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# REPRESENTATIVE PRODUCT SYSTEMS

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**Abstract.** In this paper we summarize the most relevant results with respect to the convergence of Fourier series based on representative product systems. We focus on monomial systems defined on the complete product of quaternion groups. The results clearly indicate that these systems can be considered as a natural extension of the Vilenkin systems to the non-abelian harmonic analysis.

## 1. Introduction

A more general and modern approach to the classical theory of Fourier series consists in the study of orthonormal systems defined on topological groups. This provides us with an entity in which the group operation, the topology, the measure and also some basic properties of the system are appropriately connected. A good example is the representation of the classical trigonometric system by its complex version on the torus. This theory called abstract harmonic analysis, was started by the works of A. Haar and A. Weil, and the end of its development is nowhere in sight. An exhaustive content of this theory can be found in the fundamental books of E. Hewitt and K.A. Ross (see [9, 10]).

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Another great example is the representation of the Walsh functions on the dyadic group. In 1923 N.J. Walsh [19] introduced a complete orthonormal system on the interval $[0,1)$ taking only the values 1 and $-1$, which is identical to the systems $\cos nx$, $\sin nx$ in the number of sign changes. He constructed it by recursion, but some years later R.E.A.C. Paley [11] recognized that Walsh functions are the finite product of Rademacher functions, giving a new arrangement to the Walsh functions called the Walsh–Paley system. This fact allowed N.J. Fine [3] and N. Vilenkin [18] to connect Walsh analysis with abstract harmonic analysis, representing the Walsh functions by the characters of the dyadic group. The book of F. Schipp, W.R. Wade and P. Simon (see [13]) is essential for those who want to immerse themselves in the theory of dyadic harmonic analysis.

The dyadic group is the topological group formed by the complete product of cyclic group of order 2 having the discrete topology and assigning each singleton the measure $\frac{1}{2}$. We have already mentioned that the characters of the dyadic group correspond to the Walsh functions. This structure was generalized by N. Vilenkin [18] considering the complete product of arbitrary cyclic groups. The characters of this structure ordered in the Paley's sense are called a Vilenkin system. These systems have been the subject of studies in several Hungarian research groups, and also for many researchers abroad. The book of L.E. Persson, G. Tephnadze and F. Weisz (see [12]) provides a wide and detailed compilation of the results concerning Vilenkin systems.

Following the logic of the previous constructions, it is quite reasonable to ask ourselves how different it would be to consider the topological group formed by the complete product of finite groups that are not necessarily commutative. F. Schipp proposed the study of product systems defined on these structures. However, despite the similarity with Vilenkin's structures, the characters of a finite non-abelian group are orthonormal, but they do not form a complete system. This leads us to use representation theory to find the other members and thus obtain complete systems on finite groups. Product systems based on them are called representative product systems. We can see several examples of these systems in the works of G. Gát and R. Toledo.

Right at the beginning we already realize that a representative product system can differ significantly from Vilenkin systems. Indeed, these systems can take the value 0 and they are not uniformly bounded in all cases. That is one of the reasons why we have systems where the Fourier series have a very different behavior than what we see in case of Vilenkin systems. But there are also many similarities. For instance, in [5] we proved that Paley's lemma also holds for all representative product system, which implies the existence of a subsequence of the partial sum of Fourier series that converges to the function in $L^1$-norm and almost everywhere for all integrable functions. In addition, there are concrete systems in which the convergence theorems are the same as those obtained for Vilenkin systems. This is the case for monomial systems defined on the complete product of quaternion groups.

## 2. Basic concepts in abstract harmonic analysis

Let $G$ be a set that is a group and also a topological space. $G$ is called a *topological group*, if the group operation $xy$ and the inversion $x^{-1}$ are continuous mappings. In this case the algebraic properties of the group affect the topological properties of the space and vice versa. For instance, an open basis $\mathcal{U}$ at the group identity $e$ gives an open base for $G$ by the family $\{xU \mid x \in G, U \in \mathcal{U}\}$ and also by the family $\{Ux \mid x \in G, U \in \mathcal{U}\}$.

Let $\mathcal{A}$ be the smallest $\sigma$-algebra of subsets of $G$ which contains all open subsets of $G$. The members of $\mathcal{A}$ are called the *Borel sets* of $G$. A measure $\mu$ defined on $G$ is said to be *regular* if for every open set $U$ we have

$$\mu(U) = \sup\{\mu(F) \mid F \text{ is compact and } F \subseteq U\},$$

and for all $A \in \mathcal{A}$ we have

$$\mu(A) = \inf\{\mu(U) \mid U \text{ is open and } U \supseteq A\}.$$

If $\mu(xA) = \mu(Ax) = \mu(A)$ for all $x \in G$ and $A \in \mathcal{A}$, then $\mu$ is said to be *two-sided invariant*. If $G$ is compact, then there is an unique non-negative regular measure $\mu$ on the Borel sets of $G$ which is two-sided translation invariant and $\mu(G) = 1$. This measure is called the *normalized Haar measure* of $G$.

Measurable functions on $G$ whose $p$-th power are integrable, play an important role in approximation. For $1 < p < \infty$ let $L^p(G)$ represent the set of this functions which is a Banach space with norm

$$\|f\|_p := \left( \int_G |f|^p \, d\mu \right)^{\frac{1}{p}} < \infty.$$

Since the measure $\mu$ is finite the relation

$$L^q(G) \subset L^p(G) \subset L^1(G) \qquad (1 < p < q < \infty)$$

holds. For this reason the most extensive set of functions on $G$ we consider is just $L^1(G)$. Similarly $L^\infty(G)$ represent the set of all measurable functions $f$ such that

$$\|f\|_\infty := \inf\{y \in \mathbb{R} \mid |f(x)| \le y \text{ for a.e. } x \in G\} < +\infty.$$

A *representation* $U$ of the group $G$ is a homomorphism of $G$ into the semigroup of all operators defined in some linear space $E$ over an arbitrary field $F$. That is, $U$ is a mapping $x \to U_x$ such that $U_x : E \to E$ is a linear transformation for all $x \in G$, and

$$U_{xy} = U_x U_y \qquad (x, y \in G).$$

The linear space $E$ is called the *representation space* of $U$, and let the dimension of a representation be the *dimension* of its own representation space. We can assume that $U_e$ is the identity operator on $E$.

Throughout this work suppose that the representation space of all representations is a reflexive complex Banach space which is a topological linear space under the metric and norm induced by the inner product $\langle ., . \rangle$. The representation $U$ is called *unitary* if all of operators $U_x$ are unitary, i.e. $U_x$ is a linear isometry of $E$ onto $E$. An unitary representation with dimension 1 is called a *character*, i.e. a character is a continuous complex-valued mapping $\chi : G \to \mathbb{C}$ such that

$$\chi(xy) = \chi(x)\chi(y) \quad (x, y \in G), \qquad |\chi(x)| = 1 \quad (x \in G).$$

All group have a trivial character, namely the one which is identically equal to 1.

A representation $U$ with representation space $E$ is called *irreducible* if only the spaces $\{0\}$ and $E$ are invariant under all operators $U_x$ $(x \in G)$. We can define an equivalence relation in the set of all continuous irreducible unitary representations of the group $G$ as follows: Two representations $U$ and $U'$ with representation spaces $E$ and $E'$ respectively are equivalent if there is a bounded linear isometry $T : E \to E'$ such that

$$U'_x T = T U_x \qquad (x \in G).$$

Denote by $\Sigma$ the set of all equivalence classes induced by the above relation. $\Sigma$ is called the *dual object* of the group $G$. The common dimension of all representations in the class $\sigma \in \Sigma$ is denoted by $d_\sigma$.

Let $G$ be a finite group of order $m$ and let $|\Sigma|$ denote the cardinal number of $\Sigma$. Then

a) $|\Sigma|$ is equal to the number of conjugacy class in $G$. (The system of the conjugacy classes is a partition of $G$ induced by the equivalence relation: $a \sim b$ if and only if $\exists x \in G : \; a = xbx^{-1}$).

b) if $\Sigma = \{\sigma_1, \sigma_1 \dots \sigma_{|\Sigma|}\}$, then $d^2_{\sigma_1} + d^2_{\sigma_2} + \cdots + d^2_{\sigma_{|\Sigma|}} = m$.

c) $d_{\sigma_i}$ is a divisor of $m$ for all $1 \le i \le |\Sigma|$.

d) if the group $G$ is abelian, then $|\Sigma| = m$ and all representations of $G$ are characters.

e) if the group $G$ is not abelian, then there is a representation with dimension greater than 1.

The above properties of finite groups suggests to us the construction of $d_\sigma^2$ numbers of functions for every $\sigma \in \Sigma$, which we will do as follows. Let $U^{(\sigma)}$ be a continuous irreducible representation in the class $\sigma$ of the dual object of $G$. Functions

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \qquad i, j \in \{1, \ldots, d_\sigma\}$$

are called *coordinate functions* for $U^{(\sigma)}$, where $\xi_1, \ldots, \xi_{d_\sigma}$ is a fixed orthonormal basis in the representation space of $U^{(\sigma)}$. If we do that with all the elements of $\Sigma$ we obtain a total of $m$ functions.

It is clear that a finite group $G$ of order $m$ endorsed by the discrete topology is a compact topological group. In this case the normalized Haar measure is the one that assigns $\frac{1}{m}$ to any singleton. The Weyl–Peter theorem (see [10] p. 24) ensures that the set of all coordinate functions is an orthogonal basis for $L^2(G)$, but it is not orthonormal if $G$ is not abelian. We normalize a coordinate function by multiplying it by the square root of its dimension. We arrange all normalized coordinate functions in a system denoted by $\{\varphi^s \mid 0 \le s < m\}$ assuming only that $\varphi^0 \equiv 1$ is the trivial character. For instance, consider $G = \mathcal{S}_3$ the *symmetric group* on 3 elements. This group has two characters and one representation of dimension 2. The following table contains a possible arrangement of normalized coordinate functions.

| | $e$ | (12) | (13) | (23) | (123) | (132) |
|---|---|---|---|---|---|---|
| $\varphi^0$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\varphi^1$ | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 |
| $\varphi^2$ | $\sqrt{2}$ | $-\sqrt{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ |
| $\varphi^3$ | $\sqrt{2}$ | $\sqrt{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ |
| $\varphi^4$ | 0 | 0 | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $-\frac{\sqrt{6}}{2}$ |
| $\varphi^5$ | 0 | 0 | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ |

We take now a sequence of finite groups $G_k$ of order $m_k$ ($k \in \mathbb{N}$), and suppose that each group has discrete topology and normalized Haar measure $\mu_k$. Let $\{\varphi_k^s \mid 0 \le s < m_k\}$ be a system of all normalized coordinate functions of $G_k$. Let $G$ be the compact group formed by the complete direct product of $G_k$ with the product of the topologies, operations and measures $(\mu)$. Since $G$

is compact, the dual object of $G$ is countable and the dimensions of all representations of $G$ are finite. On the other hand, all of continuous irreducible representations of $G$ are the tensor product of finite many continuous irreducible representations of different groups $G_k$. Therefore, the finite product of $\varphi_k^s$ with different values of $k$ provide us with an orthonormal system on $G$ which is complete in $L^2(G)$.

## 3. Representative product systems

Let $m := (m_k,\ k \in \mathbb{N})$ be a sequence of positive integers such that $m_k \geq 2$. According to the previous section, let's suppose that we have a finite group $G_k$ for all $k \in \mathbb{N}$ having discrete topology, normalized Haar measure $\mu_k$ and a system $\{\varphi_k^s \mid 0 \leq s < m_k\}$ which contains all normalized coordinate functions ($\varphi^0 \equiv 1$). Let $G$ be the group formed by the complete direct product of $G_k$ with the product of the topologies, operations and measures ($\mu$). Thus $G$ is a compact totally disconnected group having the normalized Haar measure $\mu$ and each $x \in G$ consist of sequences $x := (x_0, x_1, ...)$, where $x_k \in G_k$, ($k \in \mathbb{N}$). We call this sequence the *expansion of* $x$. In order to simplify the notations we use the multiplication to denote the group operation and the symbol $e$ to denote the identity of the groups.

$G$ is called a *bounded group* if the sequence $m = (m_k,\ k \in \mathbb{N})$ is bounded. On the other hand, with the sequence $m = (m_k,\ k \in \mathbb{N})$ we introduce the following notation:

$$M_0 := 1, \quad \text{and} \quad M_{k+1} := m_k M_k \qquad (k \in \mathbb{N}).$$

It is easy to see that every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k M_k, \qquad (0 \leq n_k < m_k,\ n_k \in \mathbb{N}).$$

This allows us to say that the sequence $(n_0, n_1, \ldots)$ is the *expansion of* $n$ *with respect to the sequence* $m$. We construct an orthonormal system on $G$ as follows. Let $\psi$ be the product system of $\varphi_k^s$, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G), \tag{3.1}$$

where $n$ is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, ...)$. Thus, we say that $\psi$ is the *representative product system* of $\varphi$. The system $\psi$ is orthonormal and complete in $L^2(G)$.

A representative product system can be represented on the interval $[0,1)$ taking into account the connection between the Haar integration on the complete direct product of finite groups and the Lebesgue integration on the interval $[0,1)$ (see [15]).

### 3.1. Monomial systems

Let $G$ be a finite group of order $m$ and $\{\varphi^s \mid 0 \le s < m\}$ be a system with all normalized coordinate functions on $G$. Denote $d_s$ by the dimension of the representation that corresponds to $\varphi^s$. Since all representations are unitary, we have $|\varphi^s(x)| \le \sqrt{d_s}$ for all $s = 0, 1, \ldots, m-1$ and $x \in G$. We say that the system $\{\varphi^s \mid 0 \le s < m\}$ is monomial if $\varphi^s(x) = 0$ or $|\varphi^s(x)| = \sqrt{d_s}$ for all $s = 0, 1, \ldots, m-1$ and $x \in G$.

A good example of monomial system is one formed by characters. Thus, $G$ is abelian, and $\sqrt{d_s} = 1$ for all $s = 0, 1, \ldots, m-1$. For instance, the characters of the cyclic group of order 2 form the system

$$\varphi^s(x) = (-1)^{sx} \qquad (s \in \{0,1\},\ x \in \mathcal{Z}_2).$$

In general, the characters of the cyclic group of order $m$ form the system

$$\varphi^s(x) = \exp(2\pi\imath sx/m) \qquad (s \in \{0, \ldots m-1\},\ x \in \mathcal{Z}_m,\ \imath^2 = -1).$$

Relevant examples of monomial systems on non-abelian groups can be constructed as follows. Let $m$ be an integer such that $m > 1$. Define by

$$\mathcal{Q}_m := \{[a,b] : a^{2m} = e,\ b^2 = a^m,\ bab^{-1} = a^{2m-1}\}$$

the generalized quaternion group of order $4m$. $\mathcal{Q}_m$ always has 4 characters and all of its representations are of dimension 2. If $m$ is even, then the characters of $\mathcal{Q}_m$ are $\varphi_0 = 1$ and

$$\begin{aligned}
\varphi_1(a^j) &= 1, & \varphi_1(a^j b) &= -1,\\
\varphi_2(a^j) &= (-1)^j, & \varphi_2(a^j b) &= (-1)^j,\\
\varphi_3(a^j) &= (-1)^j, & \varphi_3(a^j b) &= (-1)^{j+1}.
\end{aligned}$$

If $m$ is odd, then the characters of $\mathcal{Q}_m$ are $\varphi_0 = 1$ and

$$\begin{aligned}
\varphi_1(a^j) &= 1, & \varphi_1(a^j b) &= -1,\\
\varphi_2(a^j) &= (-1)^j, & \varphi_2(a^j b) &= (-1)^j \imath,\\
\varphi_3(a^j) &= (-1)^j, & \varphi_3(a^j b) &= (-1)^{j+1} \imath.
\end{aligned}$$

If $\alpha = \exp(\pi\imath/m)$ the first primitive $2m$-th root of unity then the mapping

$$a^j \to \begin{pmatrix} \alpha^j & 0 \\ 0 & \alpha^{-j} \end{pmatrix} \qquad a^j b \to \begin{pmatrix} 0 & \alpha^{j-m} \\ \alpha^{-j} & 0 \end{pmatrix}$$

is an unitary representation. Call this representation $U_\alpha$. Then the above characters and

$$U_\alpha, U_{\alpha^2}, U_{\alpha^3}, \dots, U_{\alpha^{m-1}}$$

complete the dual object of $\mathcal{Q}_m$. Thus, for $\mathcal{Q}_2$ we obtain the following values.

| | $e$ | $a$ | $a^2$ | $a^3$ | $b$ | $ab$ | $a^2b$ | $a^3b$ |
|---|---|---|---|---|---|---|---|---|
| $\varphi^0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\varphi^1$ | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| $\varphi^2$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| $\varphi^3$ | 1 | $-1$ | 1 | $-1$ | $-1$ | 1 | $-1$ | 1 |
| $\varphi^4$ | $\sqrt{2}$ | $\sqrt{2}\imath$ | $-\sqrt{2}$ | $-\sqrt{2}\imath$ | 0 | 0 | 0 | 0 |
| $\varphi^5$ | $\sqrt{2}$ | $-\sqrt{2}\imath$ | $-\sqrt{2}$ | $\sqrt{2}\imath$ | 0 | 0 | 0 | 0 |
| $\varphi^6$ | 0 | 0 | 0 | 0 | $\sqrt{2}$ | $-\sqrt{2}\imath$ | $-\sqrt{2}$ | $\sqrt{2}\imath$ |
| $\varphi^7$ | 0 | 0 | 0 | 0 | $-\sqrt{2}$ | $-\sqrt{2}\imath$ | $\sqrt{2}$ | $\sqrt{2}\imath$ |

In [8] we can also see a table with the values of the system related to $\mathcal{Q}_3$.

Let us now return to the complete product of finite groups having the structure described at the beginning of this section. If all the systems corresponding to the finite groups are monomial, then we say that the representative product systems defined by (3.1) is *monomial*. In the commutative case, if all finite groups are the cyclic group of order 2, then structure above is called the *dyadic group* and then the system (3.1) is called the *Walsh–Paley system*. More generally, if all finite groups are arbitrary cyclic groups, then structure above is called a *Vilenkin group* and then the system (3.1) is called a *Vilenkin system*.

## 4. Main results

Hereafter, we will assume that $G$ is the structure of the complete product of finite groups $G_k$ described at the beginning of this section, and $\{\psi_n \mid n \in \mathbb{N}\}$ is the system defined in (3.1).

### 4.1. Properties of Dirichlet kernels

For an integrable complex function $f$ on $G$ we define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int\limits_{G_m} f\overline{\psi}_k \, d\mu \quad (k \in \mathbb{N}), \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbb{N}).$$

The *Dirichlet kernels* are defined as follows:

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbb{N}).$$

It is easy to see that

$$S_n f(x) = \int_G f(y) D_n(x,y)\, d\mu(y), \tag{4.1}$$

which shows the importance of the Dirichlet kernels in the study of the convergence of Fourier series.

Define $I_0(x) := G$,

$$I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \le k < n\} \quad (x \in G, n \in \mathbb{N}^+).$$

We say that every set $I_n(x)$ is an *interval.* The set of intervals $I_n := I_n(e)$ is a countable neighborhood base at the identity $e$ of the product topology on $G$. The following lemma is very useful for the study of Dirichlet kernels.

**Lemma 4.1** (Gát and Toledo [5]). *If $n \in \mathbb{N}^+$ and $x, y \in G$, then*

$$D_n(x,y) = \sum_{k=0}^{|n|} D_{M_k}(x,y) \left( \sum_{s=0}^{n_k - 1} \varphi_k^s(x_k) \overline{\varphi}_k^s(y_k) \right) \prod_{r=k+1}^{|n|} \varphi_r^{n_r}(x_r) \overline{\varphi}_r^{n_r}(y_r),$$

*where $(n_0, n_1, \ldots)$ is the expansion of $n$, $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$, and $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots)$.*

The following lemma is known as Paley's lemma for Walsh–Paley and Vilenkin systems, and it can be also extended to any representative product system in general.

**Lemma 4.2** (Gát and Toledo [5]). *If $n \in \mathbb{N}$ and $x, y \in G$, then*

$$D_{M_k}(x,y) = \begin{cases} M_k, & \text{for } x \in I_k(y), \\ 0, & \text{for } x \notin I_k(y). \end{cases}$$

By Paley's lemma we have that the operator $S_{M_n}$ is the conditional expectation with respect to the $\sigma$-algebra generated by the sets $I_n(x)$, $x \in G$. Indeed

$$S_{M_n} f(x) = \int_G f(y) D_{M_n}(x,y)\, d\mu(y) = \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f\, d\mu.$$

Thus, by the martingale convergence theorem we obtain that $S_{M_n}f \to f$ in $L^p$-norm and also almost everywhere for all $f \in L^p(G)$ and $1 \le p < \infty$.

Another consequence of the Paley lemma is that the finite linear combinations of characteristics functions on the intervals $I_n(x)$ are the finite linear combinations of the members of the system $\psi$, and vice versa. These functions are dense in $L^1(G)$, therefore all representative product systems are complete in $L^1(G)$.

Paley's lemma holds for all representative product systems, but the behavior of Dirichlet kernels in general differs considerably from what we know for Vilenkin systems. For instance, consider the maximal value of Dirichlet kernels

$$\mathcal{D}_n := \sup_{x,\, y \in G} |D_n(x, y)| \qquad (n \in \mathbb{P}).$$

For Vilenkin systems we have $\mathcal{D}_n = n$ for all $n \in \mathbb{N}^+$, but the general case is quite different.

**Theorem 4.1** (Toledo [17]). *If $n \in \mathbb{P}$ and $A := \max\{k \in \mathbb{N} : n_k \neq 0\}$, then*

$$n \le \mathcal{D}_n \le M_{A+1}.$$

In [17] R. Toledo gave necessary and sufficient conditions to obtain the equality $\mathcal{D}_n = n$ for some $n \in \mathbb{N}$. In addition, he studied the boundedness of the sequence $\frac{\mathcal{D}_n}{n}$. We would also like to mention I. Blahota's work in [2], where he obtained further significant results related to this sequence.

## 4.2. Convergence in $L^p$-norm of Fourier series

A basic problem of Fourier analysis is to obtain the values of $p$ $(1 \le p < \infty)$ such that for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of $f$ converges to the function $f$ in $L^p$-norm. Convergence for $p = 2$ is obvious since $L^2(G)$ is a Hilbert space. For $p = 1$ the answer is negative.

**Theorem 4.2** (Toledo [14]). *For all $G$ groups there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of $f$ does not converge to the function $f$ in $L^1$-norm.*

For Vilenkin systems the convergence in norm is true for $1 < p < \infty$. However, this fact does not hols for all representative product system. The sequence

$$\Psi_k := \max_{n < M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbb{N}).$$

plays an important role here.

**Theorem 4.3** (Toledo [16]). *Suppose $G$ is the complete product of the same finite group having the same system $\varphi$ at all of their occurrences. If the sequence $\Psi$ is not bounded, then for all number $1 < p \neq 2$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of $f$ does not converge to the function $f$ in $L^p$-norm.*

The conditions of the above theorem are satisfied if $G$ is the complete product of $\mathcal{S}_3$ having the system that already appears in this paper, since in this cases $\Psi_k = \left(\frac{4}{3}\right)^k$. In [16] we can look deeper into more negative results.

It is not hard to see that $\Psi_k = 1$ for every monomial systems. In [8] the authors proved with some restrictions the converge in $L^p$-norm $(1 < p < \infty)$ for the complete product of generalized quaternion groups. This theorem is the generalization of the well known one for Vilenkin systems.

**Theorem 4.4** (Gát and Toledo [8]). *Let $G$ be the complete product of generalized quaternion groups with representative monomial systems ordered such that the first four functions in the systems are the characters. If $G$ is a bounded group, then $S_n f \to f$ in $L^p$-norm for all $f \in L^p(G)$ and $1 < p < \infty$.*

The method used to prove the above theorem is based on proving that $S_n f$ is of weak type $(1,1)$.

### 4.3. Fejér means and Cesàro means of order $\alpha$

The *Cesàro numbers of order $\alpha$* are given by the formula

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \qquad (n \in \mathbb{N})$$

where $\alpha \in \mathbb{R} \setminus \{-1,-2,\dots\}$. Then, we denote *the Cesàro means of order $\alpha$ of Fourier series* or simply $(C,\alpha)$ *means* by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f \qquad (n \in \mathbb{N}^+). \tag{4.2}$$

In addition, $\sigma_n := \sigma_n^1$ are also called *the Fejér means of Fourier series.*

Earlier in [5], the authors had already obtained the following result.

**Theorem 4.5** (Gát and Toledo [5]). *For all $G$ bounded groups $\sigma_n f \to f$ in $L^p$-norm for all $f \in L^p(G)$ and $1 \leq p < \infty$.*

This result is more interesting if we focus on the fact that there are representative product systems where $S_n f$ does not converge to $f$ in $L^p$-norm for $p \neq 2$. On the other hand, G. Gát proved the almost everywhere convergence of Fejér means.

**Theorem 4.6** (Gát [4]). *For all $G$ bounded groups $\sigma_n f \to f$ almost everywhere for all $f \in L^1(G)$.*

For Cesàro means of order $\alpha$ we obtain the following results.

**Theorem 4.7** (Gát and Toledo [7]). *Let $G$ be a bounded group, and denote*

$$\alpha_0 := \limsup_{k\to\infty} \log_{m_k} \left( \max_{0\le s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right).$$

*If $\alpha_0 < \alpha < 1$, then $\sigma_n^\alpha f \to f$ in $L^p$-norm for all $f \in L^p(G)$ and $1 \le p < \infty$.*

Note that $\alpha_0 \le \frac{1}{2}$ is always true, and $\alpha_0 = \log_6 \frac{4}{3}$ for the complete product of $\mathcal{S}_3$ having the system that already appears in this paper. In addition $\alpha_0 = 0$ for all monomial systems. Hence, we obtain immediately the next corollary as the generalization of a well known statement for Vilenkin systems.

**Corollary 4.1.** *If $G$ is a bounded group with a monomial representative product systems, then $\sigma_n^\alpha f \to f$ in $L^p$-norm for all $0 < \alpha < 1$, $f \in L^p(G)$ and $1 \le p < < \infty$.*

On the other hand, suppose we have a bounded group $G$ with $\alpha_0 > 0$. Thus, we can obtain divergence for certain values of $\alpha$.

**Theorem 4.8** (Gát and Toledo [7]). *Let $G$ be a bounded group, and denote*

$$\alpha_1 := \liminf_{k\to\infty} \log_{m_k} \left( \max_{0\le s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right).$$

*If $0 < \alpha < \alpha_1$, then there exists an $f \in L^1(G)$ such that $\sigma_n^\alpha f$ does not converge to the function $f$ in $L^1$-norm.*

### 4.4. Estimation of Fourier coefficients

First of all it is necessary to emphasize that there are representative product systems $\psi$ which are not uniformly bounded, or in other words, the sequence $\|\psi_n\|_\infty$ can be unbounded. Obviously this can only happen for non-abelian groups. This fact is important because the norm of the operators

$$T_n : L^1(G) \to \mathbb{C}, \qquad T_n f := \int_G f \overline{\psi_n} \, d\mu$$

is $\|\psi_n\|_\infty$. Therefore if this sequence is not bounded, then there is a $f \in L^1(G)$ such that $\widehat{f}(n) \not\to 0$, so the well known Riemann–Lebesgue lemma does not hold. To estimate them we introduce the concept of modulus of continuity.

Let $f \in L^p(G)$ $(1 \le p < \infty)$, $n \in \mathbb{N}$ and $I = I_n(x)$ an interval. Recall that $I_n := I_n(e)$. We define the local modulus of continuity of $f$ on $I$ by

$$\omega^{(p)}(f, I) := \sup_{h \in I_n} \left( \frac{1}{\mu(I)} \int_I |\tau_h f - f|^p \, d\mu \right)^{\frac{1}{p}}, \qquad \omega(f, I) := \omega^{(1)}(f, I),$$

and the $n$-th modulus of continuity of $f$ on $L^p$ by

$$\omega_n^{(p)}(f) := \sup_{h \in I_n} \|\tau_h f - f\|_p, \quad (n \in \mathbb{N}), \qquad \omega_n(f) := \omega_n^{(1)}(f),$$

where $\tau_h f(x) := f(x+h)$ is the right translation operator. We remark that if we use the left translation operator, we obtain identical value for the modulus of continuity, because the measure is both left and right translation invariant and $I_n$ is a normal subgroup of $G$. Notice that $\omega_n^{(p)}(f) \searrow 0$, $n \to \infty$ and the $\omega_n^{(p)}(f)$ value increases when the value of $p$ is also increases.

**Theorem 4.9** (Gát and Toledo [6]). *Let $f \in L^1(G)$, $n, \ k \in \mathbb{N}$. If $n > M_k$ then*

$$|\widehat{f}(n)| < \omega_k(f) \|\psi_n\|_\infty.$$

A function is said to be of bounded fluctuation if

$$\mathcal{F}\ell(f) := \sup_{n \in \mathbb{N}} \left( \sum_{k=0}^{M_n - 1} |\omega(f, I_n(k^*))| \right) < \infty,$$

where $k^*$ denotes the elem of $G$ with the same expansion as $k \in \mathbb{N}$, so $k^* = (k_0, k_1, \dots) \in G$ if the expansion of $k$ is $(k_0, k_1, \dots)$.

**Theorem 4.10** (Gát and Toledo [6]). *Denote by $n \in \mathbb{N}$ and $s = \max\{j \in \mathbb{N} : n_j \neq 0\}$. If $f$ is of bounded fluctuation, then*

$$|\widehat{f}(n)| \le \frac{\mathcal{F}\ell(f)}{M_s} \|\psi_n\|_\infty.$$

## 4.5. Absolute convergence of Fourier series based on characters

In [1] G. Benke proved that the Lipschitz class to which a function belongs can be identified by the best approximation characteristics of the function by trigonometric polynomials, and that functions which are easily approximated by trigonometric polynomials have absolutely convergent Fourier series. According to the above work we have some conditions for which a function has absolutely convergent Fourier series based in the system of characters of $G$ in case that the function is constant on the conjugacy classes of $G$.

Here we have an important clarification to make. Until now, we have called characters the representations with dimension 1. The concept of character may be generalized to refer to the trace of an arbitrary representation (see [10] p. 13). A construction similar to the representative product systems can be made using only the characters of the finite groups $G_k$ (see [6]). These new systems are orthogonal, but not complete. However, they are complete if we consider only functions which are constant on the conjugacy classes. Denote by $\mathcal{L}^p(G)$ the restriction of the space $L^p(G)$ for the above functions.

Denote by $\mathcal{A}$ the set of functions which have absolutely convergent Fourier series based in the system of characters of $G$. The Lipschitz class of order $\alpha$ will be denoted by Lip($\alpha$), and it is a closed subspace of the continuous functions endowed with the norm

$$\|f\|_{\mathrm{Lip}(\alpha)} := \sup_k \left[\sup_{x\in I_k} \|f(x\cdot) - f(\cdot)\|_\infty M_k^\alpha\right] + \|f\|_\infty < \infty.$$

**Theorem 4.11** (Gát and Toledo [6]). *Let $G$ be a bounded group and $f \in \mathcal{L}^2(G)$. If*

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{M_n-1} |\omega^{(2)}(f, I_n(k^*))|^2 \right)^{\frac{1}{2}} < \infty \qquad \textit{then} \qquad f \in \mathcal{A}$$

*(the notation $k^*$ has already appeared in the concept of functions of bounded fluctuation).*

**Theorem 4.12** (Gát and Toledo [6]). *Let $f : G \to \mathbb{C}$ a continuous function that is constant in the conjugacy classes of $G$ and suppose that exists a $1 \le p \le 2$ such that*

$$\sum_{n=0}^{\infty} \left( \sum_{\substack{t_i\in G_i \\ i<n}} |\omega(f, I_n(t))|^p \right)^{\frac{1}{p}} < \infty. \qquad \textit{Then} \qquad f \in \mathcal{A}.$$

**Corollary 4.2.** *Let $f : G \to \mathbb{C}$ a continuous function that is constant in the conjugacy classes of the bounded group $G$, and suppose that $\sum_{n=0}^\infty \sqrt{M_n}\omega_n(f) < < \infty$. Then $f \in \mathcal{A}$.*

**Corollary 4.3.** *Let $G$ be a bounded group and $f \in Lip(\alpha)$ for some $\alpha > \frac{1}{2}$. Then $f \in A$.*

## 5. Conclusion

The theory of abstract harmonic analysis gives us the possibility to obtain a wide variety of systems that are constant on intervals. The study of them is based on the results obtained for Vilenkin systems. However, many well-known results are not easily transferable to the non-abelian cases, or simply they are not true in general. Despite that, we can notice a great similarity with the results for monomial systems. This leads us to the conclusion that these systems are a natural extension of the Vilenkin systems, in particular, when we consider the complete product of quaternion groups.

As we can see through the results of this paper, up to now the research related to representative product systems has been focused on the study of the convergence in $L^p$-norm of the most relevant operators for bounded groups. This means that there is still a considerable amount of work to be done here. For example, it would be interesting to study the almost everywhere convergence of Fourier series. We are also faced up to the difficulties of non-bounded groups, as we are already know for Vilenkin systems. We hope this article will be of great utility for those who want to start with the study of series based on representative product systems.

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