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STABILITY OF THE BRUSSELATOR MODEL WITH DELAYED FEEDBACK CONTROL

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Abstract. The classical diffusive Brusselator model has been extensively discussed even under the presence of certain types of discrete time-delays. However, models assuming delayed feedback have only been studied under certain highly restrictive conditions due to the computational complexity. In this paper, the stability of the unique equilibrium solution of the model is examined in a more general case, leaving the mentioned conditions out of consideration. When this type of delay is introduced, the original system of differential equations (system without delay) also substantially changes. Therefore, the ordinary system is considered in the absence of delay, and then it is examined whether the stability of the equilibrium solution changes when delay is assumed. In the investigations the focus is mainly on the existence of Hopf bifurcation.

1. Introduction

The importance of oscillations in biochemical systems has been emphasized by a number of authors. One of the best known, the so-called Brusselator model, is a theoretical model of a kind of autocatalytic reaction, first studied by Prigogine and Lefever in 1968. The interesting and diverse behaviour of the

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model has led many researchers to investigate it analytically and numerically. The kinetic system associated with the diffusion Brusselator model consists of two differential equations

$$\left.\begin{aligned} \dot{X} &= A + X^2Y - (B+1)X, \\ \dot{Y} &= -X^2Y + BX \end{aligned}\right\} \tag{1.1}$$

where X and Y denote the concentrations of the interacting chemical species, $A > 0$ and $B > 0$ indicate two control parameters during the reaction process.

However, there is little research focusing on system (1.1) assuming delayed feedback control

$$\left.\begin{aligned} \dot{X} &= A + X^2Y - (B+1)X + \sigma_1 X(\cdot - \tau), \\ \dot{Y} &= -X^2Y + BX + \sigma_2 Y(\cdot - \tau) \end{aligned}\right\} \tag{1.2}$$

where σ_1, $\sigma_2 \in (0, 1)$ refer to the delayed feedback parameters, which are the strength of the feedback control. In the presence of diffusion, in [1] Alfifi studied the occurrence of Hopf bifurcation in case of $\sigma_1 = \sigma_2$ numerically, in [5] Zuo and Wei imposed a local delayed feedback control, in [2] Ji, Shen and Mao investigated the $\sigma_2 = 0$ case. So, the bifurcation analysis of the model in (1.2) in the absence and presense of diffusion is not studied in general.

This paper is devoted to the study of a kinetic system without diffusion and is organised as follows. Firstly, we consider the $\tau = 0$ case and prove that the model (1.2) has a unique equilibrium solution with positive coordinates. The stability of the constant solution and the possibility of a Hopf bifurcation are investigated. Then we discuss the same topics in the presence of delay, i.e. in case of system (1.2).

2. Stability analysis and Hopf bifurcation

2.1. The system without delay

The studied model consists the following system of differential equations

$$\left.\begin{aligned} \dot{X} &= A + X^2Y - (B+1-\sigma_1)X, \\ \dot{Y} &= -X^2Y + BX + \sigma_2 Y. \end{aligned}\right\} \tag{2.1}$$

Making the right hand sides of (2.1) equal to zero it is easy to see that X^* is in the intersection of the null-clines

$$h_1(X) := \frac{(B+1-\sigma_1)X - A}{X^2} \qquad \text{and} \qquad h_2(X) := \frac{BX}{X^2 - \sigma_2} \qquad (X > 0).$$

If

- $B+1-\sigma_1 \le 0$, then for positive X we have $h_1(X)<0$;
- $B+1-\sigma_1 > 0$, then for positive X we have

$$h_1(X) > 0 \qquad \Longleftrightarrow \qquad X > \frac{A}{B+1-\sigma_1}.$$

Since the equivalence

$$h_2(X) > 0 \qquad \Longleftrightarrow \qquad X > \sqrt{\sigma_2}$$

is valid for positive X, system (2.1) could have an equilibrium point (at least) with positive coordinates (X^*, Y^*) only if

$$X^* > \max\left\{\frac{A}{B+1-\sigma_1}, \sqrt{\sigma_2}\right\}.$$

Adding the right hand sides of (2.1) it is easy to see that

$$Y^* = \frac{(1-\sigma_1)X^* - A}{\sigma_2} \tag{2.2}$$

must hold. Thus, if $\sigma_1 \ge 1$ then system (2.1) has no feasible equilibrium, because otherwise it would follow from (2.2) that $Y^* < 0$, which is a contradiction. Finally, Y^* is positive if and only if $X^* > A/(1-\sigma_1)$ holds, then, because $B+1-\sigma_1 > 1-\sigma_1$, system (2.1) has an equilibrium solution (at least) with positive coordinates (X^*, Y^*) if and only if

$$X^* > \max\left\{\frac{A}{1-\sigma_1}, \sqrt{\sigma_2}\right\}.$$

Because

$$h_1(X^*) = h_2(X^*) \qquad \Longleftrightarrow \qquad p(X^*) = 0,$$

where

$$p(x) := a_3x^3 + a_2x^2 + a_1x + a_0 \qquad (x \in \mathbb{R}) \tag{2.3}$$

with

$$a_3 := 1-\sigma_1, \qquad a_2 := -A, \qquad a_1 := -\sigma_2(B+1-\sigma_1), \qquad a_0 := A\sigma_2,$$

due to the Descartes's rule of signs (cf. [3]) in case of $B+1-\sigma_1 > 0$ and

- $\sigma_1 < 1$ the cubic polynomial of X^* has zero or two positive roots;
- $\sigma_1 = 1$ the quadratic polynomial of X^* has exactly one positive root.

Henceforth, it is assumed that

$$\sigma_1 < 1 \tag{2.4}$$

holds. Due to the positivity of the parameters and the assumption (2.4) we have

$$p(0) = A\sigma_2 > 0, \qquad p(\sqrt{\sigma_2}) = -B\sigma_2\sqrt{\sigma_2} < 0, \quad p\left(\frac{A}{1-\sigma_1}\right) = -\frac{\sigma_2 AB}{1-\sigma_1} < 0,$$

and

$$\lim_{x\to+\infty} p(x) = +\infty \tag{2.5}$$

holds. Therefore, p has exactly two positive roots X_1 and X_2 such that

$$X_1 \in \left(0, \min\left\{\sqrt{\sigma_2}, \frac{A}{1-\sigma_1}\right\}\right) \quad \text{and} \quad X_2 \in \left(\max\left\{\sqrt{\sigma_2}, \frac{A}{1-\sigma_1}\right\}, +\infty\right).$$

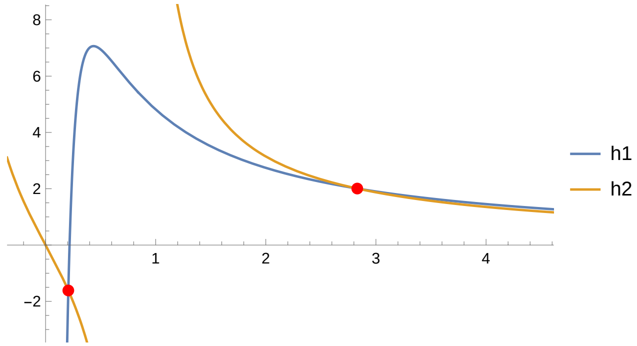

Figure 1. The case of two intersection points with parameter values $A = 1.340$, $B = 5.180$, $\sigma_1 = 0.026$, $\sigma_2 = 0.706$.

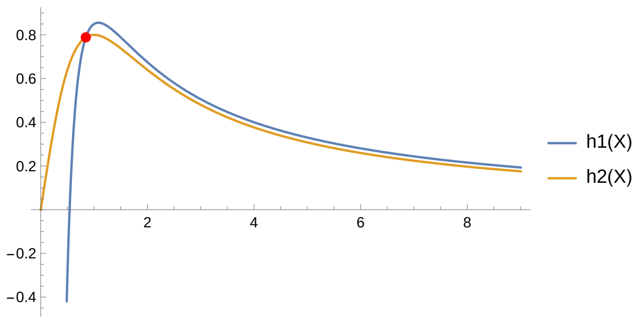

Figure 2. The case of one intersection point with parameter values $A = 1.340$, $B = 5.180$, $\sigma_1 = 1.000$, $\sigma_2 = 0.706$.

On the other hand this means that

$$Y_1 := \frac{(1-\sigma_1)X_1 - A}{\sigma_2} < 0, \qquad \text{resp.} \qquad Y_2 := \frac{(1-\sigma_1)X_2 - A}{\sigma_2} > 0,$$

i.e. the equilibrium (X_1, Y_1) is not chemically feasible, but (X_2, Y_2) is feasible (cf. Fig. 1). In the following the unique equilibrium is denoted by $(X^*, Y^*) :=$ $:= (X_2, Y_2)$ (cf. Fig. 2). This proves the following result.

Proposition 2.1. *Assume that A, B, σ_1, $\sigma_2 > 0$. Then, system (2.1) has a unique positive (chemically feasible) equilibrium (X^*, Y^*) if and only if $\sigma_1 < 1$ holds. Furthermore, X^* is the largest root of the polynomial (2.3) and Y^* can be calculated as (2.2).*

Using the idea of the position of the root X^*, we can prove an other lower bound for X^* under certain conditions. We shall use this result later.

Lemma 2.1. *If*

$$B > 1 - \sigma_1 \tag{2.6}$$

holds then the greatest positive zero of the polynomial p can be estimated as follows

$$X^* > \frac{2A}{B + 1 - \sigma_1}. \tag{2.7}$$

Proof. Assumption (2.6) implies that

$$\begin{aligned}
p\left(\frac{2A}{B+1-\sigma_1}\right) &= (1-\sigma_1)\cdot\left(\frac{2A}{B+1-\sigma_1}\right)^3 - A\cdot\left(\frac{2A}{B+1-\sigma_1}\right)^2 - \\
&\quad -\sigma_2(B+1-\sigma_1)\cdot\frac{2A}{B+1-\sigma_1} + A\sigma_2 = \\
&= \frac{8A^3(1-\sigma_1)}{(B+1-\sigma_1)^3} - \frac{4A^3}{(B+1-\sigma_1)^2} - 2A\sigma_2 + A\sigma_2 = \\
&= \frac{4A^3}{(B+1-\sigma_1)^2}\cdot\left(\frac{2(1-\sigma_1)}{B+1-\sigma_1} - 1\right) - A\sigma_2 = \\
&= \frac{4A^3}{(B+1-\sigma_1)^3}\cdot(-B+1-\sigma_1) - A\sigma_2 < 0.
\end{aligned}$$

Furthermore, since X^* is the largest positive root of p and due to (2.5) the estimation in (2.7) is valid. ■

Linearizing system (2.1) at (X^*, Y^*) we have

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = J(X^*, Y^*) \cdot \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.8}$$

where

$$J(X^*, Y^*) := \begin{bmatrix} -B - 1 + \sigma_1 + 2X^*Y^* & (X^*)^2 \\ B - 2X^*Y^* & -(X^*)^2 + \sigma_2 \end{bmatrix}$$

is the Jacobian of the right hand side of system (2.1) evaluated at the equilibrium (X^*, Y^*). The characteristic function of system (2.8) has the form

(2.9)

$$\begin{aligned} \Delta_{(2.8)}(z) &\equiv \det \begin{bmatrix} -B - 1 + \sigma_1 + 2X^*Y^* - z & (X^*)^2 \\ B - 2X^*Y^* & -(X^*)^2 + \sigma_2 - z \end{bmatrix} \equiv \\ &\equiv z^2 + d_1 z + d_0 \quad (z \in \mathbb{C}), \end{aligned}$$

where

$$\begin{aligned} d_1 &:= (X^*)^2 - \sigma_2 + B + 1 - \sigma_1 - 2X^*Y^*, \\ d_0 &:= (X^*)^2(1 - \sigma_1) - \sigma_2(B + 1 - \sigma_1 - 2X^*Y^*). \end{aligned}$$

The (unique) chemically feasible equilibrium (X^*, Y^*) is clearly asymptotically stable if and only if $d_1 > 0$ and $d_0 > 0$. Denoting the bifurcation parameter by μ, Hopf bifurcation occur at a certain value $\mu_0 \in \mathbb{R}$ if and only if $d_1(\mu_0) = 0$, $d_1'(\mu_0) \neq 0$ and $d_0(\mu_0) > 0$ hold (cf. [4]). Applying this result we prove the following

Theorem 2.2. *Assume that conditions (2.4) and (2.6), furthermore*

$$2\sigma_1 + 3\sigma_2 \neq 2 \tag{2.10}$$

hold. Then, for all values of the parameters (satisfying the assumption (2.10)), a Poincaré–Andronov–Hopf bifurcation occurs at $A = A_+^*$, where

$$A_+^* := -\frac{\sigma_2(1 + B - (\sigma_1 + \sigma_2)) + \left[X_+^*\right]^2 (-2 + 2\sigma_1 + \sigma_2)}{2X_+^*} \tag{2.11}$$

with

$$X_+^* := \sqrt{\frac{-1 + B + \sigma_1 + 2\sigma_2 + \sqrt{(-1 + B + \sigma_1)^2 + 8\sigma_2 B}}{2}}, \tag{2.12}$$

If, in addition $\sigma_1 + \sigma_2 > 1 + B$ also holds, then there is another critical value of A defined as

$$A_-^* := -\frac{\sigma_2(1 + B - (\sigma_1 + \sigma_2)) + \left[X_-^*\right]^2 (-2 + 2\sigma_1 + \sigma_2)}{2X_-^*} \tag{2.13}$$

with

$$(2.14)\qquad X^*_- := \sqrt{\frac{-1+B+\sigma_1+2\sigma_2-\sqrt{(-1+B+\sigma_1)^2+8\sigma_2 B}}{2}}$$

such that the unique positive equilibrium undergoes a Poincaré–Andronov–Hopf bifurcation when A varies and passes through $A^* \in \{A^*_-,\ A^*_+\}$.

Proof. Emphasising the dependence of the coordinates X^*, Y^* and the coefficients d_0 and d_1 on the bifurcation parameter A we introduce the notations

$$X^* = X^*(A), \quad Y^* = Y^*(A), \quad d_0 = d_0(A), \quad d_1 = d_1(A).$$

As a first step, the existence of a critical value A^* is shown, such that

$$(2.15)\qquad d_1(A^*) = [X^*(A^*)]^2 - \sigma_2 + B + 1 - \sigma_1 - 2X^*(A^*)Y^*(A^*) = 0$$

holds. Based on the equations

$$\left.\begin{aligned} X &= f_1(X,Y) := A + X^2Y - (B+1-\sigma_1)X, \\ Y &= f_2(X,Y) := -X^2Y + BX + \sigma_2 Y \end{aligned}\right\}$$

the identities

$$Y^* = \frac{(1-\sigma_1)X^* - A}{\sigma_2} \quad \text{and} \quad Y^* = \frac{BX^*}{[X^*]^2 - \sigma_2}$$

hold at the same time. Substituting $Y^* = BX^*/([X^*]^2 - \sigma_2)$ into the equation (2.15), we have that $d_1(A^*) = 0$ holds if and only if $x = X^*(A^*)$ is the root of the fourth order polynomial

$$g(x) := x^4 + x^2(1 - B - \sigma_1 - 2\sigma_2) + \sigma_2(\sigma_1 + \sigma_2 - 1 - B) \qquad (x \in \mathbb{R}).$$

Introducing $z := x^2$ it is obvious that the equation $g(x) = 0$ has a positive solution if and only if the second order polynomial

$$q(z) := z^2 + q_1 z + q_0 \qquad (z \in \mathbb{R})$$

has a positive root, where

$$q_1 := 1 - B - \sigma_1 - 2\sigma_2 \qquad \text{and} \qquad q_0 := \sigma_2(\sigma_1 + \sigma_2 - 1 - B).$$

The discriminant of q is positive due to the positivity of B and σ_2, because

$$q_1^2 - 4 \cdot q_0 = (-1 + B + \sigma_1)^2 + 8\sigma_2 B,$$

consequently q has two real roots. Studying the sign of the coefficients q_1 and q_0 it can be observed that

- $q(z) \not\equiv z^2$ under assumption (2.10);
- in case of $q_0 = 0$, the equality $B = 1 - \sigma_1 - \sigma_2$, and consequently the inequality $1 - \sigma_1 < \sigma_2$ holds, hence $q_1 = 2 - 2\sigma_1 - 3\sigma_2 < -\sigma_2 < 0$, and thus the polynomial $q(z) = z^2 + q_1 z$ has exactly one positive root, namely $-q_1$;
- similarly, if $q_1 = 0$ holds, then $B = 1 - \sigma_1 - 2\sigma_2 > 0$ implies that $q_0 < -\sigma_2 < 0$, hence the polynomial $q(z) = z^2 + q_0$ has one positive root $\sqrt{-q_0}$;
- q_0 and q_1 cannot be positive simultaneously, since the inequalities
$$1 - B - \sigma_1 - 2\sigma_2 > 0 \qquad \text{and} \qquad \sigma_2(-1 - B + \sigma_1 + \sigma_2) > 0$$
would make $-2B - \sigma_2 > 0$, which contradicts to the positivity of B and σ_2;
- in case of $q_0 < 0$, the polynomial has exactly one positive root due to the Descartes rule of signs;
- in case of $q_0 > 0$ and $q_1 < 0$, q has two positive roots due to the quadratic formula.

In summary, q has one or two positive real root(s) having the form

$$z_\pm := \frac{-q_1 \pm \sqrt{q_1^2 - 4q_0}}{2} = \frac{-1 + B + \sigma_1 + 2\sigma_2 \pm \sqrt{(-1 + B + \sigma_1)^2 + 8\sigma_2 B}}{2},$$

and $z_- < 0 < z_+$ and $z_+ > z_- > 0$ provided $q_0 \leq 0$ and $q_0 > 0$, respectively. This leads to the fact that the fourth order polynomial g has one or two positive roots, more precisely in case of

- $\sigma_1 + \sigma_2 \leq 1 + B$, the polynomial g has exactly one positive root, which is X_+^* defined in (2.12);
- $\sigma_1 + \sigma_2 > 1 + B$, g has two positive roots: X_+^*, resp. X_-^* defined in (2.12), resp. (2.14).

Henceforth, let $X^* \in \{X_+^*,\ X_-^*\}$.

On the other hand, substituting $Y^* = ((1 - \sigma_1)X^* - A^*)/\sigma_2$ into the equation $d_1 = 0$, we obtain that for the critical value A^* the equality

$$2A^* \cdot X^* + \sigma_2(1 + B - (\sigma_1 + \sigma_2)) + [X^*]^2(-2 + 2\sigma_1 + \sigma_2) = 0$$

must hold, i.e. $d_1(A)$ can be zero only if $A = A^*$ with

$$A^* = -\frac{\sigma_2(1 + B - (\sigma_1 + \sigma_2)) + [X^*]^2(-2 + 2\sigma_1 + \sigma_2)}{2X^*}.$$

Therefore, using the formulas X_+^* and X_-^* we conclude that in case of

- $\sigma_1 + \sigma_2 \leq 1 + B$, there is exactly one solution of the equation $d_1(A) = 0$ at $A = A_+^*$, where A_+^* is defined in (2.11);
- $\sigma_1 + \sigma_2 > 1 + B$, there exist two solutions of $d_1(A) = 0$: A_+^*, resp., A_-^* defined in (2.11), resp. (2.13).

Straightforward calculation shows that $d_1(A) = 0$ implies $d_0(A) > 0$, thus it is enough to study the derivative of $d_1(A)$ at $A = A^*$ with $A^* \in \{A_+^*, A_-^*\}$. The quantity $d_1'(A^*)$ is calculated using the implicit function theorem. We show that X^* is a continuously differentiable function of A on the interval $(0, +\infty)$. For this purpose, we denote the polynomial p in (2.3) by

$$P(x, A) := (1 - \sigma_1)x^3 - Ax^2 - \sigma_2(1 + B - \sigma_1)x + A\sigma_2,$$

$$(x \in (\sqrt{\sigma_2}, +\infty),\ A \in (0, +\infty))$$

thereby emphasising its dependence on x and A. Obviously, $P \in \mathfrak{C}^1(\mathbb{R}^2)$, and X^* is a real-valued function of A following from the uniqueness of the equilibrium point, furthermore for all $A \in (0, +\infty)$ the (clearly determined) point $(X^*(A), A)$ is on the zero level curve of P, i.e. $P(X^*(A), A) = 0$ holds for all $A \in (0, +\infty)$. This also means that the graph of the function X^* lies on the level curve $P(x, A) = 0$.

The derivative of P w.r.t. the first variable has the form

$$\partial_x P(x, A) = 3x^2(1 - \sigma_1) - 2Ax - \sigma_2(1 + B - \sigma_1). \tag{2.16}$$

Following from the fact that $P(X^*, A) = p(X^*) = 0$, i.e.

$$(1 - \sigma_1)[X^*]^3 - A[X^*]^2 - \sigma_2(1 + B - \sigma_1)X^* = -A\sigma_2$$

and

$$(1 - \sigma_1)[X^*]^3 - A[X^*]^2 + A\sigma_2 = \sigma_2(1 + B - \sigma_1)X^*$$

hold, the partial derivative $\partial_x P(x, A)$ can be simplified after the substitution $x = X^*$ as follows

$$\begin{aligned}
\partial_x P(X^*, A) &= 3[X^*]^2(1 - \sigma_1) - 2AX^* - \sigma_2(1 + B - \sigma_1) = \\
&= [X^*]^2(1 - \sigma_1) - AX^* - \sigma_2(1 + B - \sigma_1) + \\
&\quad +2(1 - \sigma_1)[X^*]^2 - AX^* = \\
&= \frac{(1 - \sigma_1)[X^*]^3 - A[X^*]^2 - \sigma_2(1 + B - \sigma_1)X^*}{X^*} + \\
&\quad +2(1 - \sigma_1)[X^*]^2 - AX^* = \\
&= \frac{-A\sigma_2}{X^*} + 2(1 - \sigma_1)[X^*]^2 - AX^* = \\
&= \frac{2(1 - \sigma_1)[X^*]^3 - A[X^*]^2 - A\sigma_2}{X^*}.
\end{aligned}$$

We are going to show that $\partial_x P(X^*, A) \neq 0$ holds. Indeed, $p(X^*) = 0$ and

$$2(1-\sigma_1)[X^*]^3 - A[X^*]^2 - A\sigma_2 = 0 \tag{2.17}$$

imply

$$(1-\sigma_1)[X^*]^3 + \sigma_2(1+B-\sigma_1)X^* - 2A\sigma_2 = 0.$$

Using Lemma 2.1 we have

$$(1-\sigma_1)[X^*]^3 + \sigma_2(1+B-\sigma_1)X^* - 2A\sigma_2 > (1-\sigma_1)[X^*]^3 > 0$$

which has the consequence that $\partial_x P(X^*, A)$ cannot vanish. Therefore, as a consequence of the implicit function theorem, for all $\widetilde{A} \in (0, +\infty)$ there exists a neighbourhood $K_{\widetilde{A}} \subset (0, +\infty)$ and a continuously differentiable function $\varphi \in \mathfrak{C}^1(K_{\widetilde{A}})$ such that $x = \varphi(A)$ holds on $K_{\widetilde{A}}$. Thanks to the uniqueness of X^* and the uniqueness of the function φ, $\varphi \equiv X^*$ is true on any neighbourhood $K_{\widetilde{A}}$, hence $X^* \equiv \varphi$ on $(0, +\infty)$, which has a consequence that X^* is a continuously differentiable function of A on the interval $(0, +\infty)$. The derivative of $X^*(A)$ is calculated as

$$x'(A) = -\frac{\partial_A P(x, A)}{\partial_x P(x, A)} = \frac{x^2 - \sigma_2}{(1-\sigma_1)x^2 + \sigma_2(1+B-\sigma_1)},$$

consequently $(X^*)'(A^*) = x'(A^*) > 0$.
Using the fact that

$$Y^*(A) = \frac{(1-\sigma_1)X^*(A) - A}{\sigma_2},$$

$d_1(A)$ can be written in the form

$$d_1(A) = \frac{\sigma_2 - 2(1-\sigma_1)}{\sigma_2} \cdot [X^*(A)]^2 + \frac{2A}{\sigma_2} \cdot X^*(A) + 1 + B - \sigma_1 - \sigma_2,$$

therefore the derivative of d_1 is given as follows

$$\begin{aligned} d_1'(A^*) &= 2 \cdot \frac{\sigma_2 - 2(1-\sigma_1)}{\sigma_2} \cdot (X^*(A^*))' \cdot X^*(A^*) + \\ &+ \frac{2A^*}{\sigma_2} \cdot (X^*(A^*))' + \frac{2}{\sigma_2} \cdot X^*(A^*) = \\ &= \frac{2}{\sigma_2} \cdot \frac{-P(X^*(A^*), A^*) + \sigma_2 X^*(A^*)([X^*(A^*)]^2 + 2 - 2\sigma_1 - \sigma_2)}{(1-\sigma_1)\,[X^*(A^*)]^2 + \sigma_2(1+B-\sigma_1)} = \\ &= 2X^*(A^*) \cdot \frac{[X^*(A^*)]^2 - \sigma_2 + 2(1-\sigma_1)}{(1-\sigma_1)\,[X^*(A^*)]^2 + \sigma_2(1+B-\sigma_1)} > 0. \end{aligned}$$

Hence, $d_1'(A^*) > 0$ for $A^* \in \{A_+^*,\ A_-^*\}$, which completes the proof. ∎

Example 2.3. Set $B = 8.23000$, $\sigma_1 = 0.22000$ and $\sigma_2 = 0.06000$. Then $\sigma_1 + \sigma_2 \leq 1 + B$, therefore at $A^* = 1.97594$ the unique positive equilibrium undergoes a Poincaré–Andronov–Hopf bifurcation. Fig. 3 shows the solution of system (2.1) with $A = 1.97000$ and $A = 1.98000$ and with initial conditions $X_0 = 2.70000$, $Y_0 = 3.00000$.

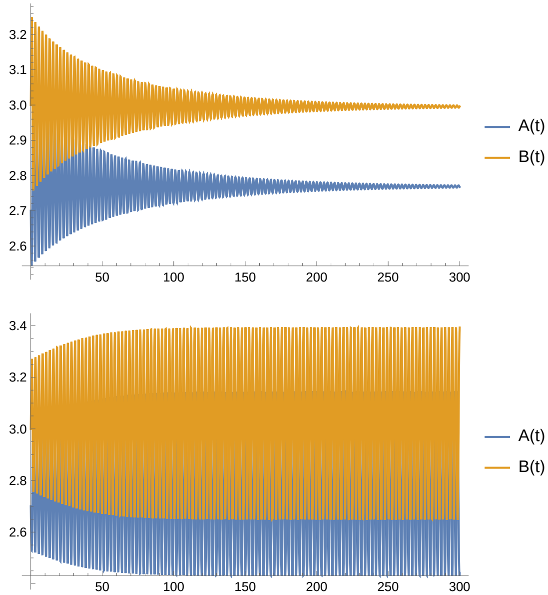

Figure 3. Solutions of system (2.1) with parameter values $B = 8.23$, $\sigma_1 = 0.22$, $\sigma_2 = 0.06$ and $A = 1.97$ (top image) resp. $A = 1.98$ (bottom image) and initial conditions $X_0 = 2.70$, $Y_0 = 3.0$.

2.2. The system with positive delay

In this section the stability of the equilibrium of the Brusselator model with delayed feedback control, i.e. system (1.2) is studied. We assume positive initial conditions

$$\Phi = (\Phi_1, \Phi_2) = \{\Phi \in \mathfrak{C}([-\tau, 0], \mathbb{R}^2_+) : \ \Phi_1(\theta) = X(\theta), \Phi_2(\theta) = Y(\theta)\},$$

where $\Phi_i(\theta) > 0$, $(\theta \in [-\tau, 0],\ i \in \{1, 2\})$.

The linearized system (1.2) at the equilibrium (X^*, Y^*) has the form

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \mathfrak{A} \begin{bmatrix} u \\ v \end{bmatrix} + \mathfrak{B} \begin{bmatrix} u(\cdot - \tau) \\ v(\cdot - \tau) \end{bmatrix},$$

where

$$\mathfrak{A} := \begin{bmatrix} -B-1+2X^*Y^* & (X^*)^2 \\ B-2X^*Y^* & -(X^*)^2 \end{bmatrix} \quad \text{and} \quad \mathfrak{B} := \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix},$$

and then the characteristic function is

$$\Delta(z;\tau) := z^2 + a_1 z + a_0 + e^{-z\tau}\cdot(b_1 z + b_0) + c\cdot e^{-2z\tau} \qquad (z\in\mathbb{C},\ \tau\geq 0), \tag{2.18}$$

with

$$\begin{aligned} a_1 &= (X^*)^2 - 2X^*Y^* + B + 1, \quad a_0 = (X^*)^2, \\ b_1 &= -(\sigma_1+\sigma_2), \quad b_0 = -\sigma_1(X^*)^2 + \sigma_2(2X^*Y^* - B - 1), \quad c = \sigma_1\sigma_2. \end{aligned}$$

Here we assume that

$$\Delta(z;0) = z^2 + (a_1+b_1)z + a_0 + b_0 + c \qquad (z\in\mathbb{C})$$

is Hurwitz-stable, i.e. all its roots have negative real part. Following from the Routh-Hurwitz stability criterion this is true if and only if the coefficients of the above polynomial are positive, i.e.

$$(X^*)^2 - \sigma_2 + B + 1 - \sigma_1 - 2X^*Y^* > 0, \tag{2.19}$$

and

$$(1-\sigma_1)(X^*)^2 + \sigma_2(-B - 1 + \sigma_1 + 2X^*Y^*) > 0 \tag{2.20}$$

hold. In the following theorem, we give a necessary and sufficient condition for the change in stability due to the delay.

Theorem 2.4. *Assume that (2.19) and (2.20) hold., i.e. the polynomial $\Delta(z;0)$ is Hurwitz stable. Then, the stability of $\Delta(z;\tau)$ changes if and only if the polynomial*

$$R(\omega) := R_0 + R_2\omega^2 + R_4\omega^4 + R_6\omega^6 + \omega^8 \qquad (\omega\in\mathbb{R})$$

with coefficients

$$\begin{aligned} R_0 &:= ([X^*]^4 - \sigma_1^2\sigma_2^2)^2 - A^2_{\sigma_1,\,\sigma_2}([X^*]^2 - \sigma_1\sigma_2)^2, \\ R_2 &:= -2A_{\sigma_2,\,\sigma_1}\cdot A_{\sigma_1,\,\sigma_2}([X^*]^2 - \sigma_1\sigma_2) + \\ &\quad +2(A^2_{1,\,1} - 2\,[X^*]^2)([X^*]^4 - \sigma_1^2\sigma_2^2) - \\ &\quad -(A_{1,\,1}\cdot A_{\sigma_1,\,\sigma_2} - (\sigma_1+\sigma_2)([X^*]^2 + \sigma_1\sigma_2))^2, \\ R_4 &:= 2([X^*]^4 - \sigma_1^2\sigma_2^2) + (A^2_{1,\,1} - 2\,[X^*]^2)^2 - A^2_{\sigma_2,\,\sigma_1} + \\ &\quad +2(\sigma_1+\sigma_2)((\sigma_1+\sigma_2)([X^*]^2 + \sigma_1\sigma_2) - A_{1,\,1}\cdot A_{\sigma_1,\,\sigma_2}), \\ R_6 &:= -4\,[X^*]^2 + 2A^2_{1,\,1} - (\sigma_1+\sigma_2)^2 \end{aligned}$$

has a nonzero real root, where

$$A_{k,l} := k \cdot [X^*]^2 + l \cdot (1 + B - 2X^*Y^*).$$

Proof. Based on our previous work (cf. [4]) the stability of the quasi-polynomial changes if and only if the polynomial

$$\begin{aligned} R(\omega) &= (a_0 - c)^2(a_0 - b_0 + c)(a_0 + b_0 + c) + \\ &+ \left[2b_0(b_0 - a_1 b_1)(a_0 - c) + 2(-2a_0 + a_1^2)(a_0 - c)(a_0 + c) - \right. \\ &\left. - (a_1 b_0 - b_1(a_0 + c))^2\right] \omega^2 + \\ &+ \left[6a_0^2 + a_1^4 - b_0^2 - a_1^2 b_1^2 + 2a_0(-2a_1^2 + b_1^2) + 2(b_1^2 - c)c\right] \omega^4 + \\ &+ \left[-4a_0 + 2a_1^2 - b_1^2\right] \omega^6 + \omega^8 = \\ &= R_0 + R_2\omega^2 + R_4\omega^4 + R_6\omega^6 + \omega^8 \qquad (\omega \in \mathbb{R}) \end{aligned}$$

has a nonzero real root ω^*, such that $F(\omega^*) < 0$ and $G(\omega^*) < 0$ hold, where

$$\begin{aligned} F(\omega) &= (-b_0^2 + (a_0 + c)^2) + (a_1^2 - 2(a_0 + c))\omega^2 + \omega^4, \\ G(\omega) &= (a_0 - c)^2 + (a_1^2 - b_1^2 - 2(a_0 - c))\omega^2 + \omega^4. \end{aligned}$$

Straightforward calculation shows that

$$R(\omega) + F(\omega) \cdot G(\omega) = (b_0 b_1 - 2a_1 c)^2 \omega^2 \qquad (\omega \in \mathbb{R}),$$

is true. Hence, if $R(\omega) = 0$, then $F(\omega) \cdot G(\omega) = (b_0 b_1 - 2a_1 c)^2 \omega^2 > 0$ for $\omega \neq 0$, i.e. $\operatorname{sgn}(F(\omega)) = \operatorname{sgn}(G(\omega))$ holds. Using the coefficients of the characteristic function in (2.18) we are going to show that $G(\omega) > 0$ for all $\omega \in \mathbb{R}$. Because

$$G(\omega) = \omega^2(a_1^2 - b_1^2) + (\omega^2 - (a_0 - c))^2 \qquad (\omega \in \mathbb{R}),$$

fulfils, it is enough to see that $a_1^2 - b_1^2 > 0$. For this purpose, we use the transformation

$$\begin{aligned} a_1^2 - b_1^2 &= (a_1 + b_1)(a_1 - b_1) = \\ &= ([X^*]^2 - 2X^*Y^* + B + 1 - \sigma_1 - \sigma_2) \cdot \\ &\quad \cdot ([X^*]^2 - 2X^*Y^* + B + 1 + \sigma_1 + \sigma_2). \end{aligned}$$

Here $a_1 - b_1 \geq a_1 + b_1$ holds following from the negativity of $b_1 = -(\sigma_1 + \sigma_2)$, and following from (2.19) $a_1 + b_1 > 0$, which completes the proof. ■

If $\omega^* > 0$ is a solution of $R(\omega) = 0$, then the possible critical values of the delay must be the solution of one of the above equations:

$$\begin{aligned}\cos(\omega^*\tau) &= \frac{-a_0b_0 + b_0c + (b_0 - a_1b_1)(\omega^*)^2}{a_0^2 - c^2 + (a_1^2 - 2a_0)(\omega^*)^2 + (\omega^*)^4} = \\ &= \frac{A_{\sigma_1,\,\sigma_2}([X^*]^2 - \sigma_1\sigma_2) + A_{\sigma_2,\,\sigma_1}[\omega^*]^2}{[X^*]^4 - \sigma_1^2\sigma_2^2 + (A_{1,\,1}^2 - 2\,[X^*]^2)[\omega^*]^2 + [\omega^*]^4},\end{aligned} \tag{2.21}$$

$$\begin{aligned}\sin(\omega^*\tau) &= -\omega^* \cdot \frac{a_1b_0 - b_1a_0 - b_1c + b_1(\omega^*)^2}{a_0^2 - c^2 + (a_1^2 - 2a_0)(\omega^*)^2 + (\omega^*)^4} = \\ &= \omega^* \cdot \frac{(A_{1,\,1}A_{\sigma_1,\,\sigma_2} - (\sigma_1 + \sigma_2)([X^*]^2 + \sigma_1\sigma_2) + (\sigma_1 + \sigma_2)[\omega^*]^2}{[X^*]^4 - \sigma_1^2\sigma_2^2 + (A_{1,\,1}^2 - 2\,[X^*]^2)[\omega^*]^2 + [\omega^*]^4}.\end{aligned}$$

Example 2.5. Set $A = 1.4000$, $B = 7.5000$, $\sigma_1 = 0.4000$, $\sigma_2 = 0.0800$. The equilibrium has the coordinates $(X^*, Y^*) \approx (2.7000,\ 2.8000)$, and the polynomial R has the form

$$R(\omega) = 2576.0700 - 1457.4300\omega^2 + 307.0200\omega^4 - 28.6635\omega^6 + \omega^8 \qquad (\omega \in \mathbb{R}),$$

which has a positive real root denoted by ω^* with approximated value $\omega^* \approx$ ≈ 3.0400, furthermore with $\tau^* \approx 1.4700$ the equation (2.21) holds, cf. Fig. 4.

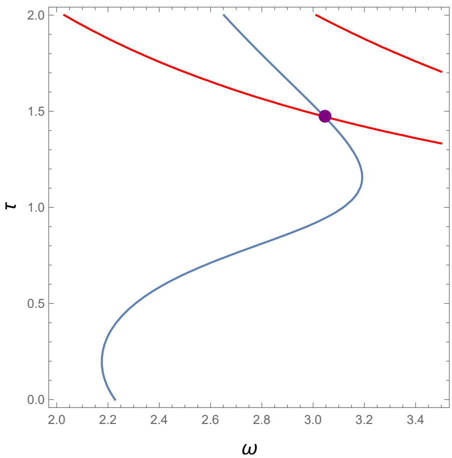

Figure 4. Level curves of $\Re(\Delta(\imath\omega;\tau)) = 0$ (blue line) and $\Im(\Delta(\omega;\tau)) = 0$ (red line) on the domain $(\omega,\tau) \in (2.0, 3.5) \times (0.0, 2.0)$. The purple point at the intersection of the lines shows the solution $(\omega^*, \tau^*) \approx (3.14, 1.47)$.

The solutions of the system (1.2) is shown in Fig. 5 with initial conditions $X_0 = 2.50$ and $Y_0 = 2.80$ and with delay parameters $\tau = 0.00$, $\tau = 1.46$ and $\tau = 1.48$, respectively.

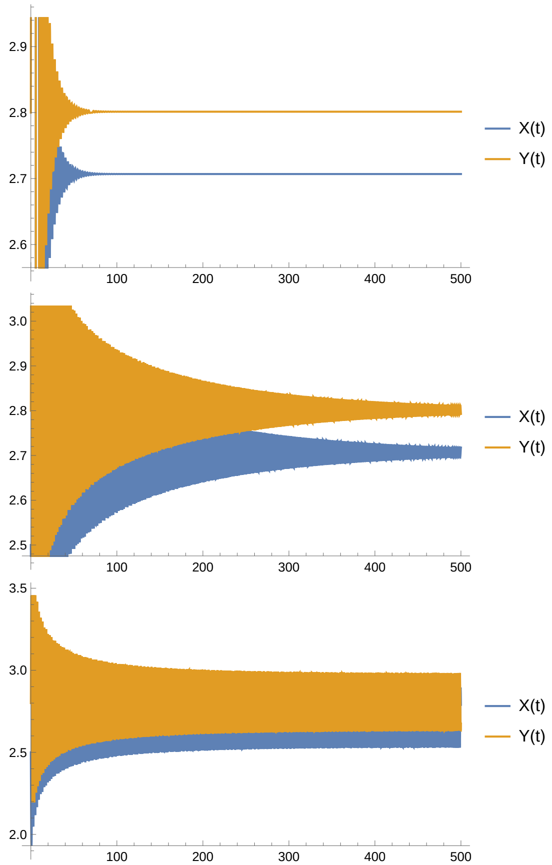

Figure 5. Solutions of system (1.2) with parameter values $A = 1.40$, $B = 7.50$, $\sigma_1 = 0.4$, $\sigma_2 = 0.08$ and initial values $X_0 = 2.50$, $Y_0 = 2.80$ with delay $\tau = 0.0$, $\tau = 1.46$ and $\tau = 1.48$, respectively.

To summarize, we have given a sufficient condition for the stability change of the equilibrium (X^*, Y^*), whereby the existence of nonzero real roots of an eighth degree polynomial (reducible to a quadratic polynomial) must be proved. The fulfilment of this condition, and thus the existence of a stability change, can be investigated even when the model parameters are not or only partially specified. Finally, a formula is given for the values of the delay at which stability change occurs.

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