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COMPARISON AND ERROR BOUNDS OF NON-PARAMETRIC IDENTIFICATION SCHEMES USING RATIONAL ORTHOGONAL BASIS FUNCTIONS

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Abstract. The study examines two extensions of the *Schi-So* algorithm, a unique approach that identifies a pole of a complex rational function using its Laguerre series expansion. The two variants of the Schi-So algorithm that are compared in the study are the *Iterative Schi-So* method which eliminates the effect of previously identified poles from the transfer function, and a recent generalization for periodic Malmquist–Takenaka systems, the *Generalized Schi-So* method, which ensures that previously found poles are not repeated in the identification process; it is achieved through an iterative parameter change of the Malmquist–Takenaka basis. In this paper these two variants have been compared both numerically and theoretically, moreover error bounds are given to series coefficients.

1. Introduction

System identification is a field between mathematics and engineering that models dynamic systems based on their measured behavior. The field has seen

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significant advancements over the last few decades. A common approach for system identification is to model the system as a black box, where explicit knowledge about the internal physics is not required. However, some general assumptions about the system's behaviour are required in order to effectively employ mathematical tools in the identification algorithms. Here single input single output, linear, time invariant (SISO LTI) systems are considered, which allows us to give a frequency domain description of the system with the transfer function formulation. Moreover, it is assumed herein that the system is realizable, causal and stable in ℓ^2 (finite energy) or ℓ^∞ (bounded-input bounded-output, BIBO) sense. With these assumptions the transfer function is an element of $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$, and the identification can be considered as an approximation problem with respect to the respective norm of these spaces [2].

For realizable systems the transfer function is a strictly proper rational function which is uniquely determined by its poles and zeros up to a constant factor. Thus the identification can be achieved by finding these parameters. Of these tasks the localization of zeros is significantly easier if we already know the location of the system's poles [8]. This observation can be utilized by the application of generalized orthogonal basis functions in $H^2(\mathbb{D})$ to the identification process, such as the Malmquist–Takenaka system and its special case the Laguerre system [13]. In the present study, we will examine the so-called Schi-So algorithm, a unique approach to system identification introduced by F. Schipp and A. Soumelidis in [12], that finds one pole at a time using Laguerre series expansion of the transfer function. In [14] this method was extended in a way that allows the algorithm to be used in an iterative manner, and in [3] the method was combined with the so-called rational Variable Projection method. With eliminating the effect of the previously found poles, this extension is capable of finding every pole of the transfer function. The Schi-So algorithm was recently generalized for so-called periodic Malmquist–Takenaka systems by T. Dózsa, F. Schipp and A. Soumelidis in [4] and for a more general class of complex analytic functions in [15]. Instead of eliminating the pole's effect, this variant uses the periodic Malmquist–Takenaka system to ensure that the same pole is not found more than once. In this article, these two variants of the Schi-So algorithm will be compared, an error bound will be given on the coefficients on which they are based on.

2. Description of considered iterative methods

2.1. The general setting for pole identification

A discrete time SISO LTI system can be expressed in the time domain via the following convolutional operation:

$$x(n) = (h * y)(n), \qquad (n \in \mathbb{Z})$$

where $x, y \in \ell^p$ $(p = 1, 2, \infty)$ are the input and output sequence of the system considered and h is called its impulse response. Taking the Z-transform (defined as the Fourier synthesis) of the equation above, we get

$$X(z) = H(z) \cdot Y(z) \qquad (z \in \mathbb{C}), \tag{2.1}$$

where H is the transfer function of the system. With the assumption of the system's stability and causality, i.e. $u \mapsto h*u$ is a bounded mapping and $u(n) = 0$ for $n > 0$, H is a meromorphic function on $\mathbb{C}$, and in fact a holomorphic on the closed complex unit disc $\overline{\mathbb{D}}$. Furthermore, if we assume realizability, i.e. the system can be represented with a finite dimensional state space model, then the transfer function is a strictly proper rational function in the following form:

$$H(z) = \frac{P(z)}{Q(z)} = c \cdot \frac{\prod_{k=0}^{\deg P - 1}(z - p_k)}{\prod_{k=0}^{\deg Q - 1}(z - q_k)} \qquad (c, p_k \in \mathbb{C},\ q_k \in \mathbb{C} \setminus \overline{\mathbb{D}}). \tag{2.2}$$

with P and Q being polynomials with no common factors and $\deg P < \deg Q$. From now on for simplicity it will assumed, that all the poles of H are simple. In this case H can be expressed with the following partial fraction decomposition

$$H(z) = \sum_{k=0}^{\deg Q - 1} \frac{A_k}{1 - \overline{a_k} z} = \sum_{k=0}^{\deg Q - 1} A_k\, r_{a_k}(z) \qquad \left(a_k = \frac{1}{\overline{q_k}}\right). \tag{2.3}$$

The function $r_a(z) := \frac{1}{1 - \overline{a} z}$ is referred to as an elementary rational function and the parameter a is called the inverse pole r_a since its reflection to the unit circle $\frac{1}{\overline{a}}$ is the pole of r_a. Since H is in $H^2(\mathbb{D})$ its poles lie outside the unit disk, thus we see that for the inverse poles $a_k \in \mathbb{D}$. It should be noted that finding the inverse poles of a rational function is equivalent to finding its poles.

The elementary rational functions are linearly independent in $H^2(\mathbb{D})$ and give rise to an orthonormal system in this Hilbert space the following way.

Definition 2.1. Let $\mathbf{a} = (a_0, a_1, a_2 \dots) \in \mathbb{D}^{\mathbb{N}}$ a series of inverse poles in $\mathbb{D}$. Then for $n \in \mathbb{N}$ we define the n-th Malmquist–Takenaka (MT) function parametrized by $\mathbf{a}$ as

$$\Phi_n^{\mathbf{a}}(z) = \sqrt{1 - |a_n|^2}\, r_{a_n}(z) \prod_{k=0}^{n-1} B_{a_k}(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n} z} \prod_{k=0}^{n-1} \frac{z - a_k}{1 - \overline{a_k} z}. \tag{2.4}$$

Here the function $B_a(z) = \frac{z - a}{1 - \overline{a} z}$ is called the Blaschke function. From our perspective it is important that for arbitrary $a \in \mathbb{D}$ the function B_a is a conform automorphism of the complex unit disk, so it maps $\mathbb{D}$ and $\mathbb{T}$ to themselves bijectively. Moreover these functions have a remarkable property:

they are isometries of $\mathbb{D}$ both with the pseudo-hyperbolic metric ρ and the hyperbolic metric δ. These metrics are defined by the respective formulae:

$$\rho(z,w) = |B_z(w)| = \left|\frac{w-z}{1-\overline{z}w}\right| \tag{2.5}$$

$$\delta(z,w) = \log\frac{1+\rho(z,w)}{1-\rho(z,w)} = 2\,\mathrm{artanh}(\rho(z,w)), \tag{2.6}$$

where $z, w \in \mathbb{D}$. The metric space $(\mathbb{D}, \delta)$ is called the Poincaré disk model of the hyperbolic plane. In fact the entire group of orientation-preserving isometries of $\mathbb{D}$ with these metrics is constituted by the Blaschke function of the form:

$$B_{(\epsilon,a)}(z) = \epsilon B_a(z) = \epsilon \cdot \frac{z-a}{1-\overline{a}z} \qquad \big((\epsilon,a) \in \mathbb{T}\times\mathbb{D},\ z \in \mathbb{D}\big)$$

with the composition as the group operation [13]. This group is sometimes referred as the Blaschke group and it is isomorphic to the indefinite projective unitary group $\mathrm{PU}(1,1)$ where the multiplication is given by the matrix product.

It is known that the spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are naturally isomorphic, which is given by the connection between the complex Taylor and Fourier series. It can be shown that for any sequence of inverse poles $\mathbf{a}$, the Malmquist–Takenaka system $\Phi_n^{\mathbf{a}}$ forms an orthonormal system in these spaces with respect to the scalar product given by

$$\langle f,g\rangle_{H^2(\mathbb{D})} = \sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\mathrm{d}\theta \tag{2.7}$$

for $f, g \in H^2(\mathbb{D})$. From (2.7) the induced norm becomes

$$\|f\|_{H^2(\mathbb{D})} = \frac{1}{2\pi}\int_0^{2\pi}\left|f(e^{i\theta})\right|^2\mathrm{d}\theta = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{|f(\zeta)|^2}{\zeta}\mathrm{d}\zeta. \tag{2.8}$$

The completeness of $\Phi_n^{\mathbf{a}}$ does hold in general, however, it can be guaranteed also for particular cases with the following theorem.

Theorem 2.1 (Szász-condition)**.** *For the inverse poles* $\mathbf{a} = (a_0, a_1, a_2 \dots) \in \mathbb{D}^{\mathbb{N}}$ *the Malmquist–Takenaka system is complete in* $H^2(\mathbb{D})$*, if and only if the limit*

$$\lim_{N\to\infty}\sum_{n=0}^{N}(1-|a_n|) \tag{2.9}$$

does not converge.

The Szász-condition relies on the fact – proved by W. Blaschke – that the infinite Blaschke-product

$$\prod_{n=0}^{\infty} \frac{a_n}{|a_n|} B_{a_n}(z)$$

is (uniformly) convergent, if and only if the limit given in (2.9) is convergent. The reader can find the detailed proof in [13].

The completeness of a Malmquist–Takenaka system means, that for arbitrary $f \in H^2(\mathbb{D})$ the abstract Fourier series of f by $\Phi_n^{\mathbf{a}}$ equals to f itself, i.e.

$$f = \sum_{n=0}^{\infty} \langle f, \Phi_n^{\mathbf{a}} \rangle \Phi_n^{\mathbf{a}} = \lim_{N \to \infty} \sum_{n=0}^{N} \langle f, \Phi_n^{\mathbf{a}} \rangle \Phi_n^{\mathbf{a}}, \tag{2.10}$$

where the limit is taken in the norm topology of $H^2(\mathbb{D})$. Due to its mentioned completeness, the Malmquist–Takenaka functions found many applications in signal processing [5, 11]. However one can observe that each $\Phi_n^{\mathbf{a}}$ is a strictly proper rational function. Building on this observation, we can claim that for $\mathbf{a} = (a_0, a_1 \dots a_{N-1} \dots)$ the subspace

$$\mathcal{R}_N^{\mathbf{a}} = \operatorname{span} \{ \Phi_n^{\mathbf{a}} \mid n = 1 \dots N-1 \}$$

where $\mathcal{R}_N^{\mathbf{a}}$ is the space of all strictly proper rational functions whose poles are exactly $1/\overline{a_0}, \dots 1/\overline{a_{N-1}}$ (generally counted with multiplicity). For the proof of this claim we direct the reader to [13]. This property makes the Malmquist–Takenaka system directly applicable in system identification and control theory.

Now we will present some of the important special cases of Malmquist–Takenaka systems. For all examples the condition given in (2.1) holds trivially.

If we chose the constant 0 sequence for inverse poles, then $B_{\mathbf{a}}(z) = z$, so the MT-coefficients of a function $f \in H^2(\mathbb{D})$ will become

$$\langle f, \Phi_n^{\mathbf{a}} \rangle = \int_0^{2\pi} f(e^{i\theta}) \overline{\Phi_n^{\mathbf{a}}(e^{i\theta})} \mathrm{d}\theta = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \mathrm{d}\theta \tag{2.11}$$

which are the ordinary complex Fourier coefficients of f.

Another special Malmquist–Takenaka system is the Laguerre system corresponding to the constant sequence $\mathbf{a} = (a, a, a \dots)$. Then the Laguerre functions are

$$L_n^a(z) = \Phi_n^{\mathbf{a}}(z) = \frac{\sqrt{1 - |a|^2}}{1 - \overline{a} z} B_a^{n-1}(z). \tag{2.12}$$

These function will bear significance in section 2.2 and 2.3.

Another interesting choice of the parameters yields the so-called periodic Malmquist–Takenaka system (PMT). In this case, the sequence of inverse poles is periodic, i.e. there exists a positive natural number p such that $a_{n+p} = a_n$ for every $n \in \mathbb{N}$. If we denote the p-th order Blaschke product by

$$B_{\mathbf{a}|p} = \prod_{k=0}^{p-1} B_{a_k}(z), \tag{2.13}$$

then we can express the n-th periodic Malmquist–Takenaka function as

$$\Phi_n^{\mathbf{a}}(z) = \Phi_m^{\mathbf{a}}(z) B_{\mathbf{a}|p}^k \qquad (n = m + kp,\ m < p). \tag{2.14}$$

These functions will be used in section 2.4 to generalize the Schi-So algorithm.

2.2. The *Schi-So* method for identifying a single pole of a rational function

In this section we will briefly summarise the Schi-So method for reconstructing a single pole of a rational function $R \in H^2(\mathbb{D})$. This method is similar in nature to the Bernoulli method for polynomial root finding, which was introduces by D. Bernoulli in 1732 (see [1]), and later proved by Euler in 1748.

Suppose R is an elementary rational function with a single inverse pole $\alpha \in \mathbb{D}$, i.e.:

$$R(z) = \frac{A}{1 - \overline{\alpha} z} \qquad (A \in \mathbb{C}).$$

Then the Laguerre coefficients of R are

$$l_n^a = \langle R, L_n^a \rangle = \overline{L_n^a(\alpha)} = A\sqrt{1 - |a|^2}\, \frac{(\overline{\alpha} - \overline{a})^n}{(1 - \overline{\alpha} a)^{n+1}}. \tag{2.15}$$

This can be directly verified by applying Cauchy's formula for L_n^a. This means that l_n^a is a geometric sequence and by taking the conjugate of the quotients, we get

$$\overline{q} = \frac{\overline{l_{n+1}^a}}{\overline{l_n^a}} = \frac{\alpha - a}{1 - \overline{a}\alpha} = B_a(\alpha). \tag{2.16}$$

Using the fact that for the composition operation the inverse of B_a is

$$\left(B_a(z)\right)^{-1}(z) = B_{-a}(z), \tag{2.17}$$

one can reconstruct the original inverse pole from R by the Laguerre coefficients through their quotients:

$$\alpha = B_{-a}(\overline{q}) = B_{-a}\left(\frac{\overline{l_{n+1}^a}}{\overline{l_n^a}}\right), \tag{2.18}$$

for every $n \in \mathbb{N}$.

This idea can be extended for the case when R has multiple poles. Let $\alpha_0, \dots \alpha_{K-1} \in \mathbb{D}$ simple inverse poles and

$$R(z) = \sum_{k=0}^{K-1} A_k r_{a_k}(z) = \sum_{k=0}^{K-1} \frac{A_k}{1 - \overline{a_k} z} \tag{2.19}$$

the partial fraction decomposition of R. Calculating the Laguerre coefficients of R we get

$$l_n^a = \sum_{k=0}^{K-1} A_k \sqrt{1-|a|^2} \, \frac{(\overline{\alpha_k} - \overline{a})^n}{(1 - \overline{\alpha_k} a)^{n+1}} = \sum_{k=0}^{K-1} A_k \frac{\sqrt{1-|a|^2}}{1 - \overline{\alpha_k} a} \overline{B_a(\alpha_k)^n}. \tag{2.20}$$

For the convergence of the quotients we have to define the following property.

Definition 2.2. Let $a \in \mathbb{D}$ a Laguerre parameter and $\alpha_0, \dots \alpha_{K-1} \in \mathbb{D}$ simple inverse poles of the rational function R. Then we say that α_d is a B_a-dominant inverse pole of R

$$\rho(a, \alpha_d) > \rho(a, \alpha_k) \tag{2.21}$$

for every $k \neq d$ index in $\{0, \dots K-1\}$, where ρ is the pseudo-hyperbolic metric defined in (2.5).

If we have chosen the parameter a such that α_d is a unique B_a-dominant inverse pole of R, then the quotient is

$$\overline{q_n} = \frac{\overline{l_{n+1}^a}}{\overline{l_n^a}} = B_a(\alpha_d) \cdot \frac{\overline{A_d} \frac{\sqrt{1-|a|^2}}{1-\overline{\alpha_d} a} + \sum_{k=0, k\neq d}^{K-1} \overline{A_k} \frac{\sqrt{1-|a|^2}}{1-\overline{\alpha_k} a} \left(\frac{B_a(\alpha_k)}{B_a(\alpha_d)} \right)^{n+1}}{\overline{A_d} \frac{\sqrt{1-|a|^2}}{1-\overline{\alpha_d} a} + \sum_{k=0, k\neq d}^{K-1} \overline{A_k} \frac{\sqrt{1-|a|^2}}{1-\alpha_k a} \left(\frac{B_a(\alpha_k)}{B_a(\alpha_d)} \right)^{n}}. \tag{2.22}$$

Because α_d is B_a-dominant

$$\left| \frac{B_a(\alpha_k)}{B_a(\alpha_d)} \right| < 1$$

for $k \neq d$, therefore we see that by taking the limit in (2.22), we get

$$B_a(\alpha_d) = \lim_{n\to\infty} \overline{q_n} = \lim_{n\to\infty} \overline{\frac{l_{n+1}^a}{l_n^a}}. \tag{2.23}$$

For this one can apply the inverse Blaschke function to recover the dominant pole of R:

$$\alpha_d = \lim_{n\to\infty} B_{-a}(\overline{q_n}). \tag{2.24}$$

We can summarize this in the following theorem.

Theorem 2.2 (Schi-So method). *Let $R \in H^2(\mathbb{D})$ be a rational function with simple inverse poles $\alpha_0, \alpha_2, \ldots, \alpha_{K-1} \in \mathbb{D}$. Let $a \in \mathbb{D}$ be fixed and consider the expansion $R(z) = \sum_{n=1}^{\infty} l_{n+1}^a L_n^a(z)$. If α_d is the dominant inverse pole of R, then for the ratio sequence*

$$q_n := \frac{l_{n+1}^a}{l_n^a}$$

we have

$$\lim_{n\to\infty} q_n = \overline{B(\alpha_d)}.$$

Moreover, the rate of convergence is $O(\gamma^n)$, where $\gamma := \max_{k\neq d} \frac{\rho(a,\alpha_k)}{\rho(a,\alpha_d)}$.

A more general version of Theorem 2.2 has been provided in the paper [12].

For the choice of the parameter $a \in \mathbb{D}$, we have a theoretical guarantee by Theorem 2.2, that the limit will converge except for a set with measure zero which is

$$\{a \in \mathbb{D} \,|\, \exists 0 \leq j, l < K, j \neq l, \rho(a, \alpha_j) = \rho(a, \alpha_l)\}.$$

This set and the regions of convergence of the Schi-So method have a special geometric interpretation about which more information can be found in [12] and [4]. For practical reasons however we need to be more careful with the choice of a. One desired property is to have a fast rate of convergence. But numerically the division of small numbers results in high absolute error which should be also considered. In [4] the authors propose the following cost function to be used for parameter a.

$$C(a) = \min_{n\geq 0} \max_{0\leq i,j<W} |q_{n+i}^a - q_{n+j}^a|, \tag{2.25}$$

where $W > 0$ is a pre-defined window size. Considering the geometrical connections of the problem, the optimization of this non-linear function can be done for example with the hyperbolic Nelder-Mead method (see [10]).

2.3. The *Iterative Schi-So* method

We can use the method described in section 2.2 to construct an iterative method for finding every pole of R. The method, as outlined in [14], hinges on the principle that after we have successfully found one inverse pole of R we can utilize the structure of the Malmquist–Takenaka functions to reduce R to a function that only depends on the remaining inverse poles. It must be emphasised, that with this algorithm we can theoretically find all poles of a rational function without *a priori* knowledge about their actual number.

Since R is a strictly proper rational function with inverse poles α_k, $(0 \leq \leq k \leq K-1)$, one can express it as the linear combination of the first K

MT functions whose parameter sequence starts with the inverse poles of R i.e. $\mathbf{a} = (\alpha_0, \alpha_1, \dots \alpha_{K-1} \dots)$:

$$R(z) = \sum_{k=0}^{K-1} c_k \Phi_n^{\mathbf{a}}(z). \tag{2.26}$$

Assuming that the location of the first m inverse poles have been already identified, one can construct the m-th order partial sum as

$$(S_m^{\mathbf{a}} R)(z) = \sum_{k=0}^{m-1} c_k \Phi_n^{\mathbf{a}}(z). \tag{2.27}$$

Note that this function only depends on the first m inverse poles, so in fact we can calculate it from R and $\alpha_0, \dots \alpha_{m-1}$. We can examine the "unknown" orthogonal complement of $S_m^{\mathbf{a}}$, which is

$$\begin{aligned} R(z) - (S_m^{\mathbf{a}} R)(z) &= \sum_{k=m}^{K-1} c_k \Phi_n^{\mathbf{a}}(z) = \\ &= B_{\mathbf{a}|m}(z) \cdot \sum_{k=m}^{K-1} c_k \frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k} z} \prod_{l=m}^{k} B_{a_l}(z), \end{aligned} \tag{2.28}$$

where $B_{\mathbf{a}|m}$ is the Blaschke product defined in (2.13). Since $|B_a(z)| = 1$ for any $z \in \mathbb{T}$ and $a \in \mathbb{D}$, therefore also $|B_{\mathbf{a}|m}| = 1$. In the context of Hardy spaces this means that the Blaschke product is an inner function. Thus we can define the function

$$R_m(z) = \overline{B_{\mathbf{a}|m}} \left(R(z) - (S_m^{\mathbf{a}} R)(z)\right) = \sum_{k=0}^{K-m-1} c_{k+m} \Phi_n^{\mathbf{a}'}(z), \tag{2.29}$$

where $\mathbf{a}' = (\alpha_m, \dots \alpha_{K-1}, \dots)$. Upon observation, we can note that R_m solely depends on the inverse poles $\alpha_m, \dots \alpha_{K-1}$, and being a strictly proper rational function, it exclusively contains the listed inverse poles.

Using (2.29) we can construct an algorithm (see **Algorithm 1**).

2.4. The *Generalized Schi-So* method

For finding every pole of a rational function we can extend the Schi-So method in a different way. This generalization was presented in the recent publication [4], and in this article we refer to it as the Generalized Schi-So method. The main suggestion of [4] is that instead of modifying the transfer function in every iteration, one can modify the parameters of the Malmquist–Takenaka system in a way that can guarantee that we do not find the poles that

Algorithm 1 Iterative Schi-So method

Choose $0 < \epsilon \ll 1$ for relative error threshold.
Acquire an appropriate discretization of R on the unit circle $\mathbb{T}$.
Let $\mathbf{p} = [\,]$ ▷ List of the already identified poles.
Let $m = 1$.
while $\epsilon <$ PRD **do**
  Choose a parameter $a \in \mathbb{D}$ for the Laguerre series expansion.
  Calculate R_m using (2.29) with the list of the found inverse poles $\mathbf{p}$.
  Calculate the Laguerre coefficients of R_m.
  Approximate the inverse pole α from the coefficients using Theorem 2.2.
  Let $\mathbf{p} = \mathbf{p} \cup \alpha$.
  Let PRD $= \frac{\|R - S_m^{\mathbf{p}} R\|_{H^2}}{\|R\|_{H^2}}$.
  Let $m = m + 1$.
end while

have already been found. This is achieved by using the more general periodic Malmquist–Takenaka system instead of the Laguerre system.

Now we are interested in its coefficients by the periodic Malmquist–Takenaka system parametrized by a periodic sequence $\mathbf{a} \in \mathbb{D}^{\mathbb{N}}$ with period p. First assume again, that $R \in H^2(\mathbb{D})$ is the elementary rational function

$$R(z) = \frac{A}{1 - \overline{\alpha} z},$$

with the periodic Malmquist–Takenaka expansion

$$R(z) = \sum_{n=0}^{\infty} c_n \Phi_n^{\mathbf{a}}(z). \tag{2.30}$$

Similarly to (2.15) we see the that by using the Cauchy formula

$$c_n = \langle R, \Phi_n^{\mathbf{a}} \rangle = \overline{\Phi_n^{\mathbf{a}}(\alpha)}. \tag{2.31}$$

So if we fix the index $m < p$ and defining $\nu_l = m + lp$ for $l \geq 0$, then using (2.14), we get

$$q_l = \frac{c_{\nu_l + 1}}{c_{\nu_l}} = \frac{\overline{\Phi_{\nu_l+1}^{\mathbf{a}}(\alpha)}}{\overline{\Phi_{\nu_l}^{\mathbf{a}}(\alpha)}} = \frac{\overline{\Phi_{m+1}^{\mathbf{a}}(\alpha) B_{\mathbf{a}|p}(\alpha)^l}}{\overline{\Phi_m^{\mathbf{a}}(\alpha) B_{\mathbf{a}|p}(\alpha)^l}} = \frac{\overline{\Phi_{m+1}^{\mathbf{a}}(\alpha)}}{\overline{\Phi_m^{\mathbf{a}}(\alpha)}}, \tag{2.32}$$

where $B_{\mathbf{a}|p}(\alpha)$ is the finite Blaschke product defined in (2.13). Before calculating the limit of (2.32) we need the following generalization of the definition 2.2.

Definition 2.3. Let $\alpha_0, \dots \alpha_{K-1} \in \mathbb{D}$ be inverse poles of a rational function $R \in H^2(\mathbb{D})$, furthermore let $\mathbf{a} = a_0, \dots a_{p-1} \in \mathbb{D}$ be the parameters of the finite Blaschke product $B_{\mathbf{a}|p}$. Then we say that an inverse pole α_d of R is $B_{\mathbf{a}|p}$-dominant, if

$$\left|B_{\mathbf{a}|p}(\alpha_d)\right| > \left|B_{\mathbf{a}|p}(\alpha_k)\right|$$

holds for every $0 \le k < K$ when $k \neq d$.

Note that this is indeed a generalization of the definition 2.2, when $p = 1$.

Now let us suppose that $R \in H^2(\mathbb{D})$ is an arbitrary rational function and α_d is a $B_{\mathbf{a}|p}$-dominant pole of R. Then, similarly to (2.22), the quotients of (2.32) can be written as:

$$\overline{q_l} = \frac{\overline{c_{\nu_l+1}}}{\overline{c_{\nu_l}}} = \frac{\Phi^{\mathbf{a}}_{m+1}(\alpha_d) + \sum_{k=0,k\neq d}^{K-1} \frac{\overline{A_k}}{\overline{A_d}} \Phi^{\mathbf{a}}_{m+1}(\alpha_k) \left(\frac{B_{\mathbf{a}|p}(\alpha_k)}{B_{\mathbf{a}|p}(\alpha_d)}\right)^l}{\Phi^{\mathbf{a}}_{m}(\alpha_d) + \sum_{k=0,k\neq d}^{K-1} \frac{\overline{A_k}}{\overline{A_d}} \Phi^{\mathbf{a}}_{m}(\alpha_k) \left(\frac{B_{\mathbf{a}|p}(\alpha_k)}{B_{\mathbf{a}|p}(\alpha_d)}\right)^l}. \tag{2.33}$$

Since α_d is $B_{\mathbf{a}|p}$-dominant, by taking the limit of (2.33) we get

$$\lambda = \lim_{k\to\infty} \overline{q_l} = \frac{\Phi^{\mathbf{a}}_{m+1}(\alpha_d)}{\Phi^{\mathbf{a}}_{m}(\alpha_d)} = \frac{\sqrt{1-|a_m|^2}}{\sqrt{1-|a_{m+1}|^2}} \cdot \frac{\alpha_d - a_m}{1-\overline{a_{m+1}}\alpha_d} = \kappa_m \frac{\alpha_d - a_m}{1-\overline{a_{m+1}}\alpha_d}$$

From (2.4) α_d can be expressed as:

$$Q_m(\lambda) = \frac{\lambda/\kappa_m + a_m}{1-\overline{a_{m+1}}\lambda/\kappa_m} = \alpha_d. \tag{2.34}$$

Note that if $\mathbf{a}$ is chosen such that $a_m \le a_{m+1}$, then Q_m is continuous on $\mathbb{D}$ because

$$\lambda = \frac{\kappa_m}{\overline{a_{m+1}}} = \frac{1}{\overline{a_{m+1}}} \cdot \frac{\sqrt{1-|a_m|^2}}{(\sqrt{1-|a_{m+1}|^2})}.$$

These can be summarized with the following theorem as also found in [4].

Theorem 2.3 (Generalized Schi-So method)**.** *Let $R \in H^2(\mathbb{D})$ be a rational function with simple inverse poles $\alpha_1, \alpha_2, \dots, \alpha_{K-1} \in \mathbb{D}$ and $\mathbf{a} = a_0, \dots a_{p-1} \in \in \mathbb{D}$ inverse poles for the periodic Malmquist–Takenaka system $\Phi^{\mathbf{a}}_n$. Consider the expansion $R(z) = \sum_{n=1}^{\infty} c_n \Phi^{\mathbf{a}}_n(z)$. If α_d is the $B_{\mathbf{a}|p}$-dominant inverse pole of R, then for the ratio sequence*

$$q_{\nu_k} := \frac{c_{\nu_k+1}}{c_{\nu_k}}$$

we have

$$\lim_{k\to\infty} Q_m(\overline{q_{\nu_k}}) = Q_m(\lim_{k\to\infty} \overline{q_{\nu_k}}) = \alpha_d.$$

Moreover, the rate of convergence is $O(\gamma^k)$, where $\gamma := \max_{k\neq d} \frac{|B_{\mathbf{a}|p}(\alpha_k)|}{|B_{\mathbf{a}|p}(\alpha_d)|}$.

From the Theorem 2.3 one can create an iterative algorithm for pole identification. Since it is up to us to determine the parameter vector $\mathbf{a} \in \mathbb{D}^p$, we choose it to contain the already found inverse poles. If α is such, then we see that

$$\left|B_{\mathbf{a}|p}(\alpha)\right| = 0, \tag{2.35}$$

from which directly follows, that α cannot by a $B_{\mathbf{a}|p}$-dominant pole. Thus, when considering the limit in Theorem 2.3, the result cannot be an inverse pole, that we have already found.

First, a parameter $b \in \mathbb{D}$ has to be chosen from which the Malmquist–Takenaka expansion can be calculated. This can be done as described at the end of section 2.2, as this step is the same as the Schi-So method itself. However for practical reasons it is also suggested, that in the $n+1$-th step of the algorithm the parameter vector should be

$$\mathbf{a} = (\alpha_0, \ldots, \alpha_{n-1}, b), \tag{2.36}$$

where α_j are the already found inverse poles and b is a parameter that needs to be optimized as before (see section 2.2). The Generalized Schi-So algorithm can be constructed with the following steps.

Algorithm 2 Generalized Schi-So method

Choose $0 < \epsilon \ll 1$ for relative error threshold.
Acquire an appropriate discretization of R on the unit circle $\mathbb{T}$.
Let $\mathbf{p} = [\,]$ ▷ List of the already identified poles.
Let $m = 1$.
while $\epsilon <$ PRD **do**
  Choose a parameter $a \in \mathbb{D}$ for the PMT series expansion.
  Define $\mathbf{b} = \mathbf{p} \cup a$.
  Calculate the PMT coefficients of R, corresponding to $\mathbf{b}$.
  Approximate the inverse pole α from the coefficients using Theorem 2.3.
  Let $\mathbf{p} = \mathbf{p} \cup \alpha$.
  Let PRD $= \frac{\|R - S_m^{\mathbf{p}} R\|_{H^2}}{\|R\|_{H^2}}$.
  Let $m = m + 1$.
end while

3. Numerical comparison

In this section we present a numerical comparison between the Iterative Schi-So and the Generalized Schi-So algorithm. All tests were run in MATLAB, using the Rational Approximation and Interpolation Toolbox (RAIT,

see [10]). For testing the Generalized Schi-So method, we have built upon the implementation provided with [4], however, as we will later see, various modifications had to be considered for higher efficiency and precision. In all tests, the parameters were chosen to ensure a comprehensive evaluation of the two algorithms. We randomly generated different fixed degree pole-zero configurations, where both the zeros and the inverse poles lie inside the unit circle (however we must note that this property is not required for the system's zeros). To ensure the significance of the result and to account for randomness, for each test scenario we run 15 tests, and then averaged the results.

3.1. Error bound on the Malmquist–Takenaka coefficients

We sampled the transfer function uniformly on the complex unit circle i.e. on the set

$$\mathbb{T}_F = \{z \in \mathbb{T} \,|\, z^F = 1\} = \{e^{2\pi i \frac{f}{F}} \,|\, f = 0, \dots F-1\}, \tag{3.1}$$

where $F > 0$ is the number of sample points, which was chosen to be $F = 1024$ in the presented tests. For the discretization of a function $R \in H^2(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$, the notation $R_F \in \mathbb{C}^F$, $R_F(f) = R(e^{2\pi i \frac{f}{F}})$ is used, and the discretization of the scalar product is defined as

$$\langle R_F, Q_F \rangle = \frac{1}{F} \sum_{f=0}^{F-1} R_F(f) \cdot \overline{Q_F(f)} \qquad (R, Q \in H^2(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})). \tag{3.2}$$

Using (3.2) we define the approximations for the true MT-coefficients of a rational function R as

$$c_{n,F} = \langle R_F, \Phi^{\mathbf{a}}_{n,F} \rangle \sim \langle R, \Phi^{\mathbf{a}}_n \rangle_{H^2} = c_n. \tag{3.3}$$

For a fixed MT system and unknown R a bound for the error can be given as follows. As for any twice continuously differentiable function, it holds for R that (as in [6]):

$$\begin{aligned} |c_{n,F} - c_n| &\leq \frac{1}{4F}\, \omega\left(\frac{d}{dt}\left(R(e^{it})\overline{\Phi^{\mathbf{a}}_n(e^{it})}\right); \frac{1}{2F}\right) \leq \\ &\leq \frac{1}{8F^2} \left\| \frac{d^2}{dt^2}\left(R(e^{it})\overline{\Phi^{\mathbf{a}}_n(e^{it})}\right)\right\|_\infty, \end{aligned} \tag{3.4}$$

where $\omega(\phi; \delta) = \sup\{|\phi(x) - \phi(y)| : |x - y| \leq \delta\}$ is the continuity modulus of a function $\phi : \mathbb{R} \to \mathbb{C}$ for $\delta > 0$. It is also possible to give further estimation of

the right hand side, because

$$\left\| \frac{d^2}{dt^2} \left(R(e^{it}) \overline{\Phi_n^{\mathbf{a}}(e^{it})} \right) \right\|_\infty \leq \|R''\|_\infty \|\Phi_n^{\mathbf{a}}\|_\infty + 2\|R'\|_\infty \|(\Phi_n^{\mathbf{a}})'\|_\infty + \|R\|_\infty \|(\Phi_n^{\mathbf{a}})''\|_\infty . \tag{3.5}$$

Note that the function $R\overline{\Phi_n^{\mathbf{a}}}$ is not complex differentiable, and because of this we need to use the submultiplicative property of the ∞-norm in order to be able to apply (3.9) later on.

Because of the linear dependence of the coefficients on R and the subadditivity of ω and the ∞-norm we can suppose that $R = r_\alpha$ is an elementary rational function with the inverse pole $\alpha \in \mathbb{D}$. It can be easily checked, that the m-th derivative of r_α is

$$r_\alpha^{(m)}(z) = \frac{m!\,\overline{\alpha}^m}{(1-\overline{\alpha}z)^{m+1}}, \tag{3.6}$$

which takes its maximum modulus on $\mathbb{T}$ at $z = \alpha/\overline{\alpha}$, thus

$$\|r_\alpha'\|_\infty = \frac{m!|\alpha|^m}{(1-|\alpha|)^{m+1}}. \tag{3.7}$$

Using (3.7) for (3.5) the following estimation can be made for the error of the coefficients of the discretized system:

$$8F^2|c_{n,F} - c_n| \leq \frac{\|(\Phi_n^{\mathbf{a}})''\|_\infty}{1-|\alpha|} + \frac{2|\alpha|\,\|(\Phi_n^{\mathbf{a}})'\|_\infty}{(1-|\alpha|)^2} + \frac{2|\alpha|^2\,\|\Phi_n^{\mathbf{a}}\|_\infty}{(1-|\alpha|)^3}. \tag{3.8}$$

This means that for a fixed system $\Phi_n^{\mathbf{a}}$ the error bound on the coefficients grows in $O(d^{-3})$, where d is the euclidean distance of α from $\mathbb{T}$.

However (3.8) holds for any function system in $H^2(\mathbb{D})$, one can utilize the fact that in these scenarios $\Phi_n^{\mathbf{a}}$ is a Laguerre system or more generally a periodic MT system, and thus we can describe the bound of the coefficient error as a function of n and the inverse poles of the MT system. For estimating the ∞-norm of the derivative of the MT functions on the complex unit circle $\mathbb{T}$, we can use the following Bernstein-type inequality [7]:

$$|Q'(z)| \leq |B'(z)| \cdot \|Q\|_\infty \qquad (z \in \mathbb{T}), \tag{3.9}$$

where Q is an arbitrary rational function and B is the Blaschke-product defined by the inverse poles of Q (counted with multiplicity).

First let's examine (3.8) for the Laguerre system, that is $\Phi_n^{\mathbf{a}} = L_n^{\mathbf{a}}$ for a parameter $a \in \mathbb{D}$. For this one has to calculate the derivative of the n-th power of a Blaschke function:

$$(B_a^n)'(z) = n(1-|a|^2) r_a^2(z) B_a^{n-1}(z). \tag{3.10}$$

Similarly to (3.7) the maximum modulus on $\mathbb{T}$ is taken at $z = a/\overline{a}$, thus

$$\|(B_a^n)'\|_\infty = \|n(1-|a|^2)r_a^2(z)\|_\infty = n\frac{1+|a|}{1-|a|}. \tag{3.11}$$

For a Laguerre function the ∞-norm can be given easily:

$$\|L_n^a\|_\infty = \|\sqrt{1-|a|^2}\, r_a B_a^n\|_\infty = \|\sqrt{1-|a|^2}\, r_a\|_\infty = \left(\frac{1+|a|}{1-|a|}\right)^{\frac{1}{2}} \tag{3.12}$$

Since L_n^a has only one inverse pole in a with multiplicity $n+1$, applying (3.9) with (3.11) and (3.12) results in

$$\|(L_n^a)'\|_\infty \le \|(B_a^{n+1})'\|_\infty \|L_n^a\|_\infty = (n+1)\left(\frac{1+|a|}{1-|a|}\right)^{\frac{3}{2}}. \tag{3.13}$$

The derivative $(L_n^a)'$ has again one inverse pole in a with multiplicity $n+2$. Applying (3.9) now with (3.13) we arrive to

$$\|(L_n^a)''\|_\infty \le \|(B_a^{n+2})'\|_\infty \|(L_n^a)'\|_\infty \le (n+2)(n+1)\left(\frac{1+|a|}{1-|a|}\right)^{\frac{5}{2}}. \tag{3.14}$$

Substituting into (3.8) yields

$$\begin{aligned} |c_{n,F} - c_n| \le \frac{1}{8F^2}\Bigg(&\frac{1}{1-|\alpha|}\left(\frac{1+|a|}{1-|a|}\right)^{\frac{1}{2}} + \\ &+ \frac{2(n+1)|\alpha|}{(1-|\alpha|)^2}\left(\frac{1+|a|}{1-|a|}\right)^{\frac{3}{2}} + \\ &+ \frac{2(n+2)(n+1)|\alpha|^2}{(1-|\alpha|)^3}\left(\frac{1+|a|}{1-|a|}\right)^{\frac{5}{2}}\Bigg). \end{aligned} \tag{3.15}$$

The case of the periodic MT system is similar to the previous one. Let $a_0, \dots a_{p-1} \in \mathbb{D}$ be the defining inverse poles for the periodic MT system $\Phi_n^{\mathbf{a}}$. Then first the derivative of the finite Blaschke product with these poles and with the corresponding multiplicities $m_0, m_1, \dots m_{p-1} \ge 0$ (where $n = m_0 + +m_1 + \dots m_{p-1}$) can be calculated as follows

$$\begin{aligned} B'_{\mathbf{a}|n}(z) &= \sum_{k=0}^{p-1} d_k \frac{1-|a_k|^2}{(1-\overline{a_k}z)^2} B_{a_k}^{m_k-1}(z) \prod_{j=0, j\neq k}^{p-1} B_{a_j}^{m_j}(z) = \\ &= B_{\mathbf{a}|n}(z) \sum_{k=0}^{p-1} m_k \frac{(1-|a_k|^2)\overline{z}}{(1-\overline{a_k}z)(\overline{z}-\overline{a_k})}. \end{aligned} \tag{3.16}$$

This means that $B'_{\mathbf{a}|n}$ has the same inverse poles a_k as $B_{\mathbf{a}|n}$, each with multiplicity m_k+1 (if $m_k \geq 1$) and that

$$|B'_{\mathbf{a}|n}(z)| = \sum_{k=0}^{p-1} m_k \frac{1-|a_k|^2}{|z-a_k|^2} \tag{3.17}$$

for any $z \in \mathbb{T}$. Although there is no general formula for finding the maximum of (3.17), it can be estimated with the dominant inverse pole a_d (i.e. $|a_d| \geq |a_k|$):

$$\|B'_{\mathbf{a}|n}\|_\infty \leq n \frac{1+|a_d|}{1-|a_d|}. \tag{3.18}$$

Similarly to (3.12), the norm of $\Phi_n^{\mathbf{a}}$ can be explicitly given:

$$\|\Phi_n^{\mathbf{a}}\|_\infty = \left(\frac{1+|a_n|}{1-|a_n|}\right)^{\frac{1}{2}} \leq \left(\frac{1+|a_d|}{1-|a_d|}\right)^{\frac{1}{2}}. \tag{3.19}$$

For the derivatives, the Bernstein inequality gives the following

$$\|(\Phi_n^{\mathbf{a}})'\|_\infty \leq \|B'_{\mathbf{a}|n+1}\|_\infty \|\Phi_n^{\mathbf{a}}\|_\infty \leq (n+1)\left(\frac{1+|a_d|}{1-|a_d|}\right)^{\frac{3}{2}}, \tag{3.20}$$

and

$$\|(\Phi_n^{\mathbf{a}})''\|_\infty \leq \|B'_{\mathbf{a}+\mathbf{1}|n+p+1}\|_\infty \|\Phi_n^{\mathbf{a}}\|_\infty \leq (n+p+1)(n+1)\left(\frac{1+|a_d|}{1-|a_d|}\right)^{\frac{5}{2}}, \tag{3.21}$$

where $B_{\mathbf{a}+\mathbf{1}|n+p+1}$ is the Blaschke product in which every inverse pole a_k has multiplicity m_k+1. Substituting into (3.8) we get the error bound for the periodic Malmquist–Takenaka system

$$\begin{aligned} |c_{n,F}-c_n| \leq \frac{1}{8F^2}\Bigg(&\frac{1}{1-|\alpha|}\left(\frac{1+|a_d|}{1-|a_d|}\right)^{\frac{1}{2}} + \\ &+ \frac{2(n+1)|\alpha|}{(1-|\alpha|)^2}\left(\frac{1+|a_d|}{1-|a_d|}\right)^{\frac{3}{2}} + \\ &+ \frac{2(n+p+1)(n+1)|\alpha|^2}{(1-|\alpha|)^3}\left(\frac{1+|a_d|}{1-|a_d|}\right)^{\frac{5}{2}}\Bigg). \end{aligned} \tag{3.22}$$

It's clear that both (3.15) and (3.22) are very similar, and in fact give the same asymptotic bound for the coefficients. The immediate consequence of these inequalities is that if one wants to increase the number of considered coefficients while maintaining the same error bound, then the number of sampled frequencies must also increase with the same ratio. It must be also mentioned that while uniform sampling is the most common in practice, nonetheless it should be noted that there are better non-uniform alternatives such as presented in [13], which preserves the orthogonality of the original system.

3.2. Comparison of the Iterative and Generalized Schi-So method

As we have seen before the two algorithms have different ways for iteratively finding new poles of the transfer function. The main difference is, that the Iterative Schi-So method modifies the transfer function in each iteration, while the Generalized Schi-So method doesn't. This means that in the former method an error in a pole approximation can propagate to the estimation of the later poles, because in this case the reduced transfer function (see (2.29)) will have different poles then the original. In contrast, the Generalized Schi-So method is more robust to errors, since if the error is small enough, then it does not effect the $B_{\mathbf{a}|p}$-dominance properties of the following poles, thus the limit of the quotients (2.33) will converge to the same poles as it would without the error.

A practical problem with which one has to be careful is the calculation and usage of the Malmquist–Takenaka coefficients. As it has been mentioned before with the uniform discretization the discretized MT system does not inherit the orthonormality property. With large number of sample points it does not cause any issues for lower order coefficients, however for large n, the numerically calculated coefficients c_n will differ heavily from the analytic case. An other issue with the MT coefficients if that they decay rapidly for a rational function. To see this let's suppose that R is an elementary rational function with the inverse pole $\alpha \in \mathbb{D}$ and $\Phi^{\mathbf{a}}_{m+1}$ is the p-periodic Malmquist–Takenaka system with parameters $\mathbf{a} = (a_0, \dots a_{p-1}, \dots)$. Then we can estimate the decay of the MT series expansion:

$$|c_n| = |\langle R, \Phi^{\mathbf{a}}_{m+lp}\rangle| = |\overline{\Phi^{\mathbf{a}}_{m+lp}(\alpha)}| = |\Phi^{\mathbf{a}}_{m+1}(\alpha)| \cdot |B_{\mathbf{a}|p}(\alpha)|^l = O(\beta^{\frac{l}{p}}), \tag{3.23}$$

where $n = m + lp$ and $\beta = |B_{\mathbf{a}|p}(\alpha)| < 1$. This means that c_n approach 0 exponentially fast (note that this estimation can be easily extended to arbitrary rational functions with inverse poles α_l, and in that case $\beta = \max_l |B_{\mathbf{a}|p}(\alpha_l)|$). From a numerical point of view this rate of convergence can cause errors when calculating the quotients. If we denote the absolute error bound of c_n with Δ_{c_n}, then by a well known formula in numerical analysis we can calculate an absolute error bound Δ_{q_n} for the quotients $q_n = c_{n+1}/c_n$:

$$\Delta_{q_n} = \frac{|c_n|\Delta_{c_{n+1}} + |c_{n+1}|\Delta_{c_n}}{|c_n|^2} = O((\Delta_{c_{n+1}} + \Delta_{c_n})\beta^{-\frac{l}{p}}). \tag{3.24}$$

As a consequence, even if one assumes that $O(\Delta_{c_n}) = O(1)$, the absolute error can grow exponentially. By Theorems 1 and 2 these issues are present in both algorithms. For choosing optimal n we would need to minimize the sum of the numerical error (3.24) and the convergence error (given in Theorem 2.2 and 2.3), however both of these depend on the poles of R, which are unknown. Solving this issue can be done heuristically by using the same principle as in

the choice of $a \in \mathbb{D}$ for the Schi-So method (see [4] and the end of section 2.2 and the function (2.25)). With predefined parameters $W > 0$ (window length) and $n_{\max}$ we can choose n_{opt} as:

$$n_{\text{opt}} = \text{argmin}_{0 \le n \le n_{\max}} \max_{0 \le i,j < W} |q_{n+i} - q_{n+j}|. \tag{3.25}$$

Here the use of the bound of indices $n_{\max}$ is important to alleviate the problem caused by the recently discussed numerical error. We suggest that its value should be determined by using a coefficient percentile function $\mathcal{P}_\mu$ (where $0 < < \mu \ll 1$ is the error percentile), defined with the equation

$$n_{\max} = \mathcal{P}_\mu(R) = \min \left\{ n \;\middle|\; \text{s.t.} \left(\sum_{k=0}^{n} |c_k|^2 \right)^{\frac{1}{2}} \ge (1-\mu)\|R\|_{H^2} \right\}. \tag{3.26}$$

From Parseval's identity, we know that $\|(c_n)_{n\in\mathbb{N}}\|_{\ell^2} = \|R\|_{H^2}$, thus with this choice of $n_{\max}$ we can dynamically guarantee that we include the most important coefficients with error smaller than μ, meanwhile excluding the error in higher order coefficients caused by the loss of orthogonality. In the tests presented in Table 1 we chose the value $\eta = 10^{-14}$.

As previously mentioned in this paper, both of the algorithms examined here are non-parametric in the sense that it doesn't require a priori knowledge about the number of poles as opposed to other popular methods. However an appropriate exit condition must be specified for both methods, on which the result can highly depend. In the tests presented in the Table 1, we used the PRD condition presented in [4], i.e.:

$$\text{PRD}(\mathbf{p}) = \frac{\|R - S_m^{\mathbf{p}} R\|_{H^2}}{\|R\|_{H^2}} < \varepsilon, \tag{3.27}$$

where $\mathbf{p}$ are the already found poles, $S_m^{\mathbf{p}} R$ is the p-th order partial sum of of the MT expansion of R and $\varepsilon > 0$ is a hyper-parameter (in these tests we chose $\varepsilon = 0.01$). We must emphasise that the error in the number of estimated poles can depend on ε. If it is chosen to be too big, we may miss some poles, while if it is too small, then an excessive number of poles might be identified.

For numerical evaluation and comparison we used the following measures:

- **Dist. (Euc.)**: The averaged euclidean distance of the real and estimated poles.
- **Dist. (ρ)**: The averaged pseudo-hyperbolic distance of the real and estimated inverse poles (see equation (2.5)).
- **$\mathbf{H^2}$-PRD** : The percent root mean squared difference of the approximation.
- **Est. $|\mathbf{P}|$**: The estimated number of poles.

(P, Z)	Method	Dist. (Euc.)	Dist. (ρ)	H^2-PRD	Est. |P|
(2,1)	Iter.	$3.9355 \cdot 10^{-13}$	$5.3717 \cdot 10^{-13}$	$5.7664 \cdot 10^{-9}$	2.0000
	Gen.	$5.2208 \cdot 10^{-9}$	$5.8517 \cdot 10^{-9}$	$1.0629 \cdot 10^{-8}$	2.0000
(3,1)	Iter.	0.0025	0.0030	$1.8735 \cdot 10^{-4}$	3.1333
	Gen.	0.0080	0.0085	0.0025	3.0000
(3,2)	Iter.	0.0037	0.0073	$7.2123 \cdot 10^{-4}$	3.3333
	Gen.	0.0146	0.0153	0.0029	3.5333
(4,2)	Iter.	0.0161	0.0165	0.0024	3.9333
	Gen.	0.0278	0.0297	0.0047	4.2666
(4,3)	Iter.	0.0035	0.0037	0.0028	4.3333
	Gen.	0.0169	0.0177	0.0043	4.7333
(5,4)	Iter.	0.0169	0.0189	0.0034	5.3333
	Gen.	0.0452	0.0530	0.0106	6.1333
(8,7)	Iter.	0.0825	0.0881	0.0092	9.8667
	Gen.	0.1222	0.1385	0.0547	9.7333
(10,9)	Iter.	0.0940	0.1003	0.0131	13.2000
	Gen.	0.1092	0.1216	0.0810	11.6666
(15,14)	Iter.	0.2078	0.2246	0.0426	16.3333
	Gen.	0.1525	0.1848	0.0810	15.6000

Table 1. Comparison of the Iterative and Generalized Schi-So algorithms for different pole-zero configurations.

As can be seen in Table 1, when the values of the transfer function were artificially generated and contained no added noise, the Iterative Schi-So method performs better in most cases. This can boil down to multiple reasons. One is that for the calculation of an inverse pole from the approximated limit of the quotient sequence by the formula (2.34) is more complicated than in the Iterative Schi-So case and thus it introduces more numerical errors. For the interested reader an error bound for (2.34) was given in [4]. An other, probably even more relevant reason is that contrary to the Laguerre system, with the periodic Malmquist–Takenaka system we only use every p-th element of the quotient sequence, and p grows in every iteration. This means that only part of the information contained in the coefficient are utilized, even though it doesn't matter from where we start the sub-sequence ν_k (see Theorem 2.3). However as expected for higher number of poles the Generalized Schi-So algorithm gains advantage. This can be attributed to the property that the previous errors in the estimation of poles are not necessarily effect the estimation of the poles found later.

In the future the performance of the two algorithms will be compared in the presence of noise. However with the assumption that the noise on the sampled transfer function is normally distributed and independent then the resulting

noise on the coefficients will be also normally distributed, and thus the expectation and variance of the elements of the quotient sequence will not be finite. Because of this other techniques have to be worked out to approximate the limit of this sequence, that are more robust to noise and other perturbations.

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