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A NEW PROOF OF A RESULT OF WIRSING FOR MULTIPLICATIVE FUNCTIONS

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In celebration of the eighty fifth birthday of Professors
Zoltán Daróczy and Imre Kátai

Abstract. In this paper we give a proof of Wirsing's result [11] for non-negative multiplicative functions.

1. Introduction

In this paper we give a new proof for the following famous result of E. Wirsing ([11], Satz 1.1).

Theorem. *Let g be a multiplicative function which assumes real nonnegative values only. Let*

$$\sum_{p\leq x}\frac{g(p)\log p}{p}\sim\alpha\log x,\quad x\to\infty, \tag{1.1}$$

hold with a constant $\alpha>0$. Furthermore, let $g(p)=O(1)$ for all primes p, and let

$$\sum_{p,k\geq 2}p^{-k}g(p^k)<\infty.$$

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Besides this, if $\alpha \le 1$, then let

$$\sum_{p^k \le x, k \ge 2} g(p^k) = O(x(\log x)^{-1}).$$

Then

$$\sum_{n \le x} g(n) \sim \frac{e^{-\gamma\alpha} x}{\Gamma(\alpha)\log x} \prod_{p \le x} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots\right)$$

as $x \to \infty$. Here γ denotes Euler's constant.

In the case of *complex-valued multiplicative functions of modulus* ≤ 1 we used a method to give proofs for results of Halász und Wirsing (see [4], [5], [6]). Here we adopt the method to the present situtation. Then the idea of the proof is as follows.

Let g as in the theorem. Define an *exponentially multiplicative function* g_0 (see [1] P. Erdős and A. Rényi) by

$$g_0(p^k) = \frac{g(p)}{k!} \quad (p \ \ prime, \quad k \in \mathbb{N}). \tag{1.2}$$

Then $g = h * g_0$ where

$$\sum_{n=1}^{\infty} \frac{|h(n)|}{n} < \infty.$$

Further, define the multiplicative function τ_α by

$$\sum_{n=1}^{\infty} \frac{\tau_\alpha(n)}{n^s} = \zeta^\alpha(s), \tag{1.3}$$

where $\zeta(s)$ is Riemann's zeta-function. Then put

$$A(x) := H(1) e^{\alpha c} \exp\left(\sum_{p \le x} \frac{g(p) - \alpha}{p}\right),$$

where $H(1) = \sum\limits_{n=1}^{\infty} \frac{h(n)}{n}$ and

$$c = \sum_p \left(\frac{1}{p} + \log(1 - \frac{1}{p})\right).$$

We define, for $1 \le u \le x$,

$$\begin{aligned} M(u) &:= 1 * (g - A(x)\tau_\alpha)(u) = \\ &= \sum_{n \le u} (g(n) - A(x)\tau_\alpha(n)) \end{aligned}$$

and prove in the first part of the paper by convolution arithmetic

$$|M(x)| \ll \frac{x}{\log x}\int_1^x \frac{|M(u)|}{u^2}du + o\left(\frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right) \quad as \quad x\to\infty.$$

Next, we define an exponentially multiplicative function $\overline{g}_0$ by

$$\overline{g}_0(p^k) = \begin{cases} g_0(p^k) & for\ p\le x,\ k\ge 1 \\ \alpha/k! & for\ p > x,\ k\ge 1 \end{cases}$$

and put $\overline{g} = h * \overline{g}_0$. Choosing

$$K_0(u) = 1 * \Lambda_{\overline{g}_0} * (\overline{g} - A(x)\tau_\alpha)(u) \tag{1.4}$$

we obtain, for $2\le u\le x$,

$$M(u) = \frac{K_0(u)}{\log u} + o\left(\frac{u}{\log u}\exp\left(\sum_{p\le u}\frac{g(p)}{p}\right)\right). \tag{1.5}$$

Now

$$\begin{aligned} \int_1^x \frac{|M(u)|}{u^2}du \quad &\le \quad \int_1^{x^\varepsilon}\frac{|M(u)|}{u^2}du + \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2}du =: \\ &=: \quad I_1 + I_2. \end{aligned} \tag{1.6}$$

Then

$$I_1 \le \sum_{n\le x^\varepsilon}\frac{|g(n) - A(x)\tau_\alpha(n)|}{n} \ll \varepsilon\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right) + \varepsilon^\alpha\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)$$

and

$$I_2 \le \frac{1}{\varepsilon}\left(\frac{1}{\log x}\int_1^x\frac{|K_0(u)|^2}{u^3}du\right)^{\frac12} + o\left(\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right).$$

Put $\sigma = 1 + \frac{1}{\log x}$. Then

$$\int_1^x \frac{|K_0(u)|^2}{u^3}du \le e^2\int_1^x\frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}}du.$$

Since $(s = \sigma + it)$

$$\int_0^\infty K_0(e^u)e^{-us}e^{-iut}dt = \frac{\overline{G}_0'(s)}{\overline{G}_0(s)}(\overline{G}(s) - A(x)\zeta^\alpha(s)),$$

where

$$\overline{G}_0(s) = \sum_{n=1}^\infty \frac{\overline{g}_0(n)}{n^s} \quad \text{and} \quad \overline{G}(s) = \sum_{n=1}^\infty \frac{\overline{g}(n)}{n^s}, \tag{1.7}$$

we conclude, by Parseval's equation,

$$\int_1^\infty \frac{|K_0(u)|^2}{u^{3+2(\sigma-1)}}du = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2 |\overline{G}(s) - A(x)\zeta^\alpha(s)|^2 \frac{dt}{|s|^2}. \tag{1.8}$$

Using ideas of G. Tenenbaum (see [9], Theorem 4.10 and Theorem 4.13, and [10]) we estimate the last integral in (1.8) by $o\left(\log x \exp\left(2\sum_{p\le x}\frac{g(p)}{p}\right)\right)$, which proves our Theorem. ■

2. Notations and some preliminaries

Let g be as in Theorem. Define the exponentially multiplicative function g_0 by (1.2). The generating function $G_0(s)$ of g_0 satisfies

$$G_0(s) = \exp\left(\sum_p \frac{g(p)}{p^s}\right). \tag{2.1}$$

If we write $g = h * g_0$ then $G(s) = H(s)G_0(s)$ with $H(s) = \sum_{n=1}^\infty h(n)n^{-s}$ and

$$h(p) = 0 \ (p \ prime) \ and \ \sum_{p,k\ge 2}\frac{|h(p^k)|}{p^k} < \infty$$

which implies

$$\sum_{n=1}^\infty \frac{|h(n)|}{n} < \infty.$$

For $\alpha > 0$ we define the multiplicative function τ_α by (1.3).

By Theorem II 5.2 and Theorem II 5.4 of Tenenbaum [9] the following holds.

Lemma 2.1. *Let* $0 < a < b$*. Then, uniformly for* $a < \alpha < b, y \geq 2$ *the relation*

$$\sum_{n\leq y} \tau_\alpha(n) = \frac{y}{\Gamma(\alpha)}(\log y)^{\alpha-1}\left\{1 + O(\frac{1}{\log y})\right\}$$

holds.

Let $s = \sigma + it$. The function $\zeta(s)\exp(-\sum_p \frac{1}{p^s})$ is holomorphic at $s = 1$ and for $\sigma > 1$,

$$\log \zeta(s) - \sum_p \frac{1}{p^s} = -c + O(|s-1|),$$

where

$$c = \sum_p \left(\frac{1}{p} + \log\Big(1 - \frac{1}{p}\Big)\right).$$

Then

$$\zeta^\alpha(s) = e^{-\alpha c} \exp\left(\sum_p \frac{\alpha}{p^s}\right)\{1 + O(|s-1|)\}. \tag{2.2}$$

Put

$$A(x) = H(1)e^{\alpha c} \exp\left(\sum_{p\leq x} \frac{g(p)-\alpha}{p}\right) \tag{2.3}$$

and $(\xi > 0)$

$$W(\xi) := A(e^\xi).$$

We observe that, because of (1.1), $W(\xi)$ is a slowly oscillating function.
Next, we put

$$M(u) := \sum_{n\leq u} g(n) - A(x)\sum_{n\leq u} \tau_\alpha(n). \tag{2.4}$$

Further, if we define the arithmetic function $L_0 g$ by $L_0 g(n) = (\log n)g(n)$, we introduce Λ_g by

$$L_0 g = g * \Lambda_g.$$

Obviously,

$$\Lambda_{g_0}(n) = \begin{cases} g(p)\log p & n = p,\ p\ \textit{prime}; \\ 0 & n\ \textit{not prime}. \end{cases} \tag{2.5}$$

and

$$\Lambda_{\tau_\alpha}(n) = \begin{cases} \alpha \log p & n = p^k,\ p\ prime; \\ 0 & n \neq p^k. \end{cases}$$

This can easily be seen from (2.1) and from the relation

$$\frac{(\zeta^\alpha(s))'}{\zeta^\alpha(s)} = \alpha \frac{\zeta'(s)}{\zeta(s)},$$

respectively.

Consider

$$\sum_{p \le x^\varepsilon} \frac{g(p)}{p} = \sum_{p \le x} \frac{g(p)}{p} - \sum_{x^\varepsilon < p \le x} \frac{g(p)}{p}.$$

Obviously, by partial summation,

$$\begin{aligned} \sum_{x^\varepsilon < p \le x} \frac{g(p)}{p} &= \sum_{x^\varepsilon < p \le x} \frac{g(p) - \alpha}{p} \cdot \frac{\log p}{\log p} + \sum_{x^\varepsilon < p \le x} \frac{\alpha}{p} = \\ &= o(1) + \alpha \log \frac{1}{\varepsilon} \end{aligned}$$

which implies the inequality

$$\sum_{n \le x^\varepsilon} \frac{g_0(n)}{n} \ll \varepsilon^\alpha \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right). \tag{2.6}$$

Now we prove

Lemma 2.2 *Let $\alpha > 1$. Then there exists a constant K so that*

$$\exp\left(\sum_{p \le y} \frac{g(p) - 1}{p} \right) \le K \exp\left(\sum_{p \le x} \frac{g(p) - 1}{p} \right)$$

for $y \le x$ and $x \ge x_0$.

Proof. Observe

$$\exp\left(\sum_{p \le y} \frac{g(p) - 1}{p} \right) \asymp \exp\left(\sum_{p \le y} \frac{g(p) - \alpha}{p} \right) (\log y)^{\alpha - 1}.$$

Putting $\xi = \log x$ we estimate $W(\xi)\xi^{\alpha-1}$ where $\alpha > 1$. Since $W(\xi)$ is slowly oscillating we have, for $\xi \ge \xi_0(\varepsilon)$,

$$W(\eta) \le (1 + \varepsilon) W(\xi) \ \ for \ \frac{\xi}{2} \le \eta \le \xi$$

and

$$W(\eta) \leq (1+\varepsilon)^{\nu} W(\xi) \ \ if \ \ \eta \in [\frac{\xi}{2^{\nu}}, \frac{\xi}{2^{\nu-1}}].$$

Thus

$$\begin{aligned} \eta^{\alpha-1} W(\eta) &\leq (\frac{\xi}{2^{\nu-1}})^{\alpha-1} (1+\varepsilon)^{\nu} W(\xi) = \\ &= \xi^{\alpha-1} W(\xi) \frac{(1+\varepsilon)^{\nu-1}}{(2^{\alpha-1})^{\nu-1}} (1+\varepsilon) \leq \\ &\leq 2\xi^{\alpha-1} W(\xi) \end{aligned}$$

and

$$\eta^{\alpha-1} W(\eta) \leq 2\xi^{\alpha-1} W(\xi), \ \ \xi_0 \leq \eta \leq \xi.$$

Therefore

$$\exp\left(\sum_{p \leq y} \frac{g(p)-1}{p}\right) \leq K \exp\left(\sum_{p \leq x} \frac{g(p)-1}{p}\right) \ \ for \ \ x_0(\varepsilon) \leq y \leq x$$

and, eventually with some larger constant, this holds for $y \leq x_0 \leq x$. ■

Remark 2.1. If $\alpha > 1$ then an easy consequence of Lemma 2.2 gives

$$\sum_{n \leq x} |h(n)| \sum_{m \leq \frac{x}{n}} g_0(m) \ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right).$$

The same estimate is valid in the case $0 < \alpha \leq 1$ since (see (3.6))

$$\sum_{n \leq x} |h(n)| \log n \ll x$$

which implies

$$\sum_{n \leq x} |h(n)| \ll \frac{x}{\log x}.$$

Therefore

$$\sum_{n \leq x} g(n) \ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} \frac{g(p)}{p}\right).$$

For applying Lemma 4.13 from [9] we choose B such that $\alpha < B$ and $|g(p)| \leq B$ for all p, and define

$$\varphi_0 := \pi \frac{B}{\alpha} \ \ and \ \ \beta = 1 - \frac{sin\varphi_0}{\varphi_0}. \tag{2.7}$$

Observe $\beta > 0$.

Further, put

$$I_1(\varepsilon) = \left\{ t : |t| \le \frac{1}{\varepsilon \log x} \right\}$$

and

$$I_2(\varepsilon) = \left\{ t : \frac{1}{\varepsilon \log x} < |t| \le \varepsilon^{-2\alpha} \right\}.$$

Then the following Lemma holds.

Lemma 2.3 *Let $\overline{G}(s)$ be defined by (1.7), where $s = \sigma + it (\sigma > 1)$. Then, for every $\varepsilon > 0$,*

$$\overline{G}(s) - A(x)\zeta^\alpha(s) \ll \varepsilon \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right) \quad for \quad t \in I_1(\varepsilon) \tag{2.8}$$

and

$$\overline{G}(s) \ll \varepsilon^{\alpha\beta} \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right) \quad for \quad t \in I_2(\varepsilon). \tag{2.9}$$

Proof. Since, by (2.2),

$$\frac{\overline{G}(s)}{A(x)\zeta^\alpha(s)} = \{1 + o(1)\} \exp\left(\sum_p \frac{\overline{g}(p)}{p^s} - \sum_{p \le x} \frac{g(p) - \alpha}{p} + \alpha c - \alpha c - \sum_p \frac{\alpha}{p^s} \right),$$

we consider

$$\exp\left(\sum_p \frac{\overline{g}(p)}{p^s} - \sum_{p \le x} \frac{g(p) - \alpha}{p} - \sum_p \frac{\alpha}{p^s} \right) =$$

$$= \exp\left(\sum_{p \le x} \frac{g(p) - \alpha}{p^s} - \sum_{p \le x} \frac{g(p) - \alpha}{p} \right).$$

By (1.1) and partial summation we conclude, as $x \to \infty$

$$\sum_{x^{\varepsilon^2} < p \le x} \frac{g(p) - \alpha}{p} = o(1), \quad \sum_{x^{\varepsilon^2} < p \le x} \frac{g(p) - \alpha}{p^s} = o(1). \tag{2.10}$$

Obviously

$$\sum_{p \le x^{\varepsilon^2}} \frac{|g(p) - \alpha|}{p} |1 - p^{-(s-1)}| \ll |s - 1| \varepsilon^2 \log x \ll \varepsilon \quad for \quad t \in I_1(\varepsilon),$$

and thus (2.8) holds.

Next we observe

$$\begin{aligned} |\overline{G}(s)| &\ll |\overline{G}_0(s)| \ll \\ &\ll \exp\left(Re\sum_{p\le x}\frac{g(p)}{p^{1+it}}\right) = \\ &= \exp\left(\sum_{p\le x}\frac{g(p)}{p}cos(t\log p)\right). \end{aligned}$$

Obviously

$$\begin{aligned} \sum_{p\le x}\frac{g(p)}{p}cos(t\log p) &\le \sum_{p\le x^\varepsilon}\frac{g(p)}{p} + \sum_{x^\varepsilon<p\le x}\frac{g(p)}{p}cos(t\log p) = \\ &= \sum_{p\le x}\frac{g(p)}{p} - \sum, \end{aligned}$$

where

$$\sum = \sum_{x^\varepsilon<p\le x}\frac{g(p)}{p}(1-cos(t\log p)).$$

With $0<\lambda<1$ we have

$$\begin{aligned} \sum &= (1-\lambda)\sum_{x^\varepsilon<p\le x}\frac{g(p)}{p} - \sum_{x^\varepsilon<p\le x}\frac{g(p)}{p}(cos(t\log p-\lambda)) \ge \\ &\ge (1-\lambda)\sum_{x^\varepsilon<p\le x}\frac{g(p)}{p} - B\sum_{x^\varepsilon<p\le x}\frac{(cos(t\log p)-\lambda)^+}{p} = \\ &= \Sigma_1 - \Sigma_2. \end{aligned}$$

By partial summation

$$\Sigma_1 \ge (1-\lambda)\alpha\log\frac{1}{\varepsilon} + O(1),$$

since (cf.(2.10))

$$\begin{aligned} \sum_{x^\varepsilon<p\le x}\frac{g(p)}{p} &= \sum_{x^\varepsilon<p\le x}\frac{g(p)-\alpha}{p} + \sum_{x^\varepsilon<p\le x}\frac{\alpha}{p} \ge \\ &\ge \alpha\log\frac{1}{\varepsilon} + O(1). \end{aligned}$$

Next we use ideas of G. Tenenbaum (see [9], Lemma 4.13 and [10]). By [9], Lemma 4.13, we have

$$\Sigma_2 = Bm\log\frac{1}{\varepsilon} + O(1),$$

where (see (2.7))

$$\begin{aligned} m &= \frac{1}{2\pi}\int_{-\varphi_0}^{\varphi_0}(cos\varphi-\lambda)d\varphi = \\ &= \frac{sin\varphi_0-\varphi_0\lambda}{\pi} = \\ &= \frac{\varphi_0}{\pi}(\frac{sin\varphi_0}{\varphi_0}-\lambda) = \\ &= \frac{\alpha}{B}(1-\lambda-\beta), \end{aligned}$$

i.e.

$$-\Sigma_2 = (-\alpha(1-\lambda)+\beta)\log\frac{1}{\varepsilon}.$$

Then, together with (2.10),

$$\sum \geq (1-\lambda)\alpha\log\frac{1}{\varepsilon} - \alpha(1-\lambda)\log\frac{1}{\varepsilon} + O(1) + \beta\alpha\log\frac{1}{\varepsilon},$$

and

$$-\sum \leq -\alpha\beta\log\frac{1}{\varepsilon} + O(1) \ll \alpha\beta\log\varepsilon,$$

which proves (2.9). ∎

Remark 2.2. A trivial estimate for $G(s)$ is given by

$$|\overline{G}(s)| \ll \exp\left(\sum_{p\leq x}\frac{g(p)}{p}\right) \quad for \quad all \quad \sigma > 1.$$

Remark 2.3. We use the following notations:

$$M^*(x) := \sum_{n\leq x} g(n), \quad m^*(x) := \sum_{n\leq x}\frac{g(n)}{n}$$

and

$$\beta^*(t) := e^{-t}M^*(e^t), \quad L^*(t) = t^{-\alpha}m^*(e^t).$$

Then the assertion of Theorem can be written as (cf. [3], p.147)

$$\begin{aligned} M^*(x) &\sim \alpha x(\log x)^{\alpha-1}L^*(\log x), \\ m^*(x) &\sim \frac{e^{\gamma\alpha}}{\Gamma(\alpha+1)}\prod_{p\leq x}\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\cdots\right). \end{aligned} \tag{2.11}$$

The function L^* is slowly oscillating. Let

$$G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

be the generating Dirichlet series of g. Then integration by parts shows that, for $s = \sigma + it, \sigma > 1$, we have

$$s^{-1}G(s) = \int_0^{\beta} \beta^*(u)e^{-u(\sigma-1)}e^{-iut}du,$$

which implies that if (2.11) holds, then

$$G(s) = \frac{\alpha L^*(\frac{1}{\sigma-1})}{(s-1)^{\alpha}} + o\left(\frac{L^*(\frac{1}{\sigma-1})}{(\sigma-1)^{\alpha}}\right)$$

as $\sigma \to 1^+$, which holds uniformly on each bounded interval $-K \leq t \leq K$.

Lemma 2.4. *Let $M(u)$ be defined by (2.4), and let $\{a_n\}$ be a sequence of positive numbers a_n such that*

$$\sum_{n\leq x} a_n = 2x \log x + O(x).$$

Then

$$\sum_{n\leq x} |M(\frac{x}{n})|a_n = 2\int_1^x |M(\frac{x}{u})| \log u du + O\left(x \exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right)\right).$$

Proof. Put $c_1 = 0$ and, for $n \geq 2$,

$$c_n = a_n - 2\int_{n-1}^{n} \log t dt.$$

Then

$$\begin{aligned} C(x) &:= \sum_{n\leq x} c_n = \sum_{n\leq x} a_n - 2\int_1^{[x]} \log t dt = \\ &= O(x). \end{aligned}$$

We write

$$\begin{aligned}\sum_{n\le x}|M(\frac{x}{n})|a_n &= \sum_{n\le x}|M(\frac{x}{n})|a_n - 2\sum_{2\le n\le x}|M(\frac{x}{n})|\int_{n-1}^{n}\log t dt + \\ &+ 2\sum_{2\le n\le x}|M(\frac{x}{n})|\int_{n-1}^{n}\log t dt - 2\int_{1}^{x}|M(\frac{x}{t})|\log t dt = \\ &= \Sigma_1 + \Sigma_2 + 2\int_{1}^{x}|M(\frac{x}{t})|\log t dt.\end{aligned}$$

By partial summation

$$\begin{aligned}|\Sigma_1| &= |\sum_{2\le n\le x-1}C(n)\{|M(\frac{x}{n})| - |M(\frac{x}{n+1})|\} + C(x)|M(\frac{x}{[x]})| + |M(x)| \ll \\ &\ll \sum_{n\le x-1}n||M(\frac{x}{n})| - |M(\frac{x}{n+1})|| + O(x) + |M(x)| \ll \\ &\ll \sum_{n\le x-1}n|M(\frac{x}{n}) - M(\frac{x}{n+1})| + O(x) + |M(x)|.\end{aligned}$$

Using

$$|M(y)| \le M^*(y) := \sum_{n\le y}|g(n) - A(x)\tau_\alpha(n)|$$

we conclude

$$\begin{aligned}|\Sigma_1| &\ll \sum_{n\le x}n\left(M^*(\frac{x}{n}) - M^*(\frac{x}{n+1})\right) + O(x) + M^*(x) = \\ &= \sum_{n\le x}M^*(\frac{x}{n}) + O(x) + M^*(x) = \\ &= \sum_{m\le x}|g(m) - A(x)\tau_\alpha(m)|\sum_{n\le \frac{x}{m}}1 + O(x) + M^*(y) = \\ &= O\left(x\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right).\end{aligned}$$

For the estimate of $\sum_2$ we observe

$$\begin{aligned} & M(\frac{x}{n})|\int_{n-1}^{n} \log t dt - \int_{n-1}^{n} |M(\frac{x}{t})| \log t dt \leq \\ \leq & \int_{n-1}^{n} |M(\frac{x}{n})| - |M(\frac{x}{t})| \log t dt \leq \\ \leq & \int_{n-1}^{n} (M^*(\frac{x}{t}) - M^*(\frac{x}{n})) \log t dt \leq \\ \leq & (n-1)(M^*(\frac{x}{n-1}) - M^*(\frac{x}{n})). \end{aligned}$$

Then

$$\begin{aligned} \Sigma_2 & \leq \sum_{n \leq x-1} n \left(M^*(\frac{x}{n}) - M^*(\frac{x}{n+1}) \right) + O(\log x) \ll \\ & \ll x \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right), \end{aligned}$$

and the assertion of Lemma 2.4 holds. ■

3. Proof of Theorem; First Step

With the function M defined in (2.4), the identity function $\mathbf{1}$ and L given by $L(u) = \log u$ $(u \geq 1)$ we use, for arithmetical functions f, the convolutions

$$\begin{aligned} L * f(x) &= \sum_{n \leq x} (\log \frac{x}{n}) f(n), \\ M * f(x) &= \sum_{n \leq x} (M(\frac{x}{n}) f(n), \\ \mathbf{1} * f(x) &= \sum_{n \leq x} f(n). \end{aligned}$$

Then the proof of Theorem 2 in Indlekofer [5] gives

(3.1) $L^2 M = M * (\Lambda_{g_0} * \Lambda_{g_0} + L_0 \Lambda_{g_0}) + (R_1 + R_2 + R_3) * \Lambda_{g_0} + L(R_1 + R_2 + R_3),$

where

$$\begin{aligned} R_1 &= L * (g - A(x)\tau_\alpha), \\ R_2 &= \mathbf{1} * (L_0 h * g_0), \\ R_3 &= -\mathbf{1} * A(x)\tau_\alpha * (\Lambda_{\tau_\alpha} - \Lambda_{g_0}). \end{aligned}$$

In the **first step** we shall prove

$$\begin{aligned} (3.2)\quad |M(x)| &\ll \\ &\ll \frac{1}{\log x}\int_1^x |M(\frac{x}{u})|du + o\left(\frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right) = \\ &= \frac{x}{\log x}\int_1^x \frac{|M(u)|}{u^2}du + o\left(\frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right) \end{aligned}$$

as $x \to \infty$.

Since $g(p) = O(1)$ we obtain by a crude estimate (see (2.5))

$$\Lambda_{g_0} * \Lambda_{g_0} + L_0\Lambda_{g_0} \ll \Lambda * \Lambda + L_0\Lambda.$$

Putting $a_n = (\Lambda * \Lambda)(n) + \Lambda(n)(\log n)$ we use Selberg's formula

$$\sum_{n\le x} a_n = 2x\log x + O(x) \tag{3.3}$$

and get, by (3.1) and Lemma 2.4, the first estimate in (3.2).

For $R_1(x)$ we obtain

$$\begin{aligned} |R_1(x)| &\ll \int_2^x \frac{\sum_{n\le u} |g(n) + A(x)\tau_\alpha(n)|}{u} du \ll \\ &\ll \exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\int_2^x \frac{du}{\log u} \ll \\ &\ll \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right). \end{aligned} \tag{3.4}$$

In the case

$$R_2(x) = \sum_{n\le x} h(n)\log n \sum_{m\le \frac{x}{n}} g_0(m)$$

we use, if $\alpha > 1$, Lemma 2.2 and conclude

$$(3.5)\qquad |R_2(x)| \ll \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\sum_{n\le x}\frac{|h(n)|\log n}{n} = = o\left(x\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right)$$

since $\sum_{n=1}^{\infty}\frac{|h(n)|}{n} < \infty$.

In the case $0 < \alpha \le 1$ we have

$$\sum_{\substack{p^k\le x\\ k\ge 2}} |h(p^k)|\log p^k = O(x)$$

and (see Postnikov [8], p. 201)

$$(3.6)\qquad \sum_{n\le x}|h(n)|\log n \ll \sum_{n\le x}|h(n)|\sum_{\substack{p^k\le x\\ k\ge 2}}|h(p^k)|\log p^k \ll \ll x\sum_{n\le x}\frac{|h(n)|}{n} \ll x.$$

Then

$$\begin{aligned} R_2(x) &= \sum_{n\le x} h(n)\log n \sum_{m\le \frac{x}{n}} g_0(m) = \\ &= \sum_{m\le x^\varepsilon} g_0(m)\sum_{n\le\frac{x}{m}} h(n)\log n + \\ &\quad + \sum_{n\le x^{1-\varepsilon}} h(n)\log n \sum_{m\le\frac{x}{n}} g_0(m) =: \\ &=: \Sigma_1 + \Sigma_2. \end{aligned}$$

By (2.6)

$$\Sigma_1 \ll x\sum_{n\le x^\varepsilon}\frac{g_0(n)}{n} \ll \varepsilon^\alpha x\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)$$

and as in (3.5),

$$\Sigma_2 \ll \frac{x}{\log x}\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\sum_{n\le x^{1-\varepsilon}}\frac{|h(n)|\log n}{n} = = o(x\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right).$$

Let us consider $R_3(x)$. We have

$$\text{(3.7)}\qquad \begin{aligned} R_3(x) &= A(x)\sum_{n\le x}\tau_\alpha(n)\sum_{m\le \frac{x}{n}}(\Lambda_{\tau_\alpha}(m)-\Lambda_{g_0(m)}) = \\ &= A(x)\{\sum_{n\le x^\varepsilon}\ldots+\sum_{x^\varepsilon<n\le x}\ldots\} =: \\ &=: A(x)(\Sigma_1'+\Sigma_2'). \end{aligned}$$

Then

$$\begin{aligned} |\Sigma_1'| &= \sum_{n\le x^\varepsilon}\tau_\alpha(n)\sum_{m\le\frac{x}{n}}|\Lambda_{\tau_\alpha}(m)-\Lambda_{g_0}(m)| \ll \\ &\ll x\sum_{n\le x^\varepsilon}\frac{\tau_\alpha(n)}{n}\ll \varepsilon^\alpha x(\log x)^\alpha, \end{aligned}$$

i.e.

$$\text{(3.8)}\qquad A(x)\Sigma_1' \ll \varepsilon^\alpha x\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right).$$

For the estimate of Σ_2' we use Lemma 2.1. Then

$$\Sigma_2' = \frac{x}{\Gamma(\alpha)}\sum_{m\le x^{1-\varepsilon}}\frac{\Lambda_{\tau_\alpha}(m)-\Lambda_{g_0}(m)}{m}(\log\frac{x}{m})^{\alpha-1}\{1+O(\frac{1}{\varepsilon\log x})\}.$$

Since

$$\sum_{m\le y}\frac{\Lambda_{\tau_\alpha}(m)-\Lambda_{g_0}(m)}{m} = \sum_{p\le y}\frac{\alpha\log p-g(p)\log p}{p}+\sum_{\substack{p^k\le y\\ k\ge 2}}\frac{\alpha\log p}{p^k} = o(\log y)+O(1)$$

we conclude

$$\text{(3.9)}\qquad \Sigma_2' = o(x(\log x)^\alpha)$$

by partial summation. By (3.7), (3.8) and (3.9)

$$R_3(x) = o\left(x \exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right).$$

Consider

$$R_1 * \Lambda_{g_0}(x) = \sum_{n\le x}\log\frac{x}{n}(g - A(x)\tau_\alpha) * \Lambda_{g_0}(n).$$

Observing

$$\sum_{n\le y}|g(n) - A(x)\tau_\alpha(n)| \sum_{p\le \frac{y}{n}} g(p)\log p \ll$$

$$\ll y\sum_{n\le y}\frac{|g(n) - A(x)\tau_\alpha(n)|}{n} \ll y\exp\left(\sum_{p\le y}\frac{g(p)}{p}\right)$$

we conclude (cf. (3.4))

$$R_1 * \Lambda_{g_0}(x) \ll x\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right). \tag{3.10}$$

Since $L_0 g_0 = g_0 * \Lambda_{g_0}$ we have

$$R_2 * \Lambda_{g_0}(x) \le \log x R_2(x) = o\left(x\log x\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right). \tag{3.11}$$

Let us return to

$$R_3 * \Lambda_{g_0}(x) = \sum_{n\le x}(A(x)\tau_\alpha * (\Lambda_{\tau_\alpha} - \Lambda_{g_0}) * \Lambda_{g_0})(n).$$

First, we observe

$$\sum_{n\le y}(\Lambda_{\tau_\alpha} * \Lambda_{g_0} - \Lambda_{g_0} * \Lambda_{g_0})(n) = O(y\log y).$$

Next, we consider

$$\sum_{n\le y}(\frac{\Lambda_{\tau_\alpha} * \Lambda_{g_0} - \Lambda_{g_0} * \Lambda_{g_0})(n)}{n} =$$

$$= \sum_{pp'\le y}\frac{g(p)\log p}{p}\left(\frac{g(p') - \alpha}{p'}\log p'\right) + \sum_{\substack{pp'^k\le y\\ k\ge 2}}\frac{g(p)\log p}{p}\frac{\log p'}{p'^k} =$$

$$= \sum_{pp'\le y}\frac{g(p)\log p}{p}\left(\frac{g(p') - \alpha}{p'}\log p'\right) + O(\log y) =: \sum + O(y).$$

We split $\sum$ into

$$\sum = \sum_{p' \le y} \ldots + \sum_{p \le y^{1-\varepsilon}} \ldots =: \Sigma_1 + \Sigma_2.$$

Then obviously

$$\Sigma_1 \ll \varepsilon (\log y)^2$$

and

$$\Sigma_2 = \sum_{p \le y^{1-\varepsilon}} \frac{g(p) \log p}{p} o(\log \frac{y}{p}) = o((\log y)^2).$$

Then, arguing as in (3.8) and (3.9) we prove

$$R_3 * \Lambda_{g_0}(x) = o\left(x \log x \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right) \right). \tag{3.12}$$

Collecting the results (3.3) with Lemma 2.1, (3.4), (3.5), (3.7), (3.10), (3.11) and (3.12) we obtain (3.2).

Remark 3.1. Let g as above. If f is multiplicative and $|f| \le g$, then

$$|\sum_{n \le x} f(n)| \ll \frac{x}{\log x} \int_2^x \frac{|\sum_{n \le u} f(n)|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right) \right)$$

and

$$|\sum_{n \le x} f(n) - A_x g(n)| \ll$$

$$\ll \frac{x}{\log x} \int_2^x \frac{|\sum_{n \le u} f(n) - A_x g(n)|}{u^2} du + o\left(\frac{x}{\log x} \exp\left(\sum_{p \le x} \frac{g(p)}{p} \right) \right)$$

if $A_x \neq 0$, $A_x = O(1)$ and

$$\sum_{p \le x} \frac{\Lambda_f - \Lambda_g}{p} = o(\log x).$$

Remark 3.2. It is easy to see that

$$\begin{aligned} LM &= \mathbf{1} * L_0(g - A(x)\tau_\alpha) + R_1 = \\ &= \mathbf{1} * \Lambda_{g_0} * (g - A(x)\tau_\alpha) + R_1 + R_2 + R_3, \end{aligned}$$

which implies, by (3.4), (3.5) and (3.10),

$$M(u) = \frac{\mathbf{1} * \Lambda_{g_0} * (g - A(x)\tau_\alpha)(u)}{\log u} + o\left(\frac{u}{\log u} \exp\left(\sum_{p\le u} \frac{g(p)}{p}\right)\right)$$

for $u \le x$. Since $g(n) = \overline{g}(n), g_0(n) = \overline{g_0}(n)$ for $n \le x$, we define (cf. (1.4))

$$K_0(u) = \mathbf{1} * \Lambda_{\overline{g}_0} * (\overline{g} - A(x)\tau_\alpha)(u) \quad for \quad u \le x.$$

Collecting the estimates (1.5), (1.6) from Section 1 we obtain

$$\begin{aligned}
\int_1^x \frac{|M(u)|}{u^2} du &\le \int_1^{x^\varepsilon} \frac{|M(u)|}{u^2} du + \int_{x^\varepsilon}^x \frac{|M(u)|}{u^2} du \ll \\
&\ll (\varepsilon + \varepsilon^\alpha) \exp\left(\sum_{p\le x} \frac{g(p)}{p}\right) + \\
&+ \frac{1}{\varepsilon}\left(\frac{1}{\log x} \int_{-\infty}^{\infty} \left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2 |\overline{G}(s) - A(x)\zeta^\alpha(s)|^2 \frac{dt}{|s|^2}\right) + \\
&+ o\left(\exp\left(\sum_{p\le x} \frac{g(p)}{p}\right)\right) \quad as \quad x \to \infty.
\end{aligned}$$

Therefore, if we show that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2 |\overline{G}(s) - A(x)\zeta^\alpha(s)|^2 \frac{dt}{|s|^2} = \\
&= o\left(\log x \exp\left(2\sum_{p\le x} \frac{g(p)}{p}\right)\right) \quad as \quad x \to \infty
\end{aligned} \tag{3.13}$$

we have finished the proof of our Theorem. ■

4. Proof of Theorem; Second Step

For the estimate of the integral in (3.13) we split the interval $(-\infty,\infty)$ into $I_1(\varepsilon)\cup I_2(\varepsilon)$ and $\{t:\varepsilon^{-2\alpha}<|t|\}$. In the first case we obtain by (2.3)

$$\begin{aligned}(4.1)\qquad &\int\limits_{|t|\le\varepsilon^{-2\alpha}}\left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2|\overline{G}(s)-A(x)\zeta^{\alpha}(s)|^2\frac{dt}{|s|^2}\ll\\ &\ll\max(\varepsilon^2,\varepsilon^{2\alpha},\varepsilon^{2\alpha\beta})\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\int\limits_{|t|\le\varepsilon^{-2\alpha}}\left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2\frac{dt}{|s|^2}\end{aligned}$$

whereas

$$\begin{aligned}(4.2)\qquad &\int\limits_{|t|>\varepsilon^{-2\alpha}}\left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2|\overline{G}(s)-A(x)\zeta^{\alpha}(s)|^2\frac{dt}{|s|^2}\ll\\ &\ll\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\int\limits_{|t|>\varepsilon^{-2\alpha}}\left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2\frac{dt}{|s|^2}.\end{aligned}$$

The integrals in (4.1) and (4.2) can be estimated by

$$\sum_{\substack{k\in\mathbb{Z}\\|k|\le[\varepsilon^{-2\alpha}]}}\frac{1}{k^2+1}\int\limits_{|t-k|\le\frac12}\left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2dt,$$

$$\sum_{\substack{k\in\mathbb{Z}\\|k|>[\varepsilon^{-\alpha}]}}\frac{1}{k^2+1}\int\limits_{|t-k|\le\frac12}\left|\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}\right|^2dt,$$

respectively. Further, we have

$$\frac{\overline{G}_0'(s)}{\overline{G}_0(s)}=-\sum_p\frac{g(p)\log p}{p^s},$$

$$-\sum_p\frac{\log p}{p^s}=\frac{\zeta'(s)}{\zeta(s)}+O(1)\quad for\quad \sigma>1.$$

Since $|f(p)|\le B$, we use $|a_n|<Bb_n$ in the proof of the Theorem 4.10 in [9] and conclude

$$\int\limits_{-\frac12}^{\frac12}\left|\frac{\overline{G}_0'}{\overline{G}_0}(1+\frac{1}{\log x}+ik+it)\right|^2dt\le 3B^2\int\limits_{-\frac12}^{\frac12}\left|\frac{\zeta'}{\zeta}(1+\frac{1}{\log x}+it)\right|^2dt\ll\log x.$$

Thus, by (4.1), (4.2) and (3.13) the proof of our Theorem is finished. ■

Remark 4.1. Let $g \geq 0$ be a multiplicative function, satisfying $g(p) \leq B$,

$$\sum_{p,k\geq 2} g(p^k)p^{-k} < \infty \quad \text{and} \quad \sum_{p^k\leq x, k\geq 2} g(p^k) = O(x(\log x)^{-1}). \tag{4.3}$$

If, for some $\alpha > 0$,

$$\sum_{n\leq x^{\varepsilon}} \frac{g(n)}{n} \ll \varepsilon^{\alpha} \exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right) \quad \textit{for every } \varepsilon > 0,$$

then we may apply the methods of the paper in the following context.

Put $f = h_1 * f_0$ where f_0 is exponentially multiplicative and $F(s) = H_1(s)F_0(s)$. Assume that

$$\sum_{p} \frac{g(p) - Ref(p)p^{it}}{p} \tag{4.4}$$

converges for some $t = t_0$. For simplicity let $t_0 = 0$.

Put

$$A_x = \frac{H_1(1)}{H(1)} \exp\left(-\sum_{p\leq x} \frac{g(p) - Ref(p)}{p}\right)$$

A_x has the form $AW(\log x)$, where A is non-zero constant and $W(\xi)$ is a non-vanishing slowly oscillating function of ξ. If (4.4) diverges for all $t \in \mathbb{R}$ then we choose $A_x = 0$.

Put $M = \mathbf{1} * (f - A_x g)$ $f = h_1 * f_0$, where f_0 is exponentially multiplicative. Then, as above,

$$LM = \mathbf{1} * L_0(f - A_x g) + R_1'$$

where $R_1' = L * (f - A_x g)$ and

$$R_1'(x) \ll \frac{x}{\log x} \exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right).$$

Further,

$$\mathbf{1} * L_0(f - A_x g) = \mathbf{1} * \Lambda_{f_0} * (f - A_x g) + R_2' + R_3'$$

$$R_2' = \mathbf{1} * \Lambda_{h_1} * (f - A_x g)$$

and

$$R_3' = \mathbf{1} * A_x g * (\Lambda_f - \Lambda_g). \tag{4.5}$$

If $A_x \neq 0$, then

$$\sum_p \frac{g(p) - Ref(p)}{p} \quad converges,$$

which implies

$$\sum_p \frac{|g(p) - f(p)|^2}{p} < \infty$$

and

$$R_3'(x) = O\left(\varepsilon^\alpha x \exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right)\right) + o\left(x \exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right)\right).$$

In any case

$$R_2'(x) = o\left(x \exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right)\right).$$

Now, in the same way as above we obtain

$$\int_{x^\varepsilon}^{x} \frac{|M(u)|}{u^2} du \ll \tag{4.6}$$

$$\ll \frac{1}{\varepsilon}\left(\frac{1}{\log x}\int_{-\infty}^{\infty}\left|\frac{F_0'(s)}{F_0(s)}\right|^2 |F(s) - A_x G(s)|^2 \frac{dt}{|s|^2}\right)^{\frac{1}{2}} + o\left(x \exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right)\right).$$

First observe

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{F_0'}{F_0}(1 + \sigma + ik + it)\right|^2 dt \ll \log x. \tag{4.7}$$

If $A_x = 0$ then

$$F(s) = o(G(\sigma)) = o\left(\exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right)\right) \quad for \quad t \in I_1(\varepsilon) \cup I_2(\varepsilon). \tag{4.8}$$

If (4.5) holds then

$$F(s) - A_x G(s) = o(G(\sigma)) = o\left(\exp\left(\sum_{p\leq x} \frac{g(p)}{p}\right)\right) \quad for \quad t \in I_1(\varepsilon) \tag{4.9}$$

and

$$\max(|F(s)|, G(s)) \ll \varepsilon^{\alpha\beta}\left(\exp\left(\sum_{p\le x}\frac{g(p)}{p}\right)\right) \quad \textit{for}\ \ t\in I_2(\varepsilon). \tag{4.10}$$

From this we conclude

Proposition. *Let g be multiplicative, $0\le g(p)\le B$,*

$$\sum_{p,k\ge 2} g(p^k)p^{-k} \quad \textit{and} \quad \sum_{p^k\le x, k\ge 2} g(p^k) = O\Big(\frac{x}{\log x}\Big).$$

If $f:\mathbb{N}\to\mathbb{C}$ is multiplicative, $|f|\le g$ satisfies (4.3) *then* (4.6) *together with* (4.7)$,\ldots,$(4.10) *holds.*

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