Annales Univ. Sci. Budapest., Sect. Comp. 56 (2024) 87–97

CANONICAL EXPANSION OF INTEGERS FOR FAMILIES OF ROOFLINE POLYNOMIALS

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Communicated by Péter Burcsi
(Received January 9, 2024; accepted April 30, 2024)

Abstract. We investigate the existence and length of canonical expansion of integers in polynomial bases where the bases are taken from some infinite families of polynomials which we call roofline polynomials. A roofline polynomial's coefficients form a sequence that is first increasing, then constant, then decreasing, with an additional symmetry condition. We prove necessary conditions for the CNS property, and in special cases we also prove that the condition is sufficient. We provide results on the length of the expansion of -1 as a function of the degree of the base. We also formulate some open problems about the canonical number system property of these families.

1. Introduction

The representation of positive integers in base B (e.g. decimal or binary) has many generalizations. The field of study that is devoted – among many other things – to these generalizations is usually referred to as numeration theory. Although it is originally a branch of elementary number theory, there is considerable interest in numeration by researchers of several fields. This is propelled by the interconnection between numeration systems and many other areas including dynamical systems, chaos and fractal geometry, combinatorics on words, automata theory or cryptography, see e.g. [3, 4, 17, 14, 15, 9].

Key words and phrases: Generalized number system, expansive polynomials.
2010 Mathematics Subject Classification: 11C08.
The Project is supported by the grant EFOP-3.6.3-VEKOP-16-2017-00001: Talent Management in Autonomous Vehicle Control Technologies. The Project is supported by the Hungarian Government and co-financed by the European Social Fund.

The present paper considers an important special case of generalized numeration systems: canonical number system (CNS) polynomials. Many algorithmic questions may be raised about the representation of polynomials in polynomial bases. A fundamental question is to decide on input polynomial P, whether all polynomials have a canonical representation in base P. Although this question can be investigated algorithmically (see e.g. [2, 6, 16, 12, 7, 8]), for many polynomials with larger degree, the algorithms run for a prohibitively long time. This occurs for "worst-case" polynomials already with degree 12 to 16. The paper considers a concrete family of polynomials for which the CNS decision algorithms fail (due to time or memory limitations). We call this the family of "roofline polynomials" because of the shape of the plot of their coefficients.

We investigate for which roofline polynomials has CNS property which implicates the existence of the expansion of -1 in base P. For those polynomials that have such a representation, we give results on the length of the representation.

The paper is built up as follows: in Section 2, we give definitions and prior results on CNS polynomials. In Section 3, we define roofline polynomials. In Section 4, we give a necessary condition for the CNS property in a special subfamily of roofline polynomials. In Section 5, we give necessary conditions in the general case. In Sections 6 and 7 we give experimental and theoretical results on the *length* of the expansion of -1. In Section 8 we summarize the results and formulate further research questions. Our main contributions in Sections 4 to 7 are that we give new necessary conditions on the representability of -1 and present results on the length of the representation and its dependence on the degree. As a tool, we introduce the "lifted" representation of P, a shift radix system which simplifies the analysis of orbits.

2. Basic concepts

Let $P = \sum_{i=0}^{n} p_i X^i \in \mathbb{Z}[X]$ be a monic integer polynomial with positive degree and $|P_0| \geq 2$. Let $D = \{0, 1, \ldots, |P_0| - 1\}$ be the *set of digits*, and let $D[X]$ be the set of polynomials with coefficients in D. We say that a polynomial $A \in \mathbb{Z}[X]$ is canonically represented in base P by $B \in D[X]$ if $A \equiv B (\bmod\ P)$. If every polynomial has a canonical representative, then P is called a canonical number system polynomial (CNS polynomial).

Example. Let $P = X - 10$, $A = 2020$, $B = 2X^3 + 2X$, $D = \{0, 1, \ldots, 9\}$. Then B is the canonical representative of A in base P.

Example. Let $P = X^2 + 4X + 5$, $A = 5$. We have $D = \{0, 1, 2, 3, 4\}$. Then $B = X^3 + 3X^2 + X$ is the canonical representative of A in base P. Note that if we set $\alpha = -2 + i$ (one of the roots of P), substituting α into B gives 5 (which is basically A). As proved in [10], P is a CNS polynomial, and it

follows that all Gaussian integers can be written in the form $\sum_{k=0}^{\ell} d_k \alpha^k$ with $d_k \in \{0, 1, 2, 3, 4\}$, $d_\ell \neq 0$.

A possible way to determine the canonical representative of a polynomial A is to reformulate the problem for matrix based numeration systems or shift radix systems as follows. Each residue class modulo P has a unique representative of degree at most $n-1$, thus we can identify the residue classes with elements of $\mathbb{Z}^n$. The operation $A \mapsto X \cdot A$ taken modulo P is a linear map on this lattice. For each polynomial $A = a_0 + a_1 x + \cdots a_{m-1} x^{m-1}$ there is a unique $d \in D$, for which there exists a C (also unique modulo P) such that $A \equiv C \cdot X + d \pmod{P}$. This d is the constant term of the canonical representation, if it exists. If we use an appropriate basis for the vectors (see e.g. [1, 3, 11], the mapping $A \mapsto C$ can be described as $\tau = \tau_r : \mathbb{Z}^n \to \mathbb{Z}^n$,

$$\mathbf{z} = (z_0, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_{n-1}, -\lfloor \mathbf{rz} \rfloor),$$

where $\mathbf{r} = (\frac{p_n}{p_0}, \frac{p_{n-1}}{p_0}, \ldots, \frac{p_1}{p_0})$ and $\mathbf{rz}$ denotes the scalar product. The CNS property of P is equivalent to the property that τ has a single periodic orbit 0, and that all orbits are eventually periodic.

It turns out that in the investigations below, it is sometimes convenient to consider a "lifted" representation of the dynamical system defined by P, by which we the shift radix system associated to $Q(X) = (X-1)P(X)$. Here, the original CNS property is equivalent to saying that each vector eventually maps to a vector of the form $(a, a, \ldots, a)$, i.e. all of whose coordinates are the same. Note that because of the negative constant term, the final coordinate in the lifted version has to be computed as $\lfloor -\mathbf{rz} \rfloor$). For any vector v in the lifted system (which represents a residue class mod Q), we have that

$$\tau_P(v \bmod P) = \tau_Q(v) \bmod P$$

The "fiber" of zero, i.e. $\{v \mid v \bmod P = 0\}$ consists of those vectors, all of whose coordinates are identical, because of the following consideration. First of all, the space of the dynamical system defined by P is essentially a projection of the "lifted" space from a given direction, and the elements representing 0 must be in this direction. This also shows if we have an element u in the "lifted" system such that $\tau_Q(u) = u$, then u must represent an element with 1 period, but if P has the CNS property, then the only 1-periodic element is 0. From the definition of map τ it is easy to see, that

$$\tau_Q(u) = u \Longleftrightarrow u = (a, a, \ldots, a) : a \in \mathbb{Z}.$$

So the direction of projection is $(1, 1, \ldots, 1)$ and the vectors $(a, a, \ldots, a) : a \in \mathbb{Z}$ have to be the representation of 0 in the "lifted" system. The results we present are about families of polynomials we call roofline polynomials. We

give necessary conditions for the existence of a canonical representative of -1 (thus also for the CNS property), show that in some cases, these are sufficient, and examine the length of the expansion, i.e. the degree of the canonical representative.

To conclude this section, we recall a theorem which can be used for the algorithmic decision of the CNS property.

Theorem 2.1. ([6]) *Suppose that $S \subseteq \mathbb{Z}^n$ has the following properties:*

1. $(1, 0, 0 \ldots, 0) \in S$,
2. $-S \subseteq S$,
3. $\tau_p(S) \subseteq S$,
4. *for all $s \in S$ there exist a positive integer k such that $\tau_p^k(s) = 0$.*

Then $p(x)$ has the CNS property and the S is called the set of witnesses of p.

In [8], an exhaustive search for all CNS polynomials up to degree 16 was performed. The authors computed the set of witnesses for each CNS polynomial. When the degree of the polynomials is even, they found that the largest witness sets belong to polynomials of a certain monic family: the constant term is 2, the leading coefficient is 1, the second highest coefficient is 2, and all other coefficients are 3: i.e. the coefficient sequence is $2, 3, 3, \ldots, 3, 2, 1$. These polynomials form the hardest cases for the CNS decision algorithms, which is a major motivation for us to consider these polynomials specifically. In the following section we define families of polynomials, of which the aforementioned ones are a special case. By extensive experimentation, we found that most (although not all) polynomials for which -1 has a representation, have the CNS property. This observation motivates the investigation of the representability of -1.

3. Roofline polynomials

To decide the CNS property the orbit of each element of set of witnesses can be computed, but this computation can be hard, if the size of the set of witnesses is too large. For example let consider the following polynomials and the size of the corresponding sets of witnesses.

This example indicates that the decision of CNS property may become expensive as the degree increases. This motivates the investigation of a special family of polynomials which we call roofline polynomials. The members of this family can be described by three parameters n, M and k; n denotes the degree of the polynomial, M is the maximal coefficient and k is a positive integer dividing $M - 2$.

polynomial	size of set of witnesses
$x^4+2x^3+3x^2+3x+2$	61
$x^6+2x^5+3x^4+3x^3+3x^2+3x+2$	2935
$x^8+2x^7+3x^6+3x^5+3x^4+3x^3+3x^2+3x+2$	241719

Definition 3.1. Let n, M, k be positive integers with $k \mid (M-2)$ and $n \geq 2\cdot(M-2)/k+1$, We define the roofline polynomial $f(n,M,k)$ as follows:

$$f(n,M,k) = 2+(2+k)x+(2+2k)x^2+\cdots+(M-k)x^{\frac{M-2}{k}-1}+Mx^{\frac{M-2}{k}}+$$
$$+Mx^{\frac{M-2}{k}+1}+\cdots+Mx^{n-\frac{M-2}{k}-1}+(M-1)x^{n-\frac{M-2}{k}}+(M-1-k)x^{n-\frac{M-2}{k}+1}+$$
$$+(M-1-2k)x^{n-\frac{M-2}{k}+2}+\cdots+(k+1)x^{n-1}+x^n,$$

or, by giving the sequence of its coefficients,

$$(2,2+k,2+2k,\ldots,M-k,M,M,\ldots,M,M-1,M-1-k,\ldots,1+k,1)$$

Note that the constant term of these polynomials is 2, then the coefficients increase by k to M, then remain constant, then decrease once by 1 and then decrease by k. The leading coefficient is 1, so roofline polynomials are monic by definition. As an example, the coefficient sequence of $f(12,14,3)$ is equal to

$$(2,5,8,11,14,14,14,14,13,10,7,4,1) \in \mathbb{Z}^{13}.$$

4. Necessary conditions for representation, $M=3$

We found that there is a simple divisibility criterion on n that governs the existence of a representative of periodic elements for $f(n,3,1)$. Specifically, if the degree is odd, then there exists a non-trivial periodic element. We formulate the latter observation in the following theorem and return to the even degree case later in Section 7.

Theorem 4.1. *Consider the roofline polynomial $f(n,3,1)$. If n is an odd integer then f can not be a CNS polynomial.*

Proof.

$$f(n,3,1) = (2,3,3,\ldots,3,2,1) \in \mathbb{Z}^{n+1}.$$

We explicitly construct a period of length 2 in the lifted representation of $f(n,3,1)$. The lifted representation of $f(n,3,1)$ defined by p as follows:

$$p(x) = (x-1)(2+3x+3x^2+\cdots+3x^{n-2}+2x^{n-1}+x^n) =$$
$$= -2+(2-3)x+(3-3)x^2+\cdots+(3-3)x^{n-2}+(3-2)x^{n-1}+$$
$$+(2-1)x^n+x^{n+1} = -2-x+x^{n-1}+x^n+x^{n+1},$$

and $p = (-2, -1, 0, 0, \ldots, 0, 1, 1, 1) \in \mathbb{Z}^{n+2}$ is the vector of coefficients of $p(x)$. Let $\mathbf{u} = (0, -1, 0, \ldots, -1, 0, -1) \in \mathbb{Z}^{n+2}$. Note that $n+1$ is even, so the final coordinate is -1. To apply the map τ_p we need the scalar product of $\mathbf{u}$ and $\mathbf{r} = \left(\frac{p_{n+1}}{p_0}, \frac{p_n}{p_0}, \ldots, \frac{p_1}{p_0}\right) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, \cdots, 0, 0, \frac{1}{2}\right)$, which is

$$\mathbf{ru} = \frac{1}{2} - \frac{1}{2} = 0,$$

and therefore,

$$\mathbf{v} = \tau_p(\mathbf{u}) = (-1, 0, -1, \ldots, 0, -1, 0, -\lfloor \mathbf{ru} \rfloor) = (-1, 0, -1, \ldots, -1, 0, -1, 0).$$

We can apply the map τ_p on $\mathbf{v}$ as before.

$$\mathbf{rv} = \frac{1}{2} + \frac{1}{2} = 1,$$

and

$$\tau_p(\mathbf{v}) = (0, -1, 0, \ldots, 0, -1, 0, -\lfloor \mathbf{rv} \rfloor) = (0, -1, 0, \ldots, 0, -1, 0, -1) = \mathbf{u}.$$

We get that $\tau_p^2(\mathbf{u}) = \mathbf{u}$, so $\mathbf{u}$ is a non-trivial periodic element and therefore the CNS property does not hold. □

5. Necessary conditions for representation, general case

We give a necessary condition for the CNS property of roofline polynomials in general.

Theorem 5.1. *If the $f(n, M, k)$ is a roofline polynomial and* $\gcd(n+1, \frac{M-2}{k} + +1) = d > 1$, *then f can not be a CNS polynomial.*

Proof. We construct a periodic element with period length d. Recall that

$$f(n, M, k) = (2, 2+k, \ldots, M-k, M, \ldots, M, M-1, M-1-k, \ldots, 1) \in \mathbb{Z}^{n+1}.$$

Similarly to the proof of Theorem 4.1, we consider the lifted representation of $f(n, M, k)$. Let $p = (x-1)f(n, M, k)$. A simple calculation shows that the coefficient sequence of p is

$$p = (-2, -k, -k, \ldots, -k, 0, 0, \ldots, 0, 1, k, k, \ldots, k, 1) \in \mathbb{Z}^{n+2}.$$

Here, the number of consecutive $-k$ (and k) values is $(M-2)/k$. Let v_i denote the following vector:

$$\mathbf{v_i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,n+1}), \text{ where } v_{i,j} = \begin{cases} -1 & \text{if } i+j-1 \text{ is a multiple of } d \\ 0 & \text{otherwise} \end{cases}$$

To apply the map τ_p we need the vector $\mathbf{r}$, which is

$$\mathbf{r} = \left(-\frac{1}{2}, -\frac{k}{2} - \frac{k}{2}, \ldots, -\frac{k}{2}, -\frac{1}{2}, 0, 0, \ldots, 0, \frac{k}{2}, \frac{k}{2} \ldots, , \frac{k}{2}\right).$$

To prove the theorem we will show that v_1 is a non-trivial periodic element with period d and that the sequence v_i is its orbit. First we compute $\tau_p(\mathbf{v_i})$, where $i \leq d-1$. The scalar product of $\mathbf{r}$ and $\mathbf{v_i}$:

$$\mathbf{rv_i} = \frac{\frac{M-2}{k}+1}{d} \cdot \left(-\frac{k}{2}\right) + \frac{\frac{M-2}{k}+1}{d} \cdot \frac{k}{2} = 0,$$

and therefore

$$\tau_p(\mathbf{v_i}) = (v_{i,2}, v_{i,3}, \ldots, v_{i,n+1}, -\lfloor \mathbf{rv_i} \rfloor) = (v_{i,2}, v_{i,3}, \ldots, v_{i,n+1}, 0) =$$

$$= (v_{i+1,1}, v_{i+1,2}, \ldots, v_{i+1,n}, v_{i+1,n+1}) = \mathbf{v_{i+1}}$$

since $v_{1,n+1} = -1$ and $v_{i,n+1} = 0$ if $1 < i < d+1$. Next, we show that $\tau_p(\mathbf{v_d}) = \mathbf{v_1}$.

$$\mathbf{rv_d} = \frac{1}{2} + \left(\frac{\frac{M-2}{k}+1}{d} - 1\right)\left(-\frac{k}{2}\right) + \frac{1}{2} + \left(\frac{\frac{M-2}{k}+1}{d} - 1\right)\frac{k}{2} = 1$$

and

$$\tau_p(\mathbf{v_d}) = (v_{d,2}, v_{d,3}, \ldots, v_{d,n+1}, -\lfloor \mathbf{rv_d} \rfloor) = (v_{d,2}, v_{d,3}, \ldots, v_{d,n+1}, -1) =$$

$$= (v_{d+1,1}, v_{d+1,2}, \ldots, v_{d+1,n}, v_{d+1,n+1}) = (v_{1,1}, v_{1,2}, \ldots, v_{1,n}, v_{1,n+1}) = \mathbf{v_1}.$$

Summarizing the calculations we get that $\tau_p^d(\mathbf{v_1}) = \mathbf{v_1}$, so the $\mathbf{v_1}$ is a non-trivial periodic element, and therefore the CNS property fails. ■

6. Experiments of the length of representation

Beyond the existence of the representation, the length of the representation is of further interest. In the following section we present theoretical results on the length of the expansion of -1. The results are based on observations that we acquired through extensive experimentation, summarized in this section. Our main concern is to examine how the length of the CNS-representation of polynomials with small height (i.e. maximum coefficient) varies as the degree of polynomials increases. The dependence of orbit lengths on the degree has relevance to the running time analysis of decision algorithms, too.

In the present section we show results on the length of representation of the polynomial -1 for the roofline family with $M = 3$. Denote by $\ell_P(A)$ the

degree of the CNS-representative of A in base P if it exists, and define it as $+\infty$ otherwise.

For roofline-polynomials with $M = 3$ in degree 6, 8, 10 and 12, we determined the maximal representation length among all A that have degree at most $\deg(P) - 1$, and coefficients in $\{-1, 0, 1\}$, i.e. the quantity

$$\ell_{\max}(P) = \max_{\substack{A \in \{-1,0,1\}[X] \\ \deg(A) \leq \deg(P)-1}} \ell_P(A) \quad .$$

For the roofline polynomials with degree 20, 30, 50, 80, 100, an exhaustive search for the maximum was beyond reach. Here, we estimated the maximum by sampling for a large number of independent choices of A. The following table summarizes our results

n	$\ell_P(-1)$	$\ell_{\max}(P)$	$\ell_{\max}(P)/n^2$
$n = 6$	33	63	1.75
$n = 8$	59	150	2.34
$n = 10$	93	242	2.42
$n = 12$	135	336	2.33
$n = 20$	383	730	1.83
$n = 30$	873	1670	1.86
$n = 50$	2453	4280	1.71

Table 1. This table gives the expansion length for -1 and the maximal expansion length for polynomials with coefficients in $\{-1, 0, 1\}$ for base polynomials in the roofline family $f(n, 3, 1)$. For $n > 12$ the maximum is not computed exhaustively, the table only gives a lower bound which is based on an extensive set of randomized inputs.

Based on the experiments, it is plausible that both $\ell_{f(n,3,1)}(-1)$ and $\ell_{\max}(f(n, 3, 1))$ grow quadratically with n. We prove this statement for $\ell_{f(n,3,1)}(-1)$ in the following section.

7. Results on the length of expansion

We note that priorly [5] considered expansion of small integers in various bases, but our results on roofline polynomials are not covered there. We consider roofline polynomials with parameters (n, M, k) where $k = 1$. The following theorem states that for a special case, the condition of Theorem 4.1 is sufficient for the expansion to exist, and we can determine the expansion length which is quadratic in the degree.

Theorem 7.1. *Let n be an even number. Then -1 has a canonical representation in base $f(n,3,1)$. The degree of the representative is n^2-n+2.*

Proof. We explicitly give the canonical representative of -1. Let

$$\begin{aligned} B &= (1+X)\cdot(1+X^{n^2/2-n/2+1})\cdot\sum_{i=0}^{n/2-2}X^{i(n+1)}+\\ &\quad +(1+X^{n^2/2-n/2+2})\cdot X^{n-1}\cdot\sum_{i=0}^{n/2-2}X^{i(n-1)}+\\ &\quad +(1+X+X^2)\cdot X^{n^2-n}+X^{n^2/2-n/2-1} \end{aligned}$$

We have to show that $B+1$ is a multiple of $f(n,3,1)$. This can be verified by hand (with a lengthy but straightforward computation) or by a computer algebra system. We note that computer algebra systems do not usually directly support polynomials with symbolic expressions in the exponents. We found that two small technical modifications to the statement makes it computable by computer algebra systems. First, it is more convenient to prove that $(X-1)\cdot(B+1)$ is a multiple of $(X-1)\cdot f(n,3,1)$. Second, if we replace occurrences of $X^{n/2}$ with Y by introducing a new indeterminate, we have a 2-variable polynomial identity *without* symbolic exponents. In this modified form, SageMath and Maple were both able to verify the identity. ■

Remark. The representative in the above proof was obtained by a careful examination of the series of representatives for small n and guessing the right general formula. This makes the proof difficult to generalize. We therefore outline another proof idea which is based on the lifted shift radix system used in Sections 4 and 5. In our ongoing research we intend to use this latter approach for proving more general theorems for roofline polynomials $f(n,M,k)$.

Sketch of alternative proof for Theorem 7.1. Consider the lifted base $p=(x-1)\cdot f(n,3,1)$. Define vectors s_j for $j=0,1,\ldots,n+1$ as follows: s_j contains exactly j coordinates that are 1 (the others being 0), and these 1s are at position $1,3,5,\ldots,2j-1$ (here positions larger than $n+1$ are interpreted modulo $n+1$). Using the formula for τ_p, one can verify that claim that the orbit of -1 consists only of vectors of the form s_i-s_j for some $i,j\in\{0,1,2,\ldots,n+1\}$. The computation is technical but does not involve guessing the right formula for the representation.

8. Summary

We presented some results about the existence and the length expansion for families of polynomials. We now formulate two open problems about roofline

polynomials. The first conjecture is backed by computational evidence for small values of n, and states that the existence of the expansion of -1 implies the CNS property. We only formulate it in a weak form for a special case, but already this challenge evaded our attempts of finding a proof. It is worth to remark that although these polynomials are almost as simple as the ones with monotone coefficient sequences discussed in [13], computational verification of the CNS property is very hard as the set of witnesses (see e.g. [1]) grows wildly with the degree ($n \leq 14$ is verified, larger values remain conjectural).

Conjecture 8.1 ($f(n,3,1)$ conjecture)**.** *Let $n \geq 4$ be an even number. The polynomial $f(n,3,1)$ i.e. $P(x) = 2 + 3x + 3x^2 + \cdots + 3x^{n-2} + 2x^{n-1} + x^n$ is a CNS polynomial.*

The intuitive meaning of our second conjecture is that there is a relatively simple law that describes how the length of expansion of -1 depends on the values of n, M, k.

Conjecture 8.2. *Let $L(n,M,k)$ be the length of the expansion of -1 for the roofline polynomial with parameters (n,M,k) if the expansion is finite and 0 otherwise (or if the parameters do not correspond to a valid polynomial). Construct the generating series $g(x,y,z) = \sum_{n=0}^{\infty}\sum_{M=0}^{\infty}\sum_{k=0}^{\infty} L(n,M,k)x^n y^M z^k$. Then $g(x,y,z)$ is a rational function in x, y and z.*

Computational evidence supports both conjectures. The second conjecture seems to hold when -1 is replaced by other integers. The dependence of the length of expansion from the starting point of the orbit is also interesting. Finally, we would like to extend the investigation to shift radix systems with non-integer coefficients. General results about expansion length could possibly lead to insight about the algorithmic hardness of the decision of the CNS property or related finiteness properties of shift radix systems.

This work is a detailed version of a MaCS 2020 presentation.

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