Annales Univ. Sci. Budapest., Sect. Comp. 56 (2024) 133–150

ON (2, 3)-SIMULTANEOUS NUMBER SYSTEMS OVER THE EISENSTEIN LATTICE

Izabella Ingrid Farkas (Budapest, Hungary)

Communicated by Attila Kovács
(Received April 18, 2024; accepted May 26, 2024)

Abstract. Various authors analysed simultaneous number systems over different lattices. This paper presents additional characterisations on number expansions over the "triple" Eisenstein lattice. The main objective is determining a small and enumerable set (so-called testing set), including the periodic points. The algorithmic exploration of these systems poses a significant challenge due to the substantial increase in the size of the digit set and the testing set, even for small parameters. Efficient algorithms are introduced to address the classification problem, and finally, we offer an algorithm for computing the last digit deletion function without storing it.

1. Introduction

Let M be an $n \times n$ integer linear operator. Let furthermore D be a finite subset of $\mathbb{Z}^n$ containing 0. The system $(\mathbb{Z}^n, M, D)$ is a *number system* (GNS) if each element x of $\mathbb{Z}^n$ has a unique, finite representation of the form

$$x = \sum_{i=0}^{m} M^i d_i \,,$$

where $d_i \in D$, $m \in \mathbb{N}$. Here M is called the *base* or *radix*, and D is the *digit set* or *alphabet*.

Key words and phrases: Simultaneous number systems, digital expansions.
2010 Mathematics Subject Classification: 11A63, 11Y16, 11Y40.
This study was supported by the ÚNKP-23-4 New National Excellence Program of the Ministry for Culture and Innovation from the source of the National Research, Development and Innovation Fund.

Since an integer similarity transformation does not change the structural properties of number expansions, it is enough to consider $\mathbb{Z}^n$ as a lattice. Some necessary conditions for the unique representation property are [10]:

1. The expansivity of the base.
2. The full residue system property of the digit set,
3. The unit condition, i.e., $\det(M - I) \neq \pm 1$.

The congruence relation here means that two elements are congruent if they belong to the same coset of the factor group $\mathbb{Z}^n/M\mathbb{Z}^n$. When the first two conditions hold, we call the system a *radix system*.

Let $\Phi : \mathbb{Z}^n \to \mathbb{Z}^n$, $\Phi(x) = M^{-1}(x - d)$ for the unique $d \in D$ satisfying $x \equiv d \pmod{M}$. Since M^{-1} is contractive and D is finite, there exists a vector norm $\|.\|$ on $\mathbb{R}^n$ such that for the induced norm $\left\|M^{-1}\right\| < 1$, and there is a constant L such that the orbit of every $x \in \mathbb{Z}^n$ eventually enters the finite set $T = \{x \in \mathbb{Z}^n \mid \|x\| < L\}$ for the repeated application of Φ, and after entering the orbit never leaves it. This means that the sequence $x, \Phi(x), \Phi^2(x), \ldots$ is eventually periodic for all $x \in \mathbb{Z}^n$. A point p is called *periodic* if $\Phi^k(p) = p$ for some $k > 0$. The orbit of p constitutes a cycle. It is known that if p is periodic then $\|p\| \leq L = \frac{Kr}{1-r}$, where $r = \left\|M^{-1}\right\| < 1$ and $K = \max_{d \in D} \|d\|$ ([9]). We denote by $\mathcal{B}_n(x, l)$ the closed ball around $x \in \mathbb{R}^n$ with radius l and by $\mathcal{N}(s)$ the number of lattice points inside $\mathcal{B}_2(0, s)$. A system is a number system if the orbit of each point goes to zero, i.e., the only periodic point is zero.

Since the testing set is finite, the finite representation property can be algorithmically decided. Unfortunately, the testing set can be enormous, depending on the eigenvalue spectrum of M [14]. We note that there exist different methods for deciding the number system property for a given base and alphabet (box type, carry propagation type [1], or via iterated function systems [15]). It is conjectured that the complexity of the general decision problem is $\mathcal{NP}$-hard.

The two most extensively studied and applied types of structured digit sets are the arithmetic type and the dense one. Dense alphabets contain elements with minimal norms from each residue class. An adjoint alphabet is a special dense one, where $D = \{0 = d_1, \ldots, d_t\}$, $t = |\det(M)|$, and each coordinate of $M^{adj} d_i$ belongs to the interval $(-t/2, t/2]$.

Block diagonal systems were defined in [11]:

Definition 1.1. Consider the direct product of the lattices $\mathbb{Z}^n = \mathbb{Z}^{\sum_i^k n_i}$ and the direct sum $M = M_1 \oplus \cdots \oplus M_k$ of the operators. The radix system $(\mathbb{Z}^n, M, D)$ is a *block diagonal system* if $(\mathbb{Z}^{n_i}, M_i, D_i)$ are radix systems $(1 \leq i \leq k \geq 2)$.

For simplicity, we denote the components of any $z \in \mathbb{Z}^n$ by $z = z_1 \bullet z_2 \bullet \bullet \ldots \bullet z_k$, where $z_i \in \mathbb{Z}^{n_i}$.

2. Simultaneous systems

Definition 2.1. A block diagonal system $SimR = (\mathbb{Z}^{kn}, M_1 \oplus M_2 \oplus \cdots \oplus \oplus M_k, D)$ is called an (n,k)-*simultaneous system* if $M_i : \mathbb{Z}^n \to \mathbb{Z}^n$ and all the digits $d_j \in D$ have the form $v \bullet v \bullet \cdots \bullet v$, $v \in \mathbb{Z}^n$.

Kátai et. al. [7] investigated the $(1,k)$-simultaneous system with mutually coprime integers $(N_1, N_2, \ldots, N_k)$, none of them are ± 1, and $D = \{\delta e\}$, $e = 1 \bullet \cdots \bullet 1$, $\delta = 1, 2, \ldots |N_1 N_2 \cdots N_k| - 1$. They showed that the system $(\mathbb{Z}^2, N_1 \oplus N_2, D)$ is a number system iff $N_1 < N_2 \le -2$ and $N_2 = N_1 + 1$. Van de Woestijne [19] investigated special polynomial homomorphic systems with canonical digit sets.

One of the main research problems regarding simultaneous systems is number system constructions with corresponding digit sets. As a first attempt, Nagy [17] applied canonical digit sets for the blocks in the lattice of Gaussian integers. He proved that, in this case, simultaneous number system constructions are not possible. Then, in the same lattice, the following structure was applied: for two blocks M_1 and M_2 with digit sets D_1 and D_2 let us consider the set

$$D = \{d_1 + M_1 d_2\}, \text{ where } d_1 \in D_1 \text{ and } d_2 \in D_2\,. \tag{2.1}$$

Clearly, D is a full residue system. For more than two blocks, the construction can be made recursively. Similar digit set construction was used in different papers [2, 3, 18, 8]. Kovács analysed the $(2,2)$-simultaneous systems in the Gaussian ring [11, 12] and presented a complete solution for the number system construction problem. The $(2,2)$-simultaneous systems in the Eisenstein lattice were also investigated by him [13]. Regarding the real quadratic fields, Krutki and Nagy ([16]) investigated the $(2,3)$-simultaneous systems numerically with (2.1) type alphabets where the blocks are integers from the ring $\mathbb{Q}[\sqrt{5}]$.

The following result was proven by Farkas and Kovács ([4]): there are infinitely many $(2,2)$-simultaneous number systems over the integers in imaginary quadratic fields.

Lemma 2.1. *If $SimR$ is a simultaneous GNS, then* $\det(M_i - M_j) = \pm 1$ *for all $i \ne j$.*

The Lemma 2.1 was proven in [11].

Lemma 2.2. *Let $S_1 = (\mathbb{Z}^2, M_1, D_1)$, $S_2 = (\mathbb{Z}^2, M_2, D_2)$, $S_3 = (\mathbb{Z}^2, M_3, D_3)$ be three radix systems with some D_1, D_2, D_3. Let $A = M_2 \oplus M_3$, $D_A = \{d \bullet d : : d = d_2 + M_2 d_3, d_2 \in D_2, d_3 \in D_3\}$ and $S_A = (\mathbb{Z}^4, A, D_A)$. If the block diagonal system $S = (\mathbb{Z}^6, M, D)$ with the alphabet*

$$D = \{d \bullet d \bullet d\ : d = d_1 + M_1 d_2 + M_1 M_2 d_3,\ \text{where } d_i \in D_i \text{ and } i \in \{1,2,3\}\} \tag{2.2}$$

is a $(2,3)$-simultaneous number system then S_1, S_2, S_3 and S_A are all number systems.

Proof. Consider the function $f : \mathbb{Z}^2 \times \mathbb{Z}^4 \to \mathbb{Z}^6$,

$$\begin{aligned} f(z_1, z_2) &= (c_m \bullet c_m \bullet c_m, c_{m-1} \bullet c_{m-1} \bullet c_{m-1}, \dots, c_1 \bullet c_1 \bullet c_1, c_0 \bullet c_0 \bullet c_0)_M = \\ &= (c_m, c_{m-1}, \dots, c_1, c_0)_{M_1} \bullet (c_m \bullet c_m, c_{m-1} \bullet c_{m-1}, \dots, c_1 \bullet c_1, c_0 \bullet c_0)_{M_A} = \\ &= x \bullet y, \end{aligned}$$

where $c_i = a_i + M_1 b_i$, $z_1 = \sum_i^m M_1^i a_i$, $z_2 = \sum_i^m M_A^i (b_i \bullet b_i)$, $a_i \in D_1$, $b_i \bullet b_i \in D_A$. The function f operates on those points which have finite expansions in S_1 and S_A, respectively. Observe that f is injective but not necessarily surjective. Let $x \in \mathbb{Z}^2$ be any point chosen, and the expansions of the points $x \bullet y \in \mathbb{Z}^6$ have to be examined. These points all have finite expansions in S. Exactly one of them is $f(z_1, 0) = \sum_i M_1^i a_i \bullet \sum_i M_A^i (a_i \bullet a_i)$, which shows that x has a finite expansion in S_1. Similarly, let $y \in \mathbb{Z}^4$ be any point chosen. Consider the (necessarily finite) expansions of the points $x \bullet (M_2 \oplus M_3) y$. Exactly one of them is $f(0, z_2) = \sum_i M_1^{i+1} b_i \bullet \sum_i M_A^i (M_2 b_i \bullet M_3 b_i)$ showing that y has finite expansion in S_A. Moreover, if S_A is a number system, then S_2 and S_3 are both number systems (see [4] Lemma 2.1). ■

Remark 2.1. The Lemma 2.2 gives the necessary condition for S to be a number system. If any of the systems S_1, S_2, S_3, S_A is not a number system then the $(2,3)$-simultaneous system is not a number system as well.

Remark 2.2. The number system properties of S_1, S_2, S_3 and A_A are insufficient for S to be a number system.

Example 2.1. Let $M_1 = \begin{pmatrix} 5 & -2 \\ 2 & 3 \end{pmatrix}$, $M_2 = \begin{pmatrix} 6 & -2 \\ 2 & 4 \end{pmatrix}$, and $M_3 = \begin{pmatrix} 6 & -3 \\ 3 & 3 \end{pmatrix}$. In this case, S_1, S_2, S_3 are number systems with adjoint digit sets, and the S_A is also a number system with digit set D_A, which is defined in Lemma 2.2, but S is not a number system since $p = 19 \bullet 0 \bullet 16 \bullet 2 \bullet 14 \bullet (-2)$.

The generalization of Lemma 2.2 is the following:

Lemma 2.3. *Let $S_i = (\mathbb{Z}^n, M_i, D_i)$ be radix systems with some D_i $(1 \le i \le k)$, and suppose that the block diagonal system $S = (\mathbb{Z}^{kn}, M, D)$ with the alphabet $D = \{d \bullet \dots \bullet d : d \in D^*\}$ is an (n,k)-simultaneous number system, where*

$$\begin{aligned} D^* = \{d : d = & d_1 + M_1(d_2 + M_2(\dots(d_{k-2} + M_{k-2}(d_{k-1} + M_{k-1} d_k)\dots)), \\ & \textit{where } d_i \in D_i \textit{ and } 1 \le i \le k\} \end{aligned}$$

Then, all the systems S_i are number systems.

Proof. The proof is reached by recursively applying the proof of Lemma 2.2. ∎

Remark 2.3. For S to be a GNS, a prerequisite is that every other system constructed in the proof must also be a number system. Specifically, this condition implies that S_A must be a GNS, as stated in Lemma 2.2.

For the rest of the paper, the $(2,3)$-simultaneous systems will be investigated in the Eisenstein lattice with the alphabet from (2.2). The objective is to identify the smallest size of the testing set and ascertain the structure of the periodic points.

3. The $(2,3)$-simultaneous systems over the Eisenstein lattice

The Eisenstein integers are complex numbers of the form $\eta = a + b\omega$, where $a, b \in \mathbb{Z}$ and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a cube root of unity. They can be considered as linear operators of the form

$$M_1 = \begin{pmatrix} a & -b \\ b & a-b \end{pmatrix} \tag{3.1}$$

acting on $\mathbb{Z}^2$. The norm of η is $N(a+b\omega) = a^2 - ab + b^2$.

Based on Lemma 2.2 and Lemma 1.1 form [13] the possible appropriate blocks have form: $M_a = \begin{pmatrix} a+1 & -b \\ b & a-b+1 \end{pmatrix}$, $M_b = \begin{pmatrix} a & -b-1 \\ b+1 & a-b-1 \end{pmatrix}$ and $M_c = \begin{pmatrix} a+1 & -b-1 \\ b+1 & a-b \end{pmatrix}$. Consider the radix systems $(\mathbb{Z}^6, M, D)$, where $M = \oplus_{i=1}^{3} M_i$ and $D = \{(d \bullet d \bullet d \ : \ d \in D^*\}$, where

$$D^* = \bigcup_{d_2 \in D_2} \bigcup_{d_3 \in D_3} M_1\ M_2\ d_3 + M_1\ d_2 + D_1\,. \tag{3.2}$$

Suppose that D_1, D_2, D_3 are adjoint digit sets belonging to the blocks of M_1, M_2, and M_3 respectively.

Lemma 3.1. *The only possible radix bases of the simultaneous Eisenstein number system are the following combinations of M_a, M_b, M_c and M_1 fulfilling Lemma 2.1:*

$$M_A = M_1 \oplus M_a \oplus M_c, \quad M_B = M_1 \oplus M_b \oplus M_c, \tag{3.3}$$

where $a, b \in \mathbb{Z}$ and M_1 was defined in (3.1).

$Type_A$ denotes the systems, where $M = M_A$ and $Type_B$ where $M = M_B$.

Theorem 3.1. *Every non-null and non-unit Eisenstein integer may serve as a block of a* $(2,3)$*-simultaneous number system except* $1\pm\omega$, $-1-\omega$, $2+\omega$, $-1-2\omega$, -2ω, ± 2, $-2-\omega$, $-2-2\omega$ *in case of* $Type_A$*, and* $1\pm\omega$, $-1-\omega$, $2\pm\omega$, 2, -2ω, $\pm 1-2\omega$, $-2-\omega$, $-2-2\omega$ *in case of* $Type_B$ *systems.*

Proof. Implied by the necessary conditions of the number system property. ■

It is known that $D \subseteq W$ where $W = \{(x,y,x,y,x,y)^T \mid x,y \in \mathbb{Z}\} \leq \mathbb{Z}^6$ and $|D| = |\det(M)|$. Since $\left\|M^{-1}\right\|_2 < 1$ almost always holds (except in a few cases), in the following, the norm $\|\cdot\|$ means 2-norm. The exceptional cases are:

- $(a,b) \in \{(-2,-3),(0,2),(1,2)\}$ in the $Type_A$ system,
- $(a,b) \in \{(-2,-3),(-1,-3),(1,2)\}$ in the $Type_B$ system.

The results presented below do not depend on the type of systems selected. The following notations are introduced:

- $K^* = \max\{\|d^*\| : d^* \bullet d^* \bullet d^* \in D\}$, $K = \max\{\|d\| : d \in D\}$,
- $r = \left\|M^{-1}\right\| = \max\{\left\|M_1^{-1}\right\|, \left\|M_2^{-1}\right\|, \left\|M_3^{-1}\right\|\}$,
- $R = \max\{\|M_1\|, \|M_2\|, \|M_3\|\}$,
- $L = \frac{K\,r}{1-r}$, $L_i = \frac{K^* r_i}{1-r_i}$ where $r_i = \left\|M_i^{-1}\right\|$, $(i = 1,2,3)$,
- κ_i the condition numbers of M_i, respectively, and $\kappa = r \cdot R$.

Example 3.2. Let $a = 2$, $b = 2$ then

$$M_A = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}, \; M_B = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

In $Type_A$ case, the values are: $\#D = 252$, $K = \sqrt{390} \approx 19.75$, $K^* \approx 11.4$, $r \approx 0.81$, $R \approx 4.85$, $L \approx 83.65$, $L_1 \approx 48.3$, $L_2 \approx 13.76$, $L_3 \approx 13.35$. In the $Type_B$ case: $\#D = 252$, $K = \sqrt{390} \approx 19.75$, $K^* \approx 11.4$, $r \approx 0.81$, $R \approx 4.85$, $L \approx 83.65$, $L_1 \approx 48.3$, $L_2 \approx 21.06$, $L_3 \approx 13.35$.

In the following, we present some simple observations.

Lemma 3.2. $\|M_i^{-1} - M_j^{-1}\| = \|M_i^{-1}\|\|M_j^{-1}\|$*, where* $i,j \in \{1,2,3\}$ *and* $i \neq j$.

Proof. The proof is straightforward. ■

Lemma 3.3. $K^* \leq \frac{1}{\sqrt{2}} \|M_1\| (\|M_2\| \|M_3\| + \|M_2\| + 1)$.

Proof. Since $\max_{d \in D_i} \|d\| = \max_{b \in [-1/2,1/2)^2} \|M_i^* \, b\| \leq \frac{\|M_i\|}{\sqrt{2}}$, for all $i \in \{1,2,3\}$ hence

$$K^* = \max_{d_i \in D_i} \|M_1 \ M_2 \ d_3 + M_1 \ d_2 + d_1\| \leq \frac{\|M_1\| (\|M_2\| \|M_3\| + \|M_2\| + 1)}{\sqrt{2}} . \blacksquare$$

Lemma 3.4.

$$L = \frac{\sqrt{3} K^* r}{1-r} \leq \sqrt{\frac{3}{2}} \frac{\|M_1\| (\|M_2\| \|M_3\| + \|M_2\| + 1) r}{1-r}.$$

Proof.

$$K = \max_{d \in D} \|d\| = \max_{(x,y,x,y,x,y) \in D} \|(x,y,x,y,x,y)\| = \sqrt{3} K^*. \qquad \blacksquare$$

Theorem 3.3. *Let $R = (\mathbb{Z}^6, M, D)$, and*

$$K_1 = \frac{\left\|M_1^{-1} \, M_2^{-1}\right\| (L_2 + K^*)}{1 - \left\|M_1^{-1}\right\|} = \frac{L_1 L_2}{K^*} \, ,$$
$$K_2 = \frac{\left\|M_2^{-1} M_3^{-1}\right\| (L_3 + K^*)}{1 - \left\|M_2^{-1}\right\|} = \frac{L_2 L_3}{K^*} \, ,$$
$$K_3 = \frac{\left\|M_1^{-1} M_3^{-1}\right\| (L_1 + K^*)}{1 - \left\|M_3^{-1}\right\|} = \frac{L_1 L_3}{K^*} .$$

If $K_1, K_2, K_3 < 1$ then R is a number system, otherwise if $K_i \geq 1$ for some $i \in \{1,2,3\}$ then the testing set is formed by $v = (v_1, v_2, v_3, v_4, v_5, v_6) \in \mathbb{Z}^6$ such that $\|v\| \leq L$ and

$$\left\|(v_1, v_2)^T - (v_3, v_4)^T\right\| < K_1 \, ,$$
$$\left\|(v_3, v_4)^T - (v_5, v_6)^T\right\| < K_2 \, ,$$
$$\left\|(v_1, v_2)^T - (v_5, v_6)^T\right\| < K_3 \, .$$

If the orbit of the points from the testing set terminates in $0 \in \mathbb{Z}^6$, then it is a number system.

Proof. Let $v = (v_1, v_2, v_3, v_4, v_5, v_6)$, $\|v\| \leq L$, and $z_1 = (v_1, v_2)$, $z_2 = (v_3, v_4)$, $z_3 = (v_5, v_6)$ be arbitrary. Then

$$
\begin{aligned}
&\|\Phi_1(z_1)-\Phi_2(z_2)\| = \\
&\left\|M_1^{-1}(z_1-d)-M_2^{-1}(z_2-d)\right\| = \\
&\left\|M_1^{-1}(z_1-d)-M_1^{-1}(z_2-d)+M_1^{-1}(z_2-d)-M_2^{-1}(z_2-d)\right\| \le \\
&\left\|M_1^{-1}((z_1-d)-(z_2-d))\right\| + \left\|(M_1^{-1}-M_2^{-1})(z_2-d)\right\| \le \\
&\left\|M_1^{-1}\right\| \|z_1-z_2\| + \left\|M_1^{-1}-M_2^{-1}\right\| \|z_2-d\| \le \\
&\left\|M_1^{-1}\right\| \|z_1-z_2\| + \left\|M_1^{-1}\,M_2^{-1}\right\| \|z_2-d\| \le \\
&\left\|M_1^{-1}\right\| \|z_1-z_2\| + \underbrace{\left\|M_1^{-1}\,M_2^{-1}\right\| (L_2+K^*)}_{(1-r_1)K_1} < r_1K_1+(1-r_1)K_1 = K_1
\end{aligned}
$$

The used inequalities are

$$\|z_2-d\| \le \|z_2\|+\|d\| \le L_2+K^* \quad \text{and} \quad \left\|M_1^{-1}-M_2^{-1}\right\| = \left\|M_1^{-1}\,M_2^{-1}\right\|.$$

Hence, if $\|z_1-z_2\| \le K_1$ then $\|\Phi(z_1)-\Phi(z_2)\| < K_1$. Similarly,

- If $\|z_3-z_2\| \le K_2$ then $\|\Phi(z_3)-\Phi(z_2)\| < K_2$.
- If $\|z_1-z_3\| \le K_3$ then $\|\Phi(z_1)-\Phi(z_3)\| < K_3$.

If $K_1, K_2, K_3 < 1$ then the testing set is $W \cap \mathcal{B}_6(0,L)\backslash\{0\}$. Let $p \in W \cap \mathcal{B}_6(0,L)\backslash\{0\}$ be a periodic point. It is known that $\Phi(p) \in W \cap \mathcal{B}_6(0,L)\backslash\{0\}$ and $p-d \in W \cap \mathcal{B}_6(0,L)\backslash\{0\}$ $(d \in D)$. Let $p-d=(x,y,x,y,x,y)$ such that $z \equiv d \pmod{M}$. Then $M_1^{-1}\begin{pmatrix}x\\y\end{pmatrix} = M_2^{-1}\begin{pmatrix}x\\y\end{pmatrix} = M_3^{-1}\begin{pmatrix}x\\y\end{pmatrix}$ for some $x, y \in \mathbb{Z}$. This is possible only for $x=y=0$. ■

Example 3.4. Let $a=2, b=2$. In this case $K_1 \approx 58.3$, $K_2 \approx 16.11$, $K_3 \approx 56.55$ and the testing set size is roughly 177 698 711 505 in $Type_A$ case and $K_1 \approx 89.21, K_2 \approx 24.65, K_3 \approx 56.55$, and the testing set size is roughly 1 046 597 011 869 in $Type_B$.

The previous example demonstrates that small a, b parameters may result in huge K_1, K_2, K_3 values. Let

$$A = \frac{\kappa(\kappa R+\kappa+r)}{\sqrt{2}(1-r)^2}. \tag{3.4}$$

Then

$$
\begin{aligned}
K_1 &\le \frac{(\|M_3\|+1)\kappa_1\kappa_2+\kappa_1\left\|M_2^{-1}\right\|}{\sqrt{2}(1-\left\|M_1^{-1}\right\|)(1-\left\|M_2^{-1}\right\|)} \le \frac{(R+1)\kappa^2+\kappa r}{\sqrt{2}(1-r)^2} = A \\
K_2 &\le \frac{\left\|M_3^{-1}\right\|\left\|M_2^{-1}\right\|\|M_1\|(\|M_2\|\|M_3\|+\|M_2\|+1)}{\sqrt{2}(1-\left\|M_3^{-1}\right\|)(1-\left\|M_2^{-1}\right\|)} \le \frac{R(\kappa^2+\kappa r+r^2)}{\sqrt{2}(1-r)^2} = A \\
K_3 &\le \frac{\left\|M_3^{-1}\right\|\left\|M_1^{-1}\right\|\|M_1\|(\|M_2\|\|M_3\|+\|M_2\|+1)}{\sqrt{2}(1-\left\|M_3^{-1}\right\|)(1-\left\|M_1^{-1}\right\|)} \le \frac{\kappa(\kappa R+\kappa+r)}{\sqrt{2}(1-r)^2} = A\,.
\end{aligned}
$$

Therefore, for every $K_1.K_2, K_3,\ r, \kappa$ there exist $A' > A$ such that

$$A'\sqrt{2}r^3 - (2A'\sqrt{2} + \kappa)r^2 + (A'\sqrt{2} - \kappa^2)r - \kappa^3 > 0 \tag{3.5}$$

Theorem 3.5. *Let $(\mathbb{Z}^6, M, D)$ be a $(2,3)$-block diagonal system over the ring of Eisenstein integers, where M is M_A or M_B from (3.3) with digits from (3.2). Suppose that $A' \in \mathbb{R}$ satisfies the condition (3.5). Then, for the possible periodic elements $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$ the following conditions hold:*

$$\begin{aligned}
&\|(\pi_1,\pi_2) - (\pi_3,\pi_4)\| < A',\\
&\|(\pi_3,\pi_4) - (\pi_5,\pi_6)\| < A',\\
&\|(\pi_5,\pi_6) - (\pi_1,\pi_2)\| < A',\\
&\|\pi\| \le A'\sqrt{3}(1/r - 1),\\
&\pi \in T,
\end{aligned}$$

where T is an effectively computable set having less than $N(A')^3$ elements.

Proof. If π is periodic, then

$$\|\pi\| \le L = \frac{Kr}{1-r} \le \frac{r\sqrt{3}\|M_1\|(\|M_2\|\|M_3\| + \|M_2\| + 1)}{\sqrt{2}(1-r)},$$

hence

$$\frac{Lr}{\sqrt{3}(1-r)} \le \frac{\kappa^3 + \kappa^2 r + \kappa r^2}{\sqrt{2}r(1-r)^2} < A'$$

from which $\|\pi\| \le L < A'\sqrt{3}(1/r - 1)$. This bound can easily be computed. Now, we construct the set T. If $\pi' = (\pi_1', \pi_2', \pi_3', \pi_4', \pi_5', \pi_6')$ is periodic then

$$\begin{aligned}
&\|(\pi_1',\pi_2') - (\pi_3',\pi_4')\| < A',\\
&\|(\pi_3',\pi_4') - (\pi_5',\pi_6')\| < A',\\
&\|(\pi_1',\pi_2') - (\pi_5',\pi_6')\| < A',
\end{aligned}$$

and

$$\begin{aligned}
&\|(\pi_1' - d_1, \pi_2' - d_2) - (\pi_3' - d_1, \pi_4' - d_2)\| < A',\\
&\|(\pi_3' - d_1, \pi_4' - d_2) - (\pi_5' - d_1, \pi_6' - d_2)\| < A',\\
&\|(\pi_1' - d_1, \pi_2' - d_2) - (\pi_5' - d_1, \pi_6' - d_2)\| < A'
\end{aligned}$$

for any digit $(d_1, d_2, d_1, d_2, d_1, d_2) \in D$. For the appropriate congruent digit d, let $x_1 = \pi_1' - d_1$, $x_2 = \pi_2' - d_2$, $x_3 = \pi_3' - d_1$, $x_4 = \pi_4' - d_2$, $x_5 = \pi_5' - d_1$, $x_6 = \pi_6' - d_2$.

Hence, $M_1(\pi_1, \pi_2) = (x_1, x_2)$, $M_2(\pi_3, \pi_4) = (x_3, x_4)$, $M_3(\pi_5, \pi_6) = (x_5, x_6)$ for some $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6) \in \mathbb{Z}^6$, where

$$\begin{aligned}
&\|(\pi_1, \pi_2) - (\pi_3, \pi_4)\| < A' \,, \\
&\|(\pi_3, \pi_4) - (\pi_5, \pi_6)\| < A' \,, \\
&\|(\pi_5, \pi_6) - (\pi_1, \pi_2)\| < A' \,.
\end{aligned}$$

Rewriting the previous equations in the Eisenstein-lattice, we get that

$$\begin{cases}
-b(\pi_2 - \pi_4) + a(\pi_1 - \pi_3) - \pi_3 & = x_1 - x_3 \\
(a-b)(\pi_2 - \pi_4) + b(\pi_1 - \pi_3) - \pi_4 & = x_2 - x_4 \\
-b(\pi_4 - \pi_6) + (a+1)\ (\pi_3 - \pi_5) + \pi_6 & = x_3 - x_5 \\
(a-b)(\pi_4 - \pi_6) + b(\pi_3 - \pi_5) + \pi_4 - \pi_5 & = x_4 - x_6 \\
-b(\pi_6 - \pi_2) + a(\pi_5 - \pi_1) + \pi_5 - \pi_6 & = x_5 - x_1 \\
(a-b)(\pi_6 - \pi_2) + b(\pi_5 - \pi_1) + \pi_5 & = x_6 - x_2 \,.
\end{cases}$$

Simplifying the above equations, we have

$$\begin{cases}
\pi_1 = bY - (a+1)X - \xi_1, \\
\pi_2 = -bX - (a-b+1)Y - \xi_2, \\
\pi_3 = \pi_1 + X, \\
\pi_4 = \pi_2 + Y, \\
\pi_5 = \pi_3 + Z = \pi_1 + X + Z, \\
\pi_6 = \pi_4 + W = \pi_2 + Y + W,
\end{cases} \tag{3.6}$$

where $\xi_1 = x_1 - x_3, \xi_2 = x_2 - x_4, \xi_3 = x_5 - x_1, \xi_4 = x_6 - x_2$,

$$Z = \frac{(a-b)X - (a^2-ab+b^2+a)Y + (b+1)\xi_4 + (a+1)\xi_1 + (a-b)(\xi_3 - \xi_2)}{a^2-ab+b^2+a+b+1},$$

$$W = \frac{(a^2-ab+b^2+a)X - (a^2-ab+b^2+b)Y + (a+1)\xi_4 + (a-b)\xi_1 - (b+1)(\xi_3 - \xi_2)}{a^2-ab+b^2+a+b+1},$$

$X^2 + Y^2 < A'^2$, $Z^2 + W^2 < A'^2$, $(X+Z)^2 + (Y+W)^2 < A'^2$, $\xi_1^2 + \xi_2^2 <$ $< A'^2, \xi_3^2 + \xi_4^2 < A'^2$, and $(\xi_3 + \xi_1)^2 + (\xi_2 + \xi_4)^2 < A'^2$.

The number of both (X, Y) and (ξ_1, ξ_2) pairs are $\mathcal{N}(A')$. The number of (ξ_3, ξ_4) pairs is less than $\mathcal{N}(A')$. Searching for all solutions, we have $\mathcal{N}(A')^3$ periodic candidates, exactly what we stated. ■

Remark 3.1. The size of the testing set depends on parameters a, b. However, using the formulas 3.6, the number of periodic candidates is less than or equal to $\mathcal{N}(A')^3$, which is independent of the parameters.

4. Algorithms

In this section, we present the classification algorithm. The decision algorithm uses the same logic.

4.1. Classification algorithm

Algorithm 1 is based on Theorem 3.3. It is known that if every $K_i < 1$ ($i \in \{1,2,3\}$), then the system is GNS. Otherwise, every point from the $\mathcal{B}_6(0,L)$ has to be checked by using the inequalities with K_i ($i \in \{1,2,3\}$).

Algorithm 1 ClassifyOne(L, K_1, K_2, K_3, M, D)

if $K_1 < 1$ and $K_2 < 1$ and $K_3 < 1$ **then**
  return $\{0 \to 0\}$
end if
$P \leftarrow \{\}$
for all $v \in \mathcal{B}_6(0,L)$ **do**
  if $\|(v_1,v_2)-(v_3,v_4)\| < K_1$ and $\|(v_5,v_6)-(v_3,v_4)\| < K_2$ and $\|(v_1,v_2)-(v_5,v_6)\| < K_3$ **then**
    $P \leftarrow P \cup \{$ FindCycle($\Phi(v), M, D$)$\}$
  end if
end for
return P

The FindCycle(v, M, D) function calculates the orbit of v and returns the cycle.

Let $z_1 = (v_1,v_2)$, $z_2 = (v_3,v_4)$ and $z_3 = (v_5,v_6)$. Algorithm 1 can be improved by applying the following restrictions generating the testing set $T_1 = \{z_1 \bullet z_2 \bullet z_3\}$ in case of small enough K_i parameters:

1. $z_1 \in B_2(0,L)$,
2. $z_2 = z_1 + t_1$, $t_1 \in B_2(0,K_1)$ and $\|(z_1, z_1+t_1)\| < L$,
3. $z_3 = z_1 + t_2$, $t_2 \in B_2(0,K_3)$ and $\|(z_1,z_2,z_3)\| < L$,
4. if $\|v_2 - v_3\| = \|t_2 - t_1\| < K_2$ then $t_2 \in B_2(t_1, K_2)$.

Algorithm 2 uses the previous optimisation steps. If the values K_1, K_2, K_3 are bigger than $L/2$, we apply Algorithm 1 because there are too many points outside of the ball.

Algorithm 2 investigates at most $\mathcal{N}(L)\mathcal{N}(K_1)\mathcal{N}(K_3)$ points in T_1. The FindCycle algorithm has at most $\#T_1 - \#D + 1$ steps.

Algorithm 2 ClassifyOneOptimised(L, K_1, K_2, K_3, M, D)

```
if max{K_1, K_2, K_3} < 1 then
    return {0 → 0}
end if
if min{K_1, K_2, K_3} > L/2 then
    return ClassifyOne(L, K_1, K_2, K_3, M, D)
end if
P ← {}
for all z_1 ∈ B_2(0, L) do
    for all t_1 ∈ B_2(0, K_1) do
        if ‖z_1 + t_1‖^2 < L^2 − ‖z_1‖^2 then
            z_2 ← z_1 + t_1
            for all t_2 ∈ B_2(0, K_3) do
                if (‖z_1 + t_2‖^2 < L^2 − ‖z_1‖^2 − ‖z_2‖^2 and ‖t_2 − t_1‖ < K_2 then
                    P ← P ∪ { FindCycle(Φ(z_1 • z_2 • (z_1 + t_2)), M, D)}
                end if
            end for
        end if
    end for
end for
return P
```

Algorithm 3 is based on Theorem 3.5, where A is from (3.4).

It generates $\mathcal{N}(A)^3$ points, producing the set T_2. Similarly to the previous case, the worst-case running time is $O(\#T_2(\#T_2 - \#D))$. The functions GetZ$(x, y, \xi_1, \xi_2, \xi_3, \xi_4)$ and GetW$(x, y, \xi_1, \xi_2, \xi_3, \xi_4)$ return the corresponding z and w values from equation system (3.6).

Figure 1 compares the testing set size for Algorithm 2 and Algorithm 3 in the $Type_A$ case. In the measurements, $b = 2$ is fixed, and $1 < a < 100$. The size of the original $B_6(0, L)$ is so huge that it could not be illustrated together with the testing set's size. The $Type_B$ case is similar.

It can be noted that the computation of K is time-consuming. Hence, the upper estimate of K was used from Lemma 3.3.

4.2. The Φ function without storing the digits

The calculation of the orbits is based on the function Φ. The classification and decision algorithms also rely on it. Based on Figure 2, it is easy to see that the digit set size can be large. Therefore, the Φ function calculation can be beneficial without storing any digit in the computer memory.

Algorithm 3 $\textsc{ClassifyTwo}(A, M, D)$

$P \leftarrow \{\}$
for all $(x, y) \in \mathcal{B}_2(0, A)$ **do**
  for all $(\xi_1, \xi_2) \in \mathcal{B}_2(0, A)$ **do**
    $\pi_1 \leftarrow by - (a+1)x - \xi_1$
    $\pi_2 \leftarrow -bx - (a-b+1)y - \xi_2$
    $t_1 \leftarrow \xi_1^2 + \xi_2^2$
    for all $(\xi_3, \xi_4) \in \mathcal{B}_2(0, A)$ **do**
      if $2\xi_1\xi_3 + 2\xi_2\xi_4 + \xi_3^2 + \xi_4^2 < A^2 - t_1$ **then**
        $z \leftarrow \textsc{GetZ}(x, y, \xi_1, \xi_2, \xi_3, \xi_4)$
        $w \leftarrow \textsc{GetW}(x, y, \xi_1, \xi_2, \xi_3, \xi_4)$
        if $z^2 + w^2 < A^2$ and $(x+z)^2 + (w+y)^2 < A^2$ **then**
          $\pi \leftarrow (\pi_1, \pi_2, \pi_1 + x, \pi_2 + y, \pi_1 + x + z, \pi_2 + y + w)$
          $P \leftarrow P \cup \{ \textsc{FindCycle}(\Phi(\pi), M, D)\}$
        end if
      end if
    end for
  end for
end for
return P

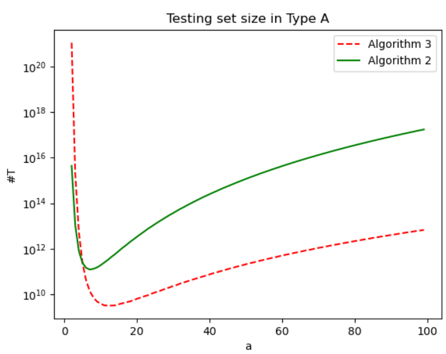

Figure 1. The testing set sizes for Algorithm 2 and Algorithm 3

Let $z = z_1 \bullet z_2 \bullet z_3 \in \mathbb{Z}^6$. The challenge is to find $d \in D$ without knowing (storing) the digit set D. We know that if $z \equiv d \pmod{M}$ then $M^* z \equiv M^* d \pmod{\det(M) I_6}$, where

$$M^* = \begin{pmatrix} \det(M_2)\det(M_3)M_1^* & 0 & 0 \\ 0 & \det(M_1)\det(M_3)M_2^* & 0 \\ 0 & 0 & \det(M_1)\det(M_2)M_3^* \end{pmatrix}.$$

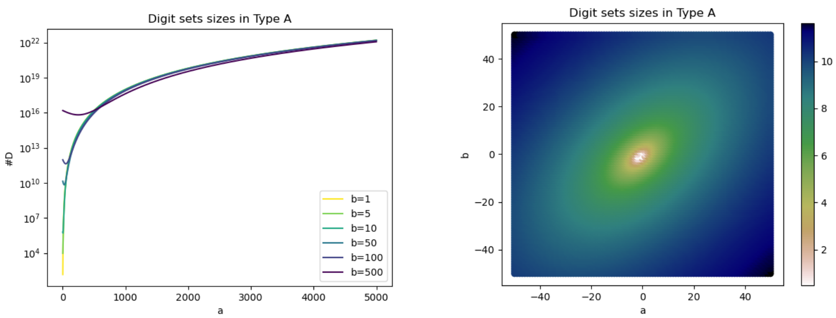

Figure 2. The left side shows the digit set size for fixed b, and the right side shows the heatmap of the digit set size with log_{10} scale for every $a, b \in [-50, 50]$ in $Type_A$ systems.

We analyse the values $M_i^* z_i \pmod{\det(M_i) I_2}$ for $i = 1, 2, 3$. If $M^* z \equiv M^* d \pmod{\det(M) I_6}$ then

$$\begin{cases} z_1 \equiv d^* \equiv b_1 \pmod{M_1} \\ z_2 \equiv d^* \equiv b_2 \pmod{M_2} \\ z_3 \equiv d^* \equiv b_3 \pmod{M_3}. \end{cases}$$

It is known that if $(\det(M_i), \det(M_j)) = 1$ fulfils for $i, j \in \{1, 2, 3\}$, $i \neq j$, then based on Chinese remaindering the values d^* can be computed by

$$d^* = M_2 M_3 x_1 b_1 + M_1 M_3 x_2 b_2 + M_1 M_2 x_3 b_3 \tag{4.1}$$

where $x_1 b_1 = A_{2,3} b_1, x_2 b_2 = A_{1,3} b_2$ and $x_3 b_3 = A_{1,2} b_3$, and further

$$\begin{cases} M_2 M_3 A_{2,3} b_1 \equiv b_1 \pmod{M_1} \\ M_1 M_3 A_{1,3} b_2 \equiv b_2 \pmod{M_2} \\ M_1 M_2 A_{1,2} b_3 \equiv b_3 \pmod{M_3}. \end{cases}$$

By equation (4.1) $d^* \bullet d^* \bullet d^* \in W$ such that $z \equiv d^* \bullet d^* \bullet d^* \pmod{M}$. There exists a corresponding digit $d' \in D^*$ such that $d^* \equiv d_1 \pmod{M_1}$, $M_1^{-1}(d^* - d_1) = d_1^* \equiv d_2 \pmod{M_2}$ and $M_2^{-1}(d_1^* - d_2) \equiv d_3 \pmod{M_3}$ where $d_i \in D_i$ $(i \in \{1, 2, 3\})$. Hence $d' = d_1 + M_1(d_2 + M_2 d_3)$ and $d = d' \bullet d' \bullet d' \in D$ such that $z \equiv d \pmod{M}$ satisfies.

Remark 4.1. The presented Φ function calculation can be generalised to every simultaneous system assuming that the determinants are co-prime.

Algorithm 4 returns the value of Φ without knowing (storing) the digit set when the determinants are co-prime. In the algorithm, M_i, D_i $(I \in \{1,2,3\})$ are the parameters of the investigated system.

Algorithm 4 $\Phi(z_1 \bullet z_2 \bullet z_3)$

$a_1 \leftarrow \textsc{SolveMod}(\det(M_2)\det(M_3)x \equiv 1, \det(M_1))$
$a_2 \leftarrow \textsc{SolveMod}(\det(M_1)\det(M_3)x \equiv 1, \det(M_2))$
$a_3 \leftarrow \textsc{SolveMod}(\det(M_2)\det(M_1)x \equiv 1, \det(M_3))$
$b_1 \leftarrow a_1 M_3^* M_2^* z_1 \pmod{M_1}$
$b_2 \leftarrow a_2 M_3^* M_1^* z_2 \pmod{M_2}$
$b_3 \leftarrow a_3 M_2^* M_1^* b_3 \pmod{M_3}$
$d^* \leftarrow M_2 M_3 b_1 + M_1 M_3 b_2 + M_1 M_2 b_3$
$d_1 \leftarrow \textsc{GetDigit}(d^*, M_1)$
$d_2 \leftarrow \textsc{GetDigit}(M_1^{-1}(d^* - d_1), M_2)$
$d_3 \leftarrow \textsc{GetDigit}(M_2^{-1}(M_1^{-1}(d^* - d_1) - d_2), M_3)$
$d \leftarrow d_1 + M_1(d_2 + M_2 d_3)$
return $M^{-1}(z_1 \bullet z_2 \bullet z_3 - d \bullet d \bullet d)$

The $\textsc{SolveMod}(ax \equiv y, b)$ function returns the solution of $ax \equiv y \pmod{b}$.

The $\textsc{GetDigit}(z, M)$ function returns d such that $d \equiv z \pmod{M}$ where d is a digit from the adjoint digit set of base M.

5. Measurements and observations

This section investigates the behaviour of the parameters L and A with fixed b and increasing a values. It can be observed that the parameter values increase drastically for every b, even for small a values. Figure 3 shows that the $Type_B$ cases are very similar to the $Type_A$ cases, so the next figures will only present the $Type_A$ case. Figure 3 shows that storing the points of the ball with radius L is hardly possible. Figure 4 shows similar measurements to Figure 3 for the parameter A. If $a < 5000$ for every investigated b, the values of A will remain below 10^4. The values of L are bigger than the A with several orders of magnitude. This implies that Algorithm 3 is preferable.

Figure 5 shows the parameter values A without extreme cases. For $a, b \in$ $\in [-100, 100]$, the values of A are changing gradually from 11.64 to 1030, except for the few extreme cases in Table 1. Those values are not shown on the heatmap. On the fourth column of the table, there is a periodic point, which proves that the system is not a number system, and on the last column is the length of the periods. No solution could be found for the missing entries in our computing environment.

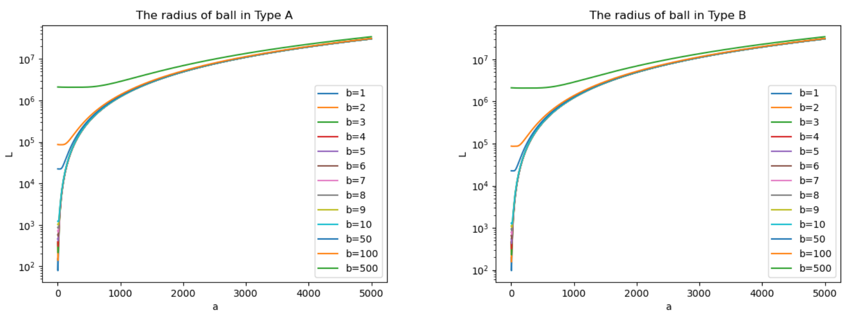

Figure 3. Approximation of L by Lemma 3.3

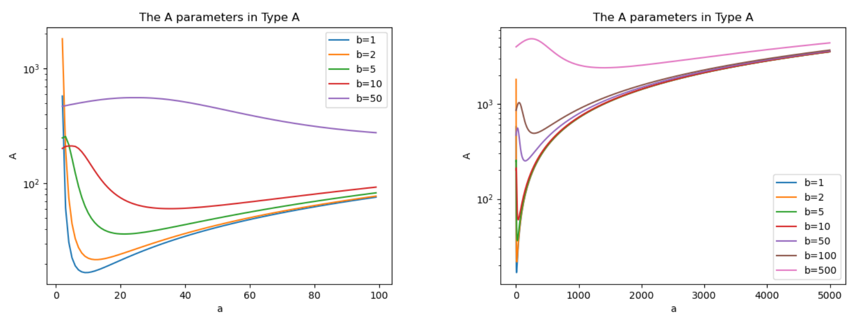

Figure 4. Parameters A for fixed $b \in \{1, 2, 5, 10, 50\}$. On the right side, the horizontal axis shows $1 < a < 5000$, while on the left side, the a values are focused on the $[2, 100)$ interval.

a	b	A	$\pi \in \mathcal{P}$	$\#\mathcal{C}_\pi$	a	b	A	$\pi \in \mathcal{P}$	$\#\mathcal{C}_\pi$
-3	-4	659.08	$(4, 5, 5, 4, 7, 7)$	3	-3	-2	946.54	$(-1, 0, -1, 1, -2, -1)$	16
-3	-3	1811.8	$(1, -1, 0, -2, 2, -2)$	3	1	3	622.44	$(1, -4, 4, -2, 2, -2)$	6
-1	2	1811.8	$(0, -2, 2, -2, 1, -1)$	3	-2	1	575.4	$(-1, 1, -2, 2, -2, 0)$	3
-2	-4	659.08	$(-5, -1, -4, 1, -7, 0)$	6	0	3	659.08		
-1	-3	1811.8	$(-2, 0, -1, 1, -2, 2)$	6	2	3	659.08		
2	2	1811.8	$(2, -2, 2, 0, 1, -1)$	6					

Table 1. Extremely large A values, where $\pi \in \mathcal{P}$ and $\#\mathcal{C}_\pi$ is the length of the period of π.

6. Conclusion

This article presents theoretical and algorithmic results for determining the testing set of $(2, 3)$-simultaneous systems. It specifies the necessary conditions for the number system property. Different conditions were presented for the

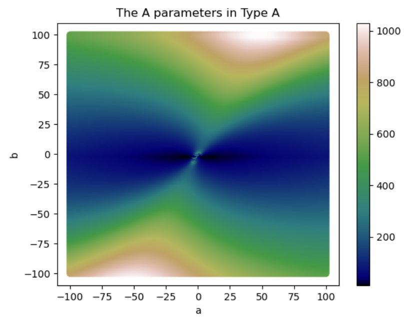

Figure 5. The heatmap of A.

location of periodic points, which helped reduce the testing set size. From the classification point of view, algorithms were provided that can solve the problem more efficiently than the previously known algorithms. From computational aspects, in most cases, the digit set size is so huge that it can not be stored in memory. A new algorithm was provided, which can compute the Φ function without storing the digit set in case the pairwise determinant of the blocks are co-primes.

References

[1] **Burcsi, P., A. Kovács and Zs. Papp-Varga,** Decision and classification algorithms for generalized number systems, *Annales Univ. Sci. Budapest., Sect. Comp.*, **28** (2008), 141–156.

[2] **Farkas, G.,** Periodic elements and number systems in $\mathbb{Q}(\sqrt{2})$, *Math. and Comp. Modelling.*, **38** (2003), 783–788.

[3] **Farkas, G. and A. Kovács,** Digital expansion in $\mathbb{Q}(\sqrt{2})$, *Annales Univ. Sci. Budapest., Sect. Comp.*, **22** (2003), 83–94.

[4] **Farkas, I.I. and A. Kovács,** Some results on simultaneous number systems in the ring of imaginary quadratic fields, *Annales Univ. Sci. Budapest., Sect. Comp.*, **51** (2020), 77–88.

[5] **Frougny, Ch. and J. Sakarovitch,** Number representation and finite automata, in: V. Berthé and M. Rigo M. (Eds.) *Combinatorics, automata and number theory*, Encyclopedia Math. Appl., **135**, 34–107, Cambridge University Press, 2010.

[6] **Germán, L. and A. Kovács,** On number system constructions, *Acta Math. Hungar.*, **115(1-2)** (2007), 15–167.

[7] **Indlekofer, K.-H., I. Kátai and P. Racskó,** Number systems and fractal geometry, in: J. Galambos and I. Kátai (Eds.) *Probability Theory and its Applications*, Kluwer Publ. (1992), 319–334.

[8] **Kátai, I.,** Number systems in imaginary quadratic fields, *Annales Univ. Sci. Budapest., Sect.Comp.*, **14** (1994), 91–103.

[9] **Kovács, A.,** On computation of attractors for invertible expanding linear operators in $\mathbb{Z}^k$, *Publ. Math. Debrecen*, **56(1-2)** (2000), 97–120.

[10] **Kovács, A.,** *Radix Expansions in Lattices*, PhD thesis, Eötvös Loránd University, Budapest, (2001), 1–98.

[11] **Kovács, A.,** Number system constructions with block diagonal bases, *RIMS Kôkyûroku Bessatsu*, **B46** (2014), 205–213.

[12] **Kovács, A.,** Algorithmic construction of simultaneous number systems in the lattice of Gaussian integers, *Annales Univ. Sci. Budapest., Sect. Comp.*, **39(1)** (2013), 279–290.

[13] **Kovács, A.,** Simultaneous number systems in the lattice of Eisenstein integers, *Annales Univ. Sci. Budapest., Sect. Comp.*, **41** (2013), 43–55.

[14] **Kovács, A.,** Generalized binary number systems, *Annales Univ. Sci. Budapest., Sect. Comp.*, **20** (2001), 195–206.

[15] **Kovács, A.,** On number expansions in lattices, *Math. and Comp. Modelling.*, **38** (2003), 909–915.

[16] **Krutki, T. and G. Nagy,** On simultaneous number systems with 3 bases, *Annales Univ. Sci. Budapest., Sect. Comp.*, **46** (2017), 153–163.

[17] **Nagy, G.,** On the simultaneous number systems of Gaussian integers, *Annales Univ. Sci. Budapest., Sect. Comp.*, **35** (2011), 223–238.

[18] **Steidl, G.,** On symmetric representation of Gaussian integers, *BIT Numerical Mathematics*, **29** (1989), 563–571.

[19] **van de Woestijne, Ch.E.,** Number systems and the Chinese remainder theorem, arXiv preprint (2011)
https://arxiv.org/pdf/1106.4219

I.I. Farkas
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University (ELTE)
H-1117 Budapest, Pázmány Péter sétány 1/C
Hungary
Ingrid.Farkas@inf.elte.hu