-\frac{p-1}{2p}}\gamma_b(s)} s^{-\frac{p+1}{2p}} ds \lesssim \\
&\lesssim \frac{\gamma_b(t)}{t\Phi^{-1}(1/t)} \frac{\Phi^{-1}(1/t)}{(1/t)^{1/p}} \frac{1}{t^{-\frac{p-1}{2p}}\gamma_b(t)} \int_0^t s^{-\frac{p+1}{2p}} ds = \\
&= \frac{\gamma_b(t)}{t\Phi^{-1}(1/t)} \frac{2p}{p-1} \frac{\Phi^{-1}(1/t)}{\gamma_b(t)} t = \frac{2p}{p-1},
\end{aligned}
$$

which proves (3.3). The proof is complete. ■

Especially for $\Phi(t) = t^p$ $(1 < p < \infty)$ in Theorem 3.1, we obtain the following result.

Corollary 3.1. *Let $1 < p < \infty$, $1 \le q \le \infty$ and b be a slowly varying function. If $f \in L_{p,q,b}(\mathbb{R}^n)$, then*

$$\|Mf\|_{p,q,b} \lesssim \|f\|_{p,q,b}.$$

If we take $b \equiv 1$ in Theorem 3.1, then the following results hold:

Corollary 3.2. ([10]) *Let Φ be a Young function with $\Phi \in \nabla_2$ and $1 \le q \le \infty$. If $f \in L_{\Phi,q}(\mathbb{R}^n)$, then*

$$\|Mf\|_{\Phi,q} \lesssim \|f\|_{\Phi,q}.$$

Remark 3.2. We refer the reader to [16] for the boundedness of the Hardy–Littlewood maximal operator on $L_{\Phi,\infty}(\mathbb{R}^n)$ in the case of $\Phi \in \Delta_2 \cap \nabla_2$. It is noteworthy that Φ does not need to satisfy $\Phi \in \Delta_2$ in Corollary 3.2 when $q = \infty$. Hence, our results improve the boundedness of the Hardy–Littlewood maximal operator in [16].

When $\Phi(t) = t^p$ $(1 < p < \infty)$ in Corollary 3.2, we get the next conclusion.

Corollary 3.3. *Let $1 < p < \infty$ and $1 \le q \le \infty$. If $f \in L_{p,q}(\mathbb{R}^n)$, then*

$$\|Mf\|_{p,q} \lesssim \|f\|_{p,q}.$$

In particular, if we consider the case $\Phi(t) = t^p$ and $1 < p = q \le \infty$, we have the boundedness of the Hardy–Littlewood maximal operator on the Lebesgue space $L_p(\mathbb{R}^n)$, see (2.2).

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Z. Hao
Hunan University of Science and Technology
School of Mathematics and Computing Science
Xiangtan
People's Republic of China
haozhiwei@hnust.edu.cn

G. Xie and D. Zhou
Central South University
School of Mathematics and Statistics
Changsha
People's Republic of China
xieguangheng@csu.edu.cn and zhoudejian@csu.edu.cn