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FURTHER STUDY OF MODULATION SPACES AS BANACH ALGEBRAS

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Abstract. This paper discusses spectral synthesis for those modulation spaces $M_s^{p,q}(\mathbf{R}^n)$ which form Banach algebras under pointwise multiplication. An important argument will be the "ideal theory for Segal algebras" by H. Reiter [15]. This paper is a continuation of our paper [5] where the case $q = 1$ is treated. As a by-product we obtain a variant of Wiener–Lévy theorem for $M_s^{p,q}(\mathbf{R}^n)$ and Fourier–Wermer algebras $\mathcal{F}L_s^q(\mathbf{R}^n)$.

1. Introduction

Spectral synthesis is one of the important topics in classical harmonic analysis. It is concerned with the question whether spectral synthesis holds for a given subset E of $\mathbf{R}^n$ and for a given Banach space $(B, \|\cdot\|_B)$. More precisely, we assume that B be a Banach space of continuous, complex-valued functions, i.e. $B \subset C_b(\mathbf{R}^n)$, with $C_c^\infty(\mathbf{R}^n)$ dense in B, and convergence in B implying pointwise convergence. For every closed subset E of $\mathbf{R}^n$, we set $I(E) = \{f \in B \mid f|_E = 0\}$ and define $J(E)$ as the closure in B of the set

$$\{f \in C_c^\infty(\mathbf{R}^n) \mid f = 0 \text{ in a neighborhoof of } E\},$$

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where $f|_E = 0$ means that $f(x) = 0$ for all $x \in E$. If $I(E) = J(E)$, then E is called a *set of spectral synthesis for* B. There are many papers devoted to study the spectral synthesis (e.g., [2], [10], [15], [16], [17]). Let $A(\mathbf{R}^n)$ be the Fourier algebra of all functions on $\mathbf{R}^n$ which are the Fourier transforms of functions in $L^1(\mathbf{R}^n)$, also denoted by $\mathcal{F}L^1(\mathbf{R}^n)$ elsewhere. A famous and pioneering result by Schwartz [18] shows that the unit sphere $S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$ is not a set of spectral synthesis for $A(\mathbf{R}^n)$, if $n \geq 3$. Surprisingly, Herz [8] showed that the unit circle S^1 is a set of spectral synthesis for $A(\mathbf{R}^2)$. Motivated by these results, Reiter [15] showed similar result in the Fourier–Beurling algebras $\mathcal{F}L^1_s(\mathbf{R}^n)$: If $n \geq 3$, then S^{n-1} is not a set of spectral synthesis for $\mathcal{F}L^1_s(\mathbf{R}^n)$, and S^1 is a set of spectral synthesis for $\mathcal{F}L^1_s(\mathbf{R}^2)$ if $0 \leq s < 1/2$, but not if $s \geq 1/2$. Moreover, Kobayashi–Sato [11] considered the spectral synthesis of S^{n-1} for the Fourier–Wermer algebra $\mathcal{F}L^q_s(\mathbf{R}^n)$: Let q' denote the conjugate exponents of q. If $1 < q < \infty$ and $s > n/q'$, then S^{n-1} $(n \geq 3)$ is not a set of spectral synthesis for $\mathcal{F}L^q_s(\mathbf{R}^n)$, and S^1 is a set of spectral synthesis for $\mathcal{F}L^q_s(\mathbf{R}^2)$ if $1 < q < 2$ and $2/q' < s < 2/q' + 1/2$, but not if $1 < q < \infty$ and $s > 2/q' + 1/2$.

The modulation spaces $M^{p,q}_s(\mathbf{R}^n)$ are one of the function spaces introduced by Feichtinger [3]. The definition of $M^{p,q}_s(\mathbf{R}^n)$ will be given in Section 2.2. The main idea for these spaces is to consider the space and the frequency variable simultaneously. In some sense, they behave like the Besov spaces $B^{p,q}_s(\mathbf{R}^n)$. But they appear to be better suited for the description of problems in the area of time-frequency analysis and are often a good substitute for the usual spaces $L^p(\mathbf{R}^n)$ or $B^{p,q}_s(\mathbf{R}^n)$ (see [4], [6]).

The aim of this paper is to understand certain sets of spectral synthesis for $M^{p,q}_s(\mathbf{R}^n)$. Our first result is:

Theorem 1.1. *Let $1 \leq p \leq 2$. Suppose that $q = 1$ and $s \geq 0$, or $1 < q \leq p'$ and $s > n/q'$. Then for any compact subset K of $\mathbf{R}^n$, K is a set of spectral synthesis for $M^{p,q}_s(\mathbf{R}^n)$ if and only if K is a set of spectral synthesis for $\mathcal{F}L^q_s(\mathbf{R}^n)$.*

Remark 1.1. The case $q = 1$ and $s \geq 0$ is given in [5].

Theorem 1.1 allows to give some concrete examples of sets of spectral synthesis for $M^{p,q}_s(\mathbf{R}^n)$ (cf. [11] or Proposition 3.1 below).

Example 1.2. (*i*) Let $1 \leq p \leq 2$, $1 < q \leq 2$ and $n/q' < s < n/q' + 1$. Then single points of $\mathbf{R}^n$ are sets of spectral synthesis for $M^{p,q}_s(\mathbf{R}^n)$.

(*ii*) Let $1 \leq p \leq 2$, $1 < q < \infty$ and $s > n/q' + 1$. Then single points of $\mathbf{R}^n$ are not sets of spectral synthesis for $M^{p,q}_s(\mathbf{R}^n)$.

Example 1.3. (*i*) Let $1 \leq p \leq 2$, $1 < q \leq p'$ and $2/q' < s < 2/q' + 1/2$. Then S^1 is a set of spectral synthesis for $M^{p,q}_s(\mathbf{R}^2)$.

(*ii*) But for $1 \leq p \leq 2$, $1 < q < 2$ and $s > 2/q' + 1/2$ S^1 is not a set of spectral synthesis for $M_s^{p,q}(\mathbf{R}^2)$.

(*iii*) Let $1 \leq p \leq 2$, $1 < q \leq p'$ and $s > n/q'$. Then S^{n-1} $(n \geq 3)$ is not a set of spectral synthesis for $M_s^{p,q}(\mathbf{R}^n)$.

As a by-product of Theorem 1.1 we obtain a variant of Wiener–Lévy theorem for $\mathcal{F}L_s^q(\mathbf{R}^n)$ and $M_s^{p,q}(\mathbf{R}^n)$.

Theorem 1.4. *Given $1 < q < \infty$, $s > n/q'$ and a real-valued function $f \in \mathcal{F}L_s^q(\mathbf{R}^n)$. Suppose that F is an analytic function on a neighborhood of $f(\mathbf{R}^n) \cup \{0\}$ with $F(0) = 0$. Then there exists $g \in \mathcal{F}L_s^q(\mathbf{R}^n)$ such that $g(x) = F(f(x))$.*

Theorem 1.5. *Given $1 < p < \infty$, $1 < q < \infty$, $s > n/q'$ and a real-valued function $f \in M_s^{p,q}(\mathbf{R}^n)$. Suppose that F is an analytic function on a neighborhood of $f(\mathbf{R}^n) \cup \{0\}$ with $F(0) = 0$. Then there exists $g \in M_s^{p,q}(\mathbf{R}^n)$ such that $g(x) = F(f(x))$.*

The organization of this paper is as follows. After a preliminary section devoted to the definitions of $M_s^{p,q}(\mathbf{R}^n)$ and $\mathcal{F}L_s^q(\mathbf{R}^n)$ we prove Theorem 1.1 in Section 3. Theorems 1.4 and 1.5 are treated in Section 4.

2. Preliminaries

The following notation will be used throughout this article. We use C to denote various positive constants which may change from line to line. We use the notation $I \lesssim J$ if I is bounded by a constant times J and we denote $I \approx J$ if $I \lesssim J$ and $J \lesssim I$. The closed ball with center $x_0 \in \mathbf{R}^n$ and radius $r > 0$ is defined by $B_r(x_0) := \{x \in \mathbf{R}^n \mid |x - x_0| \leq r\}$. Let $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ for $x \in \mathbf{R}^n$. We define for $1 \leq p < \infty$ and $s \in \mathbf{R}$

$$\|f\|_{L_s^p} := \Big(\int_{\mathbf{R}^n} \big(\langle x \rangle^s |f(x)|\big)^p dx \Big)^{\frac{1}{p}},$$

and $\|f\|_{L_s^\infty} := \text{ess.sup}_{x \in \mathbf{R}^n} \langle x \rangle^s |f(x)|$. We simply write $L^p(\mathbf{R}^n)$ instead of $L_0^p(\mathbf{R}^n)$. For $1 \leq p < \infty$, p' denotes the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. We write $C_c^\infty(\mathbf{R}^n)$ for the space of complex-valued infinitely differentiable functions on $\mathbf{R}^n$ with compact support. $\mathcal{S}(\mathbf{R}^n)$ denotes the Schwartz space of complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$ and $\mathcal{S}'(\mathbf{R}^n)$ denotes the space of tempered distributions. The Fourier transform of $f \in L^1(\mathbf{R}^n)$ is $\mathcal{F}f(\xi) = \widehat{f}(\xi) := \int_{\mathbf{R}^n} f(x) e^{-ix\xi} dx$. Similarly, the inverse Fourier transform of $h \in L^1(\mathbf{R}^n)$ is $\mathcal{F}^{-1}h(x) := (2\pi)^{-n}\widehat{h}(-x)$. Recall that

$(f*g)^\wedge = \widehat{f}\widehat{g}$ and $(fg)^\wedge = (2\pi)^{-n}(\widehat{f}*\widehat{g})$, where $(f*g)(x) = \int_{\mathbf{R}^n} f(x-y)g(y)dy$. For two Banach spaces B_1 and B_2, $B_1 \hookrightarrow B_2$ means that B_1 is continuously embedded into B_2.

2.1. Fourier–Wermer algebra

For $1 \leq q < \infty$ and $s \in \mathbf{R}$ write $\mathcal{F}L^q_s(\mathbf{R}^n) = \mathcal{F}L^q_s$ for the space of all tempered distributions with the following norm is finite:

$$\|f\|_{\mathcal{F}L^q_s} := \Big(\int_{\mathbf{R}^n} \Big(\langle\xi\rangle^s |\widehat{f}(\xi)| \Big)^q d\xi \Big)^{\frac{1}{q}}.$$

It is well known that $(\mathcal{F}L^q_s(\mathbf{R}^n), \|\cdot\|_{\mathcal{F}L^q_s})$ is a Banach space and $\mathcal{S}(\mathbf{R}^n)$ is dense in $\mathcal{F}L^q_s(\mathbf{R}^n)$. If $q = 1$ and $s \geq 0$, or $1 < q < \infty$ and $s > n/q'$, then $\mathcal{F}L^q_s(\mathbf{R}^n)$ is a multiplication algebra, i.e.,

$$\|fg\|_{\mathcal{F}L^q_s} \leq c\|f\|_{\mathcal{F}L^q_s}\|g\|_{\mathcal{F}L^q_s}, \qquad f, g \in \mathcal{F}L^q_s(\mathbf{R}^n) \tag{2.1}$$

for some $c \geq 1$. We call it the *Fourier–Wermer algebra* owing to the fact that it is the Fourier image of the convolution algebra $L^q_s(\mathbf{R}^n)$ that was studied in the early paper of Wermer [19]. Moreover, if $q = 1$ and $s \geq 0$, or $1 < q < \infty$ and $s > n/q'$, then $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$ implies $\widehat{f} \in L^1(\mathbf{R}^n)$ by the Hölder inequality. Thus the Riemann–Lebesgue lemma shows $f \in C(\mathbf{R}^n)$ and vanishes at infinity, and the inversion formula applies, giving $f(x) = \mathcal{F}^{-1}(\widehat{f})(x)$ for all $x \in \mathbf{R}^n$.

One can prove that $\mathcal{F}L^q_s(\mathbf{R}^n)$ possess approximate units.

Lemma 2.1. *Given $1 < q < \infty$ and $s > \frac{n}{q'}$, or $q = 1$ and $s \geq 0$. Then for $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$ and $\varepsilon > 0$ there exists $\phi \in C^\infty_c(\mathbf{R}^n)$ such that $\|\phi f - f\|_{\mathcal{F}L^q_s} < \varepsilon$.*

Remark 2.1. The case $q = 1$ and $s \geq 0$ is given in [16, Proposition 1.6.14].

Proof. Let $\psi \in C^\infty_c(\mathbf{R}^n)$ be such that $\psi(0) = 1$. For $0 < \lambda < 1$ we set $\psi_\lambda(x) = \psi(\lambda x)$. Since $(\widehat{\psi_\lambda} * \widehat{f})(\xi) = \int_{\mathbf{R}^n} \widehat{\psi}(\eta)\widehat{f}(\xi - \lambda\eta)d\eta$ and $1 = \psi(0) = (2\pi)^{-n}\int_{\mathbf{R}^n} \widehat{\psi}(\eta)d\eta$, we have

$$(\psi_\lambda f - f)^\wedge(\xi) = (2\pi)^{-n} \int_{\mathbf{R}^n} \widehat{\psi}(\eta)\Big(\widehat{f}(\xi - \lambda\eta) - \widehat{f}(\xi)\Big)d\eta.$$

Applying the Minkowski inequality for integrals we obtain that

$$\|\psi_\lambda f - f\|_{\mathcal{F}L^q_s} \lesssim \int_{\mathbf{R}^n} |\widehat{\psi}(\eta)| \Big\| \langle\cdot\rangle^s |\widehat{f}(\cdot - \lambda\eta) - \widehat{f}(\cdot)| \Big\|_{L^q} d\eta.$$

We note $0<\lambda<1$. By the submultiplicity of $\langle\cdot\rangle^s$ for $s\ge 0$ one has

$$\Big\|\langle\cdot\rangle^s\big|\widehat{f}(\cdot-\lambda\eta)-\widehat{f}(\cdot)\big|\Big\|_{L^q}\lesssim\langle\lambda\eta\rangle^s\|\langle\cdot-\lambda\eta\rangle^s|\widehat{f}(\cdot-\lambda\eta)|\|_{L^q}+\|\langle\cdot\rangle^s|\widehat{f}(\cdot)|\|_{L^q}\lesssim$$
$$\lesssim\langle\eta\rangle^s\|f\|_{\mathcal{F}L^q_s}.$$

Thus we can easily see $\lim_{\lambda\to 0}\|\psi_\lambda f-f\|_{\mathcal{F}L^q_s}=0$. Hence, for any $\varepsilon>0$ there exists $\phi:=\psi_{\lambda_0}$ for some $0<\lambda_0<1$ such that $\|\phi f-f\|_{\mathcal{F}L^q_s}<\varepsilon$. ∎

Furthermore, we have the following result.

Lemma 2.2. *Given $1<q<\infty$ and $s>n/q'$, or $q=1$ and $s\ge 0$. Suppose that $f\in\mathcal{F}L^q_s(\mathbf{R}^n)$ and $f(x_0)=0$ for some $x_0\in\mathbf{R}^n$. Then for $\varepsilon>0$ there exists $g\in C_c^\infty(\mathbf{R}^n)$ such that $\|f-g\|_{\mathcal{F}L^q_s}<\varepsilon$ and $g(x_0)=0$.*

Proof. Let $\varepsilon>0$. By Lemma 2.1 there exists $\phi\in C_c^\infty(\mathbf{R}^n)$ such that $\|\phi f-f\|_{\mathcal{F}L^q_s}<\varepsilon$ and $(\phi f)(x_0)=\phi(x_0)f(x_0)=0$. Take $\psi\in C_c^\infty(\mathbf{R}^n)$ with $\psi(x)=1$ on $\operatorname{supp}\phi$. Since $\phi f\in\mathcal{F}L^q_s(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$ is dense in $\mathcal{F}L^q_s(\mathbf{R}^n)$, there exists $g_0\in\mathcal{S}(\mathbf{R}^n)$ such that $\|\phi f-g_0\|_{\mathcal{F}L^q_s}<\varepsilon/3(\|\psi\|_{\mathcal{F}L^q_s}+1)$. Define $g(x):=(g_0(x)-g_0(x_0))\psi(x)\in C_c^\infty(\mathbf{R}^n)$. Then

$$\|\phi f-g_0\|_{L^\infty}\lesssim\|(\phi f-g_0)^\wedge\|_{L^1}\lesssim\|\phi f-g_0\|_{\mathcal{F}L^q_s},$$

and $g(x_0)=0$. By $\phi=\phi\psi$, $(\phi f)(x_0)=0$ and (2.1) one has

$$\|f-g\|_{\mathcal{F}L^q_s}=\|f-\phi f+\phi f\psi-(\phi f)(x_0)\psi-(g_0-g_0(x_0))\psi\|_{\mathcal{F}L^q_s}\lesssim$$
$$\lesssim\|f-\phi f\|_{\mathcal{F}L^q_s}+\|\phi f-g_0\|_{\mathcal{F}L^q_s}\|\psi\|_{\mathcal{F}L^q_s}+|g_0(x_0)-(\phi f)(x_0)|\|\psi\|_{\mathcal{F}L^q_s}\lesssim$$
$$\lesssim\|f-\phi f\|_{\mathcal{F}L^q_s}+2\|\phi f-g_0\|_{\mathcal{F}L^q_s}\|\psi\|_{\mathcal{F}L^q_s}<\varepsilon.$$ ∎

Corollary 2.1. *Given $1<q<\infty$ and $s>n/q'$, or $q=1$ and $s\ge 0$, $f\in\mathcal{F}L^q_s(\mathbf{R}^n)$ and $x_0\in\mathbf{R}^n$. Then for $\varepsilon>0$ there exists $g\in C_c^\infty(\mathbf{R}^n)$ such that $g(x_0)=f(x_0)$ and $\|f-g\|_{\mathcal{F}L^q_s}<\varepsilon$.*

Proof. It suffices to prove the case $x_0=0$; the other case is easy to see by considering $F(x):=f(x+x_0)$ instead. Let $\varepsilon>0$. Then the proof of Lemma 2.1 implies that there exists $\phi\in C_c^\infty(\mathbf{R}^n)$ with $\phi(0)=1$ and $\|\phi f-f\|_{\mathcal{F}L^q_s}<\varepsilon/2$. Set $f_0(x):=(f(x)-f(0))\phi(x)\in\mathcal{F}L^q_s(\mathbf{R}^n)$. Then $f_0(0)=0$. By Lemma 2.2 there exists $g_0\in C_c^\infty(\mathbf{R}^n)$ such that $g_0(0)=0$ and $\|f_0-g_0\|_{\mathcal{F}L^q_s}<\varepsilon/2$. Now set $g(x):=g_0(x)+f(0)\phi(x)$. Then $g(0)=f(0)$ and $\|f-g\|_{\mathcal{F}L^q_s}<\varepsilon$. ∎

2.2. Modulation spaces

Let $1\le p,q\le\infty$, $s\in\mathbf{R}$ and $\varphi\in\mathcal{S}(\mathbf{R}^n)$ be such that

$$\operatorname{supp}\varphi\subset[-1,1]^n\quad\text{and}\quad\sum_{k\in\mathbf{Z}^n}\varphi(\xi-k)=1\quad(\xi\in\mathbf{R}^n).\tag{2.2}$$

Then $M_s^{p,q}(\mathbf{R}^n) = M_s^{p,q}$ consists of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|_{M_s^{p,q}} = \Big(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq} \Big(\int_{\mathbf{R}^n} |\varphi(D-k)f(x)|^p dx \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}}$$

is finite, with obvious modifications if p or $q = \infty$. Here $\varphi(D-k)f(x) = = \mathcal{F}^{-1}(\varphi(\cdot - k)\widehat{f}(\cdot))(x)$. It is well known that $M_s^{p,q}(\mathbf{R}^n)$ is a multiplication algebra if $s > n/q'$, or $q = 1$ and $s \geq 0$. We also recall a few basic properties of the function in $M_s^{p,q}(\mathbf{R}^n)$. Next we state some auxiliary lemmata.

Lemma 2.3 ([5], [7]). *Given $1 \leq p < \infty$, $q = 1$ and $s \geq 0$, or $1 < q < \infty$ and $s > n/q'$. Then $\|fg\|_{M_s^{p,q}} \lesssim \|f\|_{M_s^{\infty,q}} \|g\|_{M_s^{p,q}}$ for $f, g \in \mathcal{S}(\mathbf{R}^n)$.*

Lemma 2.4. *Let $1 \leq p < \infty$. Suppose that $q = 1$ and $s \geq 0$, or $1 < q < \infty$ and $s > n/q'$. If $f \in M_s^{p,q}(\mathbf{R}^n)$, then for any $\varepsilon > 0$ there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that $\|\phi f - f\|_{M_s^{p,q}} < \varepsilon$.*

Proof. The case $p = q = 1$ and $s = 0$ was considered in Bhimani–Ratnakumar [1, Proposition 3.14] and their method still applies for the remaining cases, with only trifling changes (see [5] for more details). ∎

A useful inclusion relation (cf. [12], [13]) is stated next.

Lemma 2.5. *Let $1 \leq p \leq 2$. Suppose that $q = 1$ and $s \geq 0$, or $1 < q \leq p'$ and $s > n/q'$. Then we have $M_s^{p,q}(\mathbf{R}^n) \hookrightarrow \mathcal{F}L_s^q(\mathbf{R}^n)$.*

Proof. We give the proof only for $1 < p \leq 2$; the case $p = 1$ is similar. Given $f \in M_s^{p,q}(\mathbf{R}^n)$ and $\varphi \in C_c^\infty(\mathbf{R}^n)$ as in (2.2) there exists $N \in \mathbf{N}$ with

$$\chi_{k+[-1,1]^n}(\xi) = \sum_{|\ell|\leq N} \varphi(\xi - (k+\ell))\chi_{k+[-1,1]^n}(\xi) \quad (\xi \in \mathbf{R}^n)$$

for all $k \in \mathbf{Z}^n$. By the Hölder inequality with $1/(p'/q) + 1/(p'/(p'-q)) = 1$ and the Hausdorff–Young inequality we see

$$\begin{aligned}
\|f\|_{\mathcal{F}L_s^q}^q &\leq \sum_{k\in\mathbf{Z}^n} \int_{k+[-1,1]^n} \langle\xi\rangle^{sq} |\widehat{f}(\xi)|^q d\xi \lesssim \\
&\lesssim \sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq} \int_{k+[-1,1]^n} \Big| \sum_{|\ell|\leq N} \varphi(\xi-(k+\ell))\widehat{f}(\xi) \Big|^q d\xi \lesssim \\
&\lesssim \sum_{|\ell|\leq N} \sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq} \int_{k+[-1,1]^n} |\varphi(\xi-(k+\ell))\widehat{f}(\xi)|^q d\xi \lesssim \\
&\lesssim \sum_{|\ell|\leq N} \sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq} \|\varphi(\cdot-(k+\ell))\widehat{f}(\cdot)\|_{L^{p'}}^q \lesssim \\
&\lesssim \sum_{|\ell|\leq N} \sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq} \|\varphi(D-(k+\ell))f\|_{L^p}^q \lesssim \|f\|_{M_s^{p,q}}^q .
\end{aligned}$$

∎

Lemma 2.6. *Let* $1 \le p < \infty$*. Suppose that* $q = 1$ *and* $s \ge 0$*, or* $1 < q < \infty$ *and* $s > n/q'$*. Set* $(\mathcal{F}L^q_s)_c := \{f \in \mathcal{F}L^q_s(\mathbf{R}^n) \mid \operatorname{supp} f \text{ is compact}\}$*. Then we have* $(\mathcal{F}L^q_s)_c \hookrightarrow M^{p,q}_s(\mathbf{R}^n)$*.*

Proof. Let $f \in (\mathcal{F}L^q_s)_c$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$ be such that $\phi(x) = 1$ on $\operatorname{supp} f$. By Lemma 2.3 we have

$$\|f\|_{M^{p,q}_s} = \|f\phi\|_{M^{p,q}_s} \lesssim \|f\|_{M^{\infty,q}_s} \|\phi\|_{M^{p,q}_s}.$$

Moreover, let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ as in (2.2). Then there exists $N \in \mathbf{N}$ such that

$$\varphi(\xi - k) = \sum_{|\ell| \le N} \varphi(\xi - k)\varphi(\xi - (k + \ell)) \quad (\xi \in \mathbf{R}^n)$$

for all $k \in \mathbf{Z}^n$. Then the Hausdorff–Young and the Hölder inequality show that

$$\begin{aligned}\|f\|^q_{M^{\infty,q}_s} &\lesssim \sum_{k \in \mathbf{Z}^n} \langle k\rangle^{sq} \|\varphi(\cdot - k)\widehat{f}\|^q_{L^1} \le \\ &\le \sum_{k \in \mathbf{Z}^n} \langle k\rangle^{sq} \Big(\sum_{|\ell| \le N} \|\varphi(\cdot - (k+\ell))\|_{L^{q'}} \|\varphi(\cdot - k)\widehat{f}\|_{L^q}\Big)^q \lesssim \\ &\lesssim \sum_{k \in \mathbf{Z}^n} \int_{k + [-1,1]^n} \langle \xi\rangle^{sq} |\varphi(\xi - k)\widehat{f}(\xi)|^q d\xi \lesssim \|f\|^q_{\mathcal{F}L^q_s},\end{aligned}$$

which implies the desired result.

Remark 2.2. There is another characterization of $M^{p,q}_s$ using the short-time Fourier transform, i.e., for $\phi \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$, we set:

$$V_\phi f(x, \xi) := \langle f(t), \phi(t - x)e^{it\xi}\rangle = \int_{\mathbf{R}^n} f(t)\overline{\phi(t - x)}e^{-it\xi} dt,$$

$$\|f\|_{M^{p,q}_s} \approx \Big(\int_{\mathbf{R}^n} \langle \xi\rangle^{sq} \Big(\int_{\mathbf{R}^n} |V_\phi f(x, \xi)|^p dx\Big)^{\frac{q}{p}} d\xi\Big)^{\frac{1}{q}},$$

i.e. this defines an equivalent norm, and in addition

$$V_\phi f(x, \xi) = (2\pi)^{-n} e^{-ix\xi} V_{\widehat{\phi}} \widehat{f}(\xi, -x) = (2\pi)^{-n} e^{-ix\xi} (f * M_\xi \phi^*)(x), \tag{2.3}$$

where $\phi^*(x) = \overline{\phi(-x)}$ (see [6, Lemma 3.1.1]).

3. Spectral synthesis

Throughout this section, X stands for $M^{p,1}_s(\mathbf{R}^n)$ $(1 \le p < \infty,\ s \ge 0)$, $M^{p,q}_s(\mathbf{R}^n)$ $(1 \le p \le 2,\ 1 < q < \infty,\ s > n/q')$, $\mathcal{F}L^1_s(\mathbf{R}^n)$ $(s \ge 0)$ or $\mathcal{F}L^q_s(\mathbf{R}^n)$ $(1 < q < \infty,\ s > n/q')$. Moreover, the closure of $X_0 \subset X$ in X will be denoted by $\overline{X_0}^{\|\cdot\|_X}$.

Definition 3.1. Let I be a linear subspace of X. Then I is called an *ideal* in X if $fg \in I$ whenever $f \in X$ and $g \in I$. Moreover, if an ideal I in X is a closed subset of X, then I is called a *closed ideal* in X. For a subset S of X, the set $\bigcap_{\lambda\in\Lambda} I_\lambda$ is called the ideal generated by S, where $\{I_\lambda\}_{\lambda\in\Lambda}$ denoted the set of all ideals in X containing S.

Definition 3.2. Let I be a closed ideal in X. Then the *zero-set* of I is defined by $Z(I) := \bigcap\limits_{f\in I} f^{-1}(\{0\})$ with $f^{-1}(\{0\}) := \{x \in \mathbf{R}^2 \mid f(x) = 0\}$.

We note that $Z(I)$ is a closed subset of X if I is a closed ideal in X. In fact, if $f \in X$, then f is continuous on $\mathbf{R}^n$ and thus $f^{-1}(\{0\})$ is a closed subset of $\mathbf{R}^n$. We will write $f|_E = 0$ if $f(x) = 0$ for all $x \in E$.

Lemma 3.1. *Let E be a closed subset of $\mathbf{R}^n$. Then $I(E) := \{f \in X \mid f|_E = 0\}$ is a closed ideal in X with $E = Z(I(E))$.*

Proof. We give the proof only for the case $X = M^{p,q}_s(\mathbf{R}^n)$; the same applies to other cases. It is clear that $I(E)$ is an ideal in $M^{p,q}_s(\mathbf{R}^n)$. To see $I(E)$ is closed, let $f \in M^{p,q}_s(\mathbf{R}^n)$, $\{f_n\}_{n=1}^\infty \subset I(E)$ and $\|f_n - f\|_{M^{p,q}_s} \to 0$ $(n \to \infty)$. Since

$$\|f_n - f\|_{L^\infty} \lesssim \|f_n - f\|_{M^{\infty,1}} \lesssim \|f_n - f\|_{M^{p,1}} \lesssim \|f_n - f\|_{M^{p,q}_s},$$

we see that $\{f_n\}_{n=1}^\infty$ converges pointwise to f on $\mathbf{R}^n$. Since $f_n|_E = 0$, we have $f|_E = 0$, and thus $f \in I(E)$. Hence $I(E)$ is closed. Next we prove $E = Z(I(E))$. Since $E \subset Z(I(E))$ is clear, we show $Z(I(E)) \subset E$. Suppose $x_0 \notin E$. Since E is closed and $C^\infty_c(\mathbf{R}^n) \subset M^{p,1}_s(\mathbf{R}^n)$, there exists $f \in M^{p,1}_s(\mathbf{R}^n)$ such that $f(x_0) = 1$ and $f|_E = 0$. Then $f \in I(E)$ and $f(x_0) \neq 0$. Thus $x_0 \notin Z(I(E))$, which implies the desired result. ■

Definition 3.3. Let $E \subset \mathbf{R}^n$ be closed and $I(E)$ be the set defined in Lemma 3.1. Define $J(E)$ by the closed ideal in X generated by

$$J_0(E) := \{f \in X \mid f(x) = 0 \text{ in a neighborhood of } E\}.$$

Then E is called a *set of spectral synthesis* for X if $I(E) = J(E)$.

Remark 3.1. $J(E)$ is the smallest closed ideal I' with $Z(I') = E$ (cf. [11]).

3.1. Spectral synthesis for $\mathcal{F}L^q_s$

Proposition 3.1. *(i) Let $1 < q \leq 2$ and $n/q' < s < n/q' + 1$. Then single points of $\mathbf{R}^n$ are sets of spectral synthesis for $\mathcal{F}L^q_s(\mathbf{R}^n)$.*

(ii) Let $1 < q < \infty$ and $s > n/q' + 1$. Then single points of $\mathbf{R}^n$ are not sets of spectral synthesis for $\mathcal{F}L^q_s(\mathbf{R}^n)$.

Remark 3.2. It is well-known that single points of $\mathbf{R}^n$ are sets of spectral synthesis for $\mathcal{F}L^1_s(\mathbf{R}^n)$, if $0 \le s < 1$ (see [16, Theorem 2.7.9]).

To prove Proposition 3.1, we use a modification of [16, Lemma 6.3.6].

Lemma 3.2. *Let $1 < q \le 2$. Suppose that $\psi^{(1)}, \psi^{(2)} \in C_c^\infty(\mathbf{R}^n)$ be such that* supp $\psi^{(j)} \subset B(0,R)$ *for some $R > 0$ $(j = 1,2)$. Set $\psi := \psi^{(1)} * \psi^{(2)}$. Then for $n/q' < s < n/q' + 1$ and $\vartheta \in \mathbf{R}^n$*

$$\big\|\langle\cdot\rangle^s\big(\widehat{\psi}(\cdot - \vartheta) - \widehat{\psi}(\cdot)\big)\big\|_{L^q} \le C_\psi\Big(|\vartheta|^{s-\frac{n}{q'}} \max_{|t|\le R} |e^{i\vartheta t} - 1|^{1-(s-\frac{n}{q'})} + |\vartheta|^s\Big).$$

Proof. We give the proof only for $1 < q < 2$: the case $q = 2$ is similar. We first note that since $\widehat{\psi} = \widehat{\psi^{(1)}} \cdot \widehat{\psi^{(2)}}$ we have

$$\begin{aligned}&\widehat{\psi}(\xi - \vartheta) - \widehat{\psi}(\xi) = \\ &= \Big(\widehat{\psi^{(1)}}(\xi - \vartheta) - \widehat{\psi^{(1)}}(\xi)\Big)\widehat{\psi^{(2)}}(\xi - \vartheta) + \widehat{\psi^{(1)}}(\xi)\Big(\widehat{\psi^{(2)}}(\xi - \vartheta) - \widehat{\psi^{(2)}}(\xi)\Big).\end{aligned}$$

Then the Hölder inequality with $1/(2/q) + 1/(2/(2-q)) = 1$ and the Plancherel theorem show that

$$\begin{aligned}&\big\|\langle\xi\rangle^s\big(\widehat{\psi}(\xi - \vartheta) - \widehat{\psi}(\xi)\big)\big\|_{L^q(\mathbf{R}^n_\xi)} \lesssim \\ &\lesssim \big\|\widehat{\psi^{(1)}}(\xi - \vartheta) - \widehat{\psi^{(1)}}(\xi)\big\|_{L^2(\mathbf{R}^n_\xi)}\big\|\langle\xi\rangle^s\widehat{\psi^{(2)}}(\xi - \vartheta)\big\|_{L^{\frac{2q}{2-q}}(\mathbf{R}^n_\xi)} + \\ &\quad + \big\|\widehat{\psi^{(2)}}(\xi - \vartheta) - \widehat{\psi^{(2)}}(\xi)\big\|_{L^2(\mathbf{R}^n_\xi)}\big\|\langle\xi\rangle^s\widehat{\psi^{(1)}}(\xi - \vartheta)\big\|_{L^{\frac{2q}{2-q}}(\mathbf{R}^n_\xi)} = \\ &= \|\mathcal{F}_{x\to\xi}[(e^{i\vartheta x} - 1)\psi^{(1)}(x)](\xi)\|_{L^2(\mathbf{R}^n_\xi)}\langle\vartheta\rangle^s\big\|\langle\cdot\rangle^s\widehat{\psi^{(2)}}\big\|_{L^{\frac{2q}{2-q}}} + \\ &\quad + \|\mathcal{F}_{x\to\xi}[(e^{i\vartheta x} - 1)\psi^{(2)}(x)](\xi)\|_{L^2(\mathbf{R}^n_\xi)}\langle\vartheta\rangle^s\big\|\langle\cdot\rangle^s\widehat{\psi^{(1)}}\big\|_{L^{\frac{2q}{2-q}}}.\end{aligned}$$

Note $|e^{i\vartheta x} - 1| \le \min\{2, |\vartheta x|\}$. The Plancherel theorem shows

$$\|\mathcal{F}_{x\to\xi}[(e^{i\vartheta x} - 1)\psi^{(j)}(x)](\xi)\|_{L^2(\mathbf{R}^n_\xi)} = (2\pi)^{\frac{n}{2}}\|(e^{i\vartheta x} - 1)\psi^{(j)}(x)\|_{L^2(\mathbf{R}^n_x)}$$

for $j = 1,2$. Since supp $\psi^{(j)} \subset B_R(0)$ $(j = 1,2)$, we obtain

$$\begin{aligned}&\|\mathcal{F}_{x\to\xi}[(e^{i\vartheta x} - 1)\psi^{(j)}(x)](\xi)\|_{L^2(\mathbf{R}^n_\xi)}(1 + |\vartheta|^s) \lesssim \\ &\lesssim \Big(\max_{|x|\le R}|e^{i\vartheta x} - 1|\Big)^{1-(s-\frac{n}{q'})}\big\||e^{i\vartheta x} - 1|^{s-\frac{n}{q'}}|\psi^{(j)}(x)|\big\|_{L^2(\mathbf{R}^n_x)} + \|\psi^{(j)}\|_{L^2}|\vartheta|^s \lesssim \\ &\lesssim |\vartheta|^{s-\frac{n}{q'}}\Big(\max_{|x|\le R}|e^{i\vartheta x} - 1|\Big)^{1-(s-\frac{n}{q'})}\big\||x|^{s-\frac{n}{q'}}|\psi^{(j)}(x)|\big\|_{L^2(\mathbf{R})^n_x} + |\vartheta|^s,\end{aligned}$$

which yields the desired inequality. ∎

Lemma 3.3. *Let $1 < q \le 2$, $n/q' < s < n/q' + 1$, $f \in \mathcal{F}L^1_s(\mathbf{R}^n)$, $x_0 \in \mathbf{R}^n$ and $\varepsilon > 0$. Then there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that (i) $\|(f - f(x_0))\phi\|_{M_s^{1,q}} < \varepsilon$. (ii) $\phi(x) = 1$ in some neighborhood of x_0.*

Proof. Note that there exist $\psi^{(1)}, \psi^{(2)} \in C_c^\infty(\mathbf{R}^n)$ such that supp $\psi^{(j)} \subset\subset B_2(0)$ $(j = 1, 2)$ and $\psi := \psi^{(1)} * \psi^{(2)}$ satisfies $\psi(x) = 1$ on $B_1(0)$ and supp $\psi \subset\subset B_5(0)$ (see [5]). For $\lambda > 5$, we set $\psi_\lambda(x) := \psi(\lambda x)$ and $h^\lambda(x) := (f(x) - f(x_0))\psi_\lambda(x - x_0)$. If $x \in$ supp h^λ, then $\psi(x - x_0) = 1$. Thus $h^\lambda(x) = h^\lambda(x)\psi(x - x_0)$. We consider the case $x_0 = 0$; the other case can be treated by considering $\psi(x - x_0)$ instead of $\psi(x)$. Let $g_0(t) := e^{-|t|^2/2}$ $(t \in \mathbf{R}^n)$. Then (2.3) shows

$$\|h^\lambda\|_{M_s^{1,q}} = \Big\| \|\langle\xi\rangle^s V_{g_0} h^\lambda(x,\xi)\|_{L^1(\mathbf{R}^n_x)} \Big\|_{L^q(\mathbf{R}^n_\xi)} \approx$$
$$\approx \Big\| \|\langle\xi\rangle^s V_{\widehat{g_0}} \widehat{h^\lambda}(\xi,-x)\|_{L^1(\mathbf{R}^n_x)} \Big\|_{L^q(\mathbf{R}^n_\xi)}.$$

Since $\widehat{g_0} = (2\pi)^{\frac{n}{2}} g_0$, $g_0^* = g_0$ and $h^\lambda(x) = h^\lambda(x)\psi(x)$, we obtain by (2.3)

$$V_{\widehat{g_0}} \widehat{h^\lambda}(\xi,-x) = (2\pi)^{-\frac{n}{2}} V_{g_0}(\widehat{h^\lambda} * \widehat{\psi})(\xi,-x) = (2\pi)^{-\frac{n}{2}} e^{ix\xi}(\widehat{h^\lambda} * \widehat{\psi} * M_{-x} g_0)(\xi).$$

The Minkowski inequality for integrals and the Young inequality yield

$$\|h^\lambda\|_{M_s^{1,q}} = (2\pi)^{-n} \Big\| \|\langle\xi\rangle^s V_{\widehat{g_0}} \widehat{h^\lambda}(\xi,-x)\|_{L^1(\mathbf{R}^n_x)} \Big\|_{L^q(\mathbf{R}^n_\xi)} \lesssim$$
$$\lesssim \Big\| \|\langle\xi\rangle^s (\widehat{h^\lambda} * \widehat{\psi} * M_{-x} g_0)(\xi)\|_{L^q(\mathbf{R}^n_\xi)} \Big\|_{L^1(\mathbf{R}^n_x)} \le$$
$$\le \Big\| \|\langle\xi\rangle^s \widehat{h^\lambda}(\xi)\|_{L^q(\mathbf{R}^n_\xi)} \|\langle\xi\rangle^s (\widehat{\psi} * M_{-x} g_0)(\xi)\|_{L^1(\mathbf{R}^n_\xi)} \Big\|_{L^1(\mathbf{R}^n_x)} =$$
$$= \|\langle\cdot\rangle^s \widehat{h^\lambda}\|_{L^q} \|\psi\|_{M_s^{1,1}}.$$

Since $\widehat{h^\lambda}(\xi) = (2\pi)^{-n}(\widehat{\psi_\lambda} * \widehat{f})(\xi) - f(0)\widehat{\psi_\lambda}(\xi)$ and $\widehat{\psi_\lambda}(\xi) = \lambda^{-n}\widehat{\psi}(\xi/\lambda)$, we have

$$\widehat{h^\lambda}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \widehat{f}(\eta)\widehat{\psi_\lambda}(\xi - \eta)d\eta - \frac{1}{(2\pi)^n}\Big(\int_{\mathbf{R}^n} \widehat{f}(\eta)d\eta\Big)\widehat{\psi_\lambda}(\xi) =$$
$$= \frac{1}{(2\pi\lambda)^n} \int_{\mathbf{R}^n} \widehat{f}(\eta)\Big(\widehat{\psi}\Big(\frac{\xi - \eta}{\lambda}\Big) - \widehat{\psi}\Big(\frac{\xi}{\lambda}\Big)\Big)d\eta.$$

Then the Minkowski inequality for integrals and Lemma 3.2 choosing $\vartheta = \eta/\lambda$ and $R = 4$ yield that

$$\|\langle\cdot\rangle^s \widehat{h^\lambda}\|_{L^q(\mathbf{R}^n)} \le \lambda^{-n} \int_{\mathbf{R}^n} \Big\| \langle\cdot\rangle^s \Big(\widehat{\psi}\Big(\frac{\cdot - \eta}{\lambda}\Big) - \widehat{\psi}\Big(\frac{\cdot}{\lambda}\Big)\Big)\Big\|_{L^q} |\widehat{f}(\eta)| d\eta \lesssim$$
$$\lesssim \lambda^{-n+s+\frac{n}{q}} \int_{\mathbf{R}^n} \Big\| \langle\cdot\rangle^s \Big(\widehat{\psi}\Big(\cdot - \frac{\eta}{\lambda}\Big) - \widehat{\psi}(\cdot)\Big)\Big\|_{L^q} |\widehat{f}(\eta)| d\eta \lesssim$$

$$\lesssim \lambda^{s-\frac{n}{q'}} \int_{\mathbf{R}^n} \Big(\Big|\frac{\eta}{\lambda}\Big|^{s-\frac{n}{q'}} \Big(\max_{|t|\le 4} |e^{i\frac{\eta}{\lambda}t} - 1| \Big)^{1-(s-\frac{n}{q'})} + \Big|\frac{\eta}{\lambda}\Big|^{s} \Big) |\widehat{f}(\eta)| d\eta \lesssim$$

$$\lesssim \int_{\mathbf{R}^n} \Big(\Big(\max_{|t|\le 4} |e^{i\frac{\eta}{\lambda}t} - 1| \Big)^{1-(s-\frac{n}{q'})} + \lambda^{-\frac{n}{q'}} \Big) \langle \eta \rangle^s |\widehat{f}(\eta)| d\eta.$$

We observe that $\big(\max_{|t|\le 4} |e^{i\frac{\eta}{\lambda}t} - 1|\big)^{1-(s-\frac{n}{q'})} + \lambda^{-\frac{n}{q'}} \le 3$ and

$$\Big(\max_{|t|\le 4} |e^{i\frac{\eta}{\lambda}t} - 1| \Big)^{1-(s-\frac{n}{q'})} + \lambda^{-\frac{n}{q'}} \to 0 \quad (\lambda \to \infty).$$

Since $f \in \mathcal{F}L^1_s(\mathbf{R}^n)$, we see that $\|\langle\cdot\rangle^s \widehat{h^\lambda}\|_{L^q} \to 0$ $(\lambda \to \infty)$. Therefore, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that $\|h^{\lambda_0}\|_{M^{1,q}_s} < \varepsilon$. Hence, by putting $\phi(x) = \psi_{\lambda_0}(x)$, we have the desired result. ■

Remark 3.3. Let $1 < q \le 2$, $n/q' < s < n/q' + 1$, $f \in \mathcal{F}L^1_s(\mathbf{R}^n)$ and $x_0 \in \mathbf{R}^n$. Since $M^{1,q}_s(\mathbf{R}^n) \hookrightarrow \mathcal{F}L^q_s(\mathbf{R}^n)$, Lemma 3.3 implies that for any $\varepsilon > 0$, there exists $\phi \in C^\infty_c(\mathbf{R}^n)$ such that (i) $\|(f - f(x_0))\phi\|_{\mathcal{F}L^q_s} < \varepsilon$. (ii) $\phi(x) = 1$ in some neighborhood of x_0.

3.1.1. The proof of Proposition 3.1

(i) It suffices to show $I(\{x_0\}) \subset \overline{J_0(\{x_0\})}^{\|\cdot\|_{\mathcal{F}L^q_s}} = J(\{x_0\})$. Let $f \in I(\{x_0\})$ and $\varepsilon > 0$. By Lemma 2.2 there exists $g \in C^\infty_c(\mathbf{R}^n)$ such that $g(x_0) = 0$ and $\|f - g\|_{\mathcal{F}L^q_s} < \varepsilon/2$. Moreover, by Remark 3.3 there exists $\phi \in C^\infty_c(\mathbf{R}^n)$ such that $\|(g - g(x_0))\phi\|_{\mathcal{F}L^q_s} = \|g\phi\|_{\mathcal{F}L^q_s} < \varepsilon/2$ and $\phi(x) = 1$ on a neighborhood of x_0. Let $\tau = (1 - \phi)g$. Then $\tau \in \mathcal{F}L^q_s(\mathbf{R}^n)$ and $\tau(x) = 0$ on a neighborhood of x_0. Therefore $\tau \in J_0(\{x_0\})$ and $\|f - \tau\|_{\mathcal{F}L^q_s} \le \|f - g\|_{\mathcal{F}L^q_s} + \|\phi g\|_{\mathcal{F}L^q_s} < \varepsilon$, which implies $f \in \overline{J_0(\{x_0\})}^{\|\cdot\|_{\mathcal{F}L^q_s}}$.

(ii) Let $x_0 = (x^{(0)}_1, \cdots, x^{(0)}_n) \in \mathbf{R}^n$. Contrary to our claim, suppose that $\{x_0\}$ is a set of spectral synthesis for $\mathcal{F}L^q_s(\mathbf{R}^n)$. Let $\phi \in C^\infty_c(\mathbf{R}^n)$ be such that $\operatorname{supp} \phi \subset B_1(x_0)$ and $\phi(x) = 1$ on $B_{1/2}(x_0)$, and define

$$f(x) = \big((x_1 - x^{(0)}_1) + \cdots + (x_n - x^{(0)}_n)\big) g(x), \quad x = (x_1, \cdots, x_n) \in \mathbf{R}^n.$$

Then $f \in I(\{x_0\})$. Since $I(\{x_0\}) = J(\{x_0\})$, we see that for any $\varepsilon > 0$, there exists $f_\varepsilon \in \mathcal{F}L^q_s(\mathbf{R}^n)$ such that $f_\varepsilon(x) = 0$ in a neighborhood of x_0 and $\|f - f_\varepsilon\|_{\mathcal{F}L^q_s} < \varepsilon$. On the other hand, $f, f_\varepsilon \in L^1(\mathbf{R}^n) \cap \mathcal{F}L^1(\mathbf{R}^n)$. Applying the Fourier inversion formula and the Hölder inequality one has

$$\Big|\frac{\partial f}{\partial x_1}(x_0) - \frac{\partial f_\varepsilon}{\partial x_1}(x_0)\Big| = (2\pi)^{-n} \Big| \int_{\mathbf{R}^n} i\xi_1 (\widehat{f}(\xi) - \widehat{f}_\varepsilon(\xi)) e^{ix\xi} d\xi \Big| \le$$

$$\le (2\pi)^{-n} \int_{\mathbf{R}^n} \langle \xi \rangle^{1-s} \langle \xi \rangle^s |\widehat{f}(\xi) - \widehat{f}_\varepsilon(\xi)| d\xi \le (2\pi)^{-n} \varepsilon \|\langle\cdot\rangle^{1-s}\|_{L^{q'}}.$$

Since $\frac{\partial f}{\partial x_1}(x_0) = 1$, $\frac{\partial f_\varepsilon}{\partial x_1}(x_0) = 0$ and $\varepsilon > 0$ is arbitrary, this gives a contradiction. Hence, $\{x_0\}$ is not a set of spectral synthesis for $\mathcal{F}L^q_s(\mathbf{R}^n)$.

To prove Theorem 1.1, we prepare several lemmas. In the following, $(\mathcal{F}L^q_s)_c$ denotes the space defined in Lemma 2.6.

Lemma 3.4. *Let $q = 1$ and $s \geq 0$, or $1 < q < \infty$ and $s > n/q'$. Suppose that I is a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. Then $I = \overline{I \cap (\mathcal{F}L^q_s)_c}^{\|\cdot\|_{\mathcal{F}L^q_s}}$.*

Proof. We need only consider $I \subset \overline{I \cap (\mathcal{F}L^q_s)_c}^{\|\cdot\|_{\mathcal{F}L^q_s}}$. Let $f \in I$ and $\varepsilon > 0$. By Lemma 2.1 there exists $\phi \in C^\infty_c(\mathbf{R}^n)$ satisfying $\|\phi f - f\|_{\mathcal{F}L^q_s} < \varepsilon$. Since $C^\infty_c(\mathbf{R}^n) \subset \mathcal{F}L^q_s(\mathbf{R}^n)$ and I is an ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$, we get $\phi f \in I \cap (\mathcal{F}L^q_s)_c$, and thus $I \subset \overline{I \cap (\mathcal{F}L^q_s)_c}^{\|\cdot\|_{\mathcal{F}L^q_s}}$. ∎

Lemma 3.5. *Let $1 \leq p \leq 2$. Suppose that $q = 1$ and $s \geq 0$, or $1 < q \leq p'$ and $s > n/q'$. For closed ideals I and I' in $\mathcal{F}L^q_s(\mathbf{R}^n)$,*

(i) $I \cap M^{p,q}_s(\mathbf{R}^n)$ is a closed ideal in $M^{p,q}_s(\mathbf{R}^n)$.

(ii) If $I \cap M^{p,q}_s(\mathbf{R}^n) = I' \cap M^{p,q}_s(\mathbf{R}^n)$, then we have $I = I'$.

Proof. (i) Recall that $M^{p,q}_s(\mathbf{R}^n)$ is a multiplication algebra. Moreover $M^{p,q}_s(\mathbf{R}^n) \hookrightarrow \mathcal{F}L^q_s(\mathbf{R}^n)$ (see Lemma 2.5) and I is a ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. Thus $I \cdot M^{p,q}_s \subset \mathcal{F}L^q_s \cdot I \subset I$. Hence $I \cap M^{p,q}_s$ is an ideal in $M^{p,q}_s(\mathbf{R}^n)$. To see that $I \cap M^{p,q}_s(\mathbf{R}^n)$ is closed, let $f \in \overline{I \cap M^{p,q}_s(\mathbf{R}^n)}^{\|\cdot\|_{M^{p,q}_s}}$. Then there exists $\{f_n\}_{n=1}^\infty \subset I \cap M^{p,q}_s(\mathbf{R}^n)$ such that $\|f_n - f\|_{M^{p,q}_s} \to 0$ $(n \to \infty)$. Thus Lemma 2.5 gives $\|f_n - f\|_{\mathcal{F}L^q_s} \to 0$ $(n \to \infty)$. Moreover, since $M^{p,q}_s(\mathbf{R}^n)$ is complete and I is closed in $\mathcal{F}L^q_s(\mathbf{R}^n)$, we have $f \in I \cap M^{p,q}_s(\mathbf{R}^n)$, which shows $I \cap M^{p,q}_s(\mathbf{R}^n)$ is closed.

(ii) Since $I \cap M^{p,q}_s \cap (\mathcal{F}L^q_s)_c = I' \cap M^{p,q}_s \cap (\mathcal{F}L^q_s)_c$ and $(\mathcal{F}L^q_s)_c \hookrightarrow M^{p,q}_s(\mathbf{R}^n)$ (see Lemma 2.6), one has $I \cap (\mathcal{F}L^q_s)_c = I' \cap (\mathcal{F}L^q_s)_c$. Thus Lemma 3.4 yields $I = I'$. ∎

Proposition 3.2. *Let $1 \leq p \leq 2$. Suppose that $q = 1$ and $s \geq 0$, or $1 < q \leq p'$ and $s > n/q'$. For any closed ideal I_M in $M^{p,q}_s(\mathbf{R}^n)$, the ideal $I_F := \overline{I_M}^{\|\cdot\|_{\mathcal{F}L^q_s}}$ in $\mathcal{F}L^q_s(\mathbf{R}^n)$ satisfies $I_M = I_F \cap M^{p,q}_s(\mathbf{R}^n)$.*

Proof. We start by observing that $I'_F := \overline{I_M \cap (\mathcal{F}L^q_s)_c}^{\|\cdot\|_{\mathcal{F}L^q_s}}$ is a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. In fact, for $f \in I'_F$ and $g \in \mathcal{F}L^q_s(\mathbf{R}^n)$ there exists $\{f_n\}_{n=1}^\infty$ in $I_M \cap (\mathcal{F}L^q_s)_c$ such that $\|f - f_n\|_{\mathcal{F}L^q_s} \to 0$ $(n \to \infty)$. Since $f_n \in (\mathcal{F}L^q_s)_c$, there exists $\psi_n \in C^\infty_c(\mathbf{R}^n)$ such that $\psi_n(x) = 1$ on supp f_n. Then we have $\psi_n g \in (\mathcal{F}L^q_s)_c \hookrightarrow M^{p,q}_s(\mathbf{R}^n)$. Therefore, $\psi_n g \cdot f_n \in I_M$, and thus $f_n g = f_n \cdot \psi_n g \in I_M \cap (\mathcal{F}L^q_s)_c$. Furthermore, since $\|fg - f_n g\|_{\mathcal{F}L^q_s} \lesssim \|f - f_n\|_{\mathcal{F}L^q_s} \|g\|_{\mathcal{F}L^q_s} \to 0$ $(n \to \infty)$, and thus $fg \in I'_F$. Hence, I'_F is an ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. We next prove $I_F = I'_F$. It suffices to prove $I_F \subset I'_F$. Given $f \in I_F$ and $\varepsilon > 0$ there exists $g \in I_M$ such that $\|f - g\|_{\mathcal{F}L^q_s} < \varepsilon$. By Lemma 2.4 there exists

$\phi \in C_c^\infty(\mathbf{R}^n)$ such that $\|g - \phi g\|_{M_s^{p,q}} < \varepsilon$. We note that $\phi g \in I_M \cap (\mathcal{F}L_s^q)_c$ by $\phi g \in I_M \subset M_s^{p,q}(\mathbf{R}^n) \hookrightarrow \mathcal{F}L_s^q(\mathbf{R}^n)$. Since $\|f - \phi g\|_{\mathcal{F}L_s^q} \lesssim \|f - g\|_{\mathcal{F}L_s^q} + +\|\phi g - g\|_{M_s^{p,q}} \lesssim \varepsilon$, we obtain $I_F \subset I'_F$. Finally, we prove $I_M = I_F \cap M_s^{p,q}(\mathbf{R}^n)$. It suffices to show $I_F \cap M_s^{p,q}(\mathbf{R}^n) \subset I_M$. Let $f \in I_F \cap M_s^{p,q}(\mathbf{R}^n)$ and $\varepsilon > 0$. By Lemma 2.4 there exists $\phi \in C_c^\infty(\mathbf{R}^n)$ such that $\|f - \phi f\|_{M_s^{p,q}} < \varepsilon$. Take $\varphi \in \in C_c^\infty(\mathbf{R}^n)$ with $\varphi(x) = 1$ on supp ϕ. Since $f \in I_F$ there exists $h \in I_M \cap (\mathcal{F}L_s^q)_c$ such that $\|f - h\|_{\mathcal{F}L_s^q} < \varepsilon/(\|\varphi\|_{M_s^{p,q}} \|\phi\|_{\mathcal{F}L_s^q})$. Then $\phi h \in I_M$. Note that the proof of Lemma 2.6 implies $\mathcal{F}L_s^q(\mathbf{R}^n) \hookrightarrow M_s^{\infty,q}(\mathbf{R}^n)$. Thus

$$\begin{aligned}
\|f - \phi h\|_{M_s^{p,q}} &\le \|f - \phi f\|_{M_s^{p,q}} + \|\varphi\phi(f-h)\|_{M_s^{p,q}} \lesssim \\
&\lesssim \|f - \phi f\|_{M_s^{p,q}} + \|\varphi\|_{M_s^{p,q}} \|\phi(f-h)\|_{M_s^{\infty,q}} \lesssim \\
&\lesssim \|f - \phi f\|_{M_s^{p,q}} + \|\varphi\|_{M_s^{p,q}} \|\phi(f-h)\|_{\mathcal{F}L_s^q} \lesssim \\
&\lesssim \|f - \phi f\|_{M_s^{p,q}} + \|\varphi\|_{M_s^{p,q}} \|\phi\|_{\mathcal{F}L_s^q} \|f - h\|_{\mathcal{F}L_s^q}.
\end{aligned}$$

Therefore $f \in \overline{I_M}^{\|\cdot\|_{M_s^{p,q}}} = I_M$. Hence $I_F \cap M_s^{p,q}(\mathbf{R}^n) \subset I_M$. ■

Remark 3.4. Let I_M and I'_M be closed ideals in $M_s^{p,q}(\mathbf{R}^n)$, and I_F be the closure of I_M in $\mathcal{F}L_s^q(\mathbf{R}^n)$. If the closure of I'_M in $\mathcal{F}L_s^q(\mathbf{R}^n)$ is equal to I_F, then Proposition 3.2 implies that $I_M = I'_M$.

Combining these results, we obtain the "ideal theory for Segal algebras".

Theorem 3.3. *Let $1 \le p \le 2$. Suppose that $q = 1$ and $s \ge 0$, or $1 < q \le p'$ and $s > n/q'$. Let $\mathcal{I}_F$ be the set of all closed ideals in $\mathcal{F}L_s^q(\mathbf{R}^n)$, and $\mathcal{I}_M$ be the set of all closed ideals in $M_s^{p,q}(\mathbf{R}^n)$. Then the map $\iota : \mathcal{I}_F \to \mathcal{I}_M$, $\iota(I_F) = = I_F \cap M_s^{p,q}(\mathbf{R}^n)$ $(I_F \in \mathcal{I}_F)$ is bijective. More precisely, we have $\iota^{-1}(I_M) = = \overline{I_M}^{\|\cdot\|_{\mathcal{F}L_s^q}}$ and $\iota(\overline{I_M}^{\|\cdot\|_{\mathcal{F}L_s^q}}) = \overline{I_M}^{\|\cdot\|_{\mathcal{F}L_s^q}} \cap M_s^{p,q}(\mathbf{R}^n)$ for $I_M \in \mathcal{I}_M$.*

3.1.2. The proof of Theorem 1.1

For a closed subset K of $\mathbf{R}^n$, we set $I_F(K) := \{f \in \mathcal{F}L_s^q(\mathbf{R}^n) \mid f|_K = 0\}$ and $I_M(K) := \{f \in M_s^{p,q}(\mathbf{R}^n) \mid f|_K = 0\}$. Moreover, we define $J_F(K)$ by the closure of $\{f \in \mathcal{F}L_s^q(\mathbf{R}^n) \mid f(x) = 0$ in a neighborhood of $K\}$ in $\mathcal{F}L_s^q(\mathbf{R}^n)$, and $J_M(K)$ by the closure of $\{f \in M_s^{p,q}(\mathbf{R}^n) \mid f(x) = 0$ in a neighborhood of $K\}$ in $M_s^{p,q}(\mathbf{R}^n)$. Then Theorem 3.3 shows $I_F(K) = J_M(K)$ if and only if $I_F(K) = = J_F(K)$. Hence, K is a set of spectral synthesis for $M_s^{p,q}(\mathbf{R}^n)$ if and only if K is a set of spectral synthesis for $\mathcal{F}L_s^q(\mathbf{R}^n)$.

4. Wiener–Lévy theorem

We only prove Theorem 1.4 because a slight change in the proof of Theorem 1.4 shows Theorem 1.5 (cf. Remark 4.1). We first recall the following lemma and prepare a local version of Theorem 1.4.

Lemma 4.1 ([14, Theorem 4.13])**.** *Let $1 < p < \infty$, $1 \le q \le \infty$ and $s > n/q'$. Let μ be a complex measure on* $\mathbf{R}$ *such that*

$$\int_{\mathbf{R}} (1+|\xi|)^{1+(s+n/q)(1+\frac{1}{s-n/q'})} d|\mu|(\xi) < \infty$$

and such that $\mu(\mathbf{R}) = 0$. Let F be the inverse Fourier transform of μ. Then $F(f) \in M_s^{p,q}(\mathbf{R}^n)$ holds for all real-valued $f \in M_s^{p,q}(\mathbf{R}^n)$.

Lemma 4.2. *Given $1 < q < \infty$, $s > n/q'$, a real-valued function $f \in \mathcal{F}L_s^q(\mathbf{R}^n)$ and a compact subset $K \subset \mathbf{R}^n$. Suppose that $F \in \mathcal{S}(\mathbf{R})$ and $F(0) = 0$. Then there exists $g \in \mathcal{F}L_s^q(\mathbf{R}^n)$ such that $g(x) = F(f(x))$ for all $x \in K$.*

Proof. Take a real-valued function $\tau \in C_c^\infty(\mathbf{R}^n)$ with $\tau(x) = 1$ on K. Then $\tau f \in \mathcal{F}L_s^q(\mathbf{R}^n)$. Since

$$(\mathcal{F}L_s^q)_c = \{f \in M_s^{p,q}(\mathbf{R}^n) \mid \text{supp } f \text{ is compact}\} \tag{4.1}$$

(cf. [9, Lemma A.1], [13, Lemma 1]), we have $\tau f \in M_s^{p,q}(\mathbf{R}^n)$ and thus $F(\tau f) \in M_s^{p,q}(\mathbf{R}^n)$ by Lemma 4.1. Note that $F \in \mathcal{S}(\mathbf{R})$ and $\text{supp}(\tau f)$ is compact. Thus $\text{supp}(F(\tau f))$ is compact. By (4.1) we have $F(\tau f) \in \mathcal{F}L_s^q(\mathbf{R}^n)$. Now set $g = F(\tau f)$. Then $g \in \mathcal{F}L_s^q(\mathbf{R}^n)$ and $g(x) = F(\tau(x)f(x)) = F(f(x))$ $(x \in K)$. ∎

4.1. The proof of Theorem 1.4

Since F is analytic on a neighborhood of 0 with $F(0) = 0$, there exists $\varepsilon_0 > 0$ such that $F(z)$ has the power series representation $F(z) = \sum_{j=1}^\infty c_j z^j$ $(|z| < \varepsilon_0)$. Take $\phi \in C_c^\infty(\mathbf{R}^n)$ such that $\|f - \phi f\|_{\mathcal{F}L_s^q} < \varepsilon/c$ for any ε with $0 < \varepsilon < \varepsilon_0$ (see Lemma 2.1), where c is the constant as in (2.1). Now we set $g_0(x) := \sum_{j=1}^\infty c_j \big(f(x) - \phi(x)f(x)\big)^j$. Then $g_0 \in \mathcal{F}L_s^q(\mathbf{R}^n)$ and $g_0(x) = F(f(x) - \phi(x)f(x)) = F(f(x))$ $(x \notin \text{supp } \phi)$. On the other hand, let $\tau_0, \tau_1 \in C_c^\infty(\mathbf{R}^n)$ be such that $\tau_0(x) = 1$ on $\text{supp } \phi$ and $\tau_1(x) = 1$ on $\text{supp } \tau_0$. Take $\psi \in C_c^\infty(\mathbf{R}^n)$ with $\psi(y) = 1$ for all $y \in \{\tau_1(x)f(x) \mid x \in \text{supp } \tau_1\}$. We note $f \in L^\infty(\mathbf{R}^n)$. Moreover one has $G := \psi F \in C_c^\infty(\mathbf{R})$, $G(0) = 0$ and

$$F(f(x)) = \psi(\tau_1(x)f(x))F(\tau_1(x)f(x)) = G(\tau_1(x)f(x)) \quad (x \in \text{supp } \tau_0).$$

By Lemma 4.2 with $K = \text{supp } \tau_0$ and $\tau_1 f \in \mathcal{F}L_s^q(\mathbf{R}^n)$, there exists exists $g_1 \in \mathcal{F}L_s^q(\mathbf{R}^n)$ such that $g_1(x) = F(f(x))$ on $\text{supp } \tau_0$. Set $g(x) := (1 - \tau_0(x))g_0(x) + \tau_0(x)g_1(x)$. Then $g \in \mathcal{F}L_s^q(\mathbf{R}^n)$. If $x \in \text{supp } \phi$, then $\tau_0(x) = 1$ and $g_1(x) = F(f(x))$. Thus $g(x) = F(f(x))$. Moreover, if $x \in \text{supp } \tau_0 \setminus \text{supp } \phi$, then $g_0(x) = F(f(x) - \phi(x)f(x)) = F(f(x))$ and $g_1(x) = F(f(x))$. Thus $g(x) = F(f(x))$. If $x \notin \text{supp } \tau_0$, then $\tau_0(x) = 0$, $g_0(x) = F(f(x))$ and $g(x) = F(f(x))$.

Remark 4.1. Applying Lemma 2.4 and Lemma 4.1 to $\tau_1 f$ for a real-valued function $f \in M_s^{p,q}(\mathbf{R}^n)$, we can prove Theorem 1.5 similarly.

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