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SOME OLD AND NEW PROBLEMS ON ARITHMETICAL FUNCTIONS

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Dedicated to Professors
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and Bui Minh Phong on their birthdays

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Abstract. I, together with my friends, or alone formulated some open problems in the last fifty years. Some of that problems are solved, some others remained open. Here we shall present some new open problems.

1. Introduction

Notation.

(1) In the following let $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of primes, positive integers, integers, rational , real and complex numbers, respectively.

(2) We denote by $\mathcal{A}, \mathcal{A}^*, \mathcal{M}, \mathcal{M}^*$ the set of all additive, completely additive, complex-valued multiplicative, completely multiplicative functions, respectively.

(3) $\omega(n), \Omega(n), \tau(n), \varphi(n), \sigma(n)$ are typical arithmetical functions.

(4) $p(n)$= smallest prime divisor, $P(n)$ = largest prime divisor of n.

(5) $e(x) = e^{2\pi ix}$, $\Phi(x)$=Gaussian distribution function.

(6) $\pi(x) = \sharp\{p \leq x | p \in \mathcal{P}\}$, $\pi(x, k, \ell) = \sharp\{p \leq x | p \in \mathcal{P}, p \equiv \ell \pmod{k}\}$.

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2. On a theorem of H. Daboussi

2.1. H. Daboussi [1] proved that if $f \in \mathcal{M}$, $|f(n)| \leq 1$, then

$$S(x) := \sum_{n \leq x} f(n)e(n\alpha) = o(x) \quad \text{as} \quad x \to \infty \tag{2.1}$$

holds for every irrational α.

I gave a simple proof for it [15] by using a variant of Turán-Kubilius inequality, namely the following:

Let

$$\mathcal{P}_1 = \{p_1, \cdots, p_k\} \subseteq \mathcal{P}, \quad p_1 < \cdots < p_k \leq x,$$

$$L = \sum_{j=1}^{k} \frac{1}{p_j} \quad \textit{and} \quad \omega_{\mathcal{P}_1}(n) := \sum_{\substack{p|n \\ p \in \mathcal{P}_1}} 1,$$

Then

$$\sum_{n \leq x} \Big(\omega_{\mathcal{P}_1}(n) - L\Big)^2 \leq cxL, \tag{2.2}$$

where c is an absolute constant.

Hence, by using the Cauchy-Schwarz inequality, and that $\frac{1}{x} \sum\limits_{m \leq x} e(m\beta) \to 0$ as $x \to \infty$ for every irrational β, (2.1) follows.

By using this method we proved

Theorem 1. (J. M. De Koninck and I. Kátai [8]) *Let $\mathcal{P}_1 \subseteq \mathcal{P}$, $\sum_{p \in \mathcal{P}_1} \frac{1}{p} = \infty$. Let $\mathcal{B}$ be the set of those function $f : \mathbb{N} \to \mathcal{U}$, where $\mathcal{U} := \{z \in \mathbb{C} | |z| \leq 1\}$, for which*

$$f(pm) = f(p)f(m) \quad \textit{if} \quad p \in \mathcal{P}_1, (p, m) = 1.$$

Moreover, let $a : \mathbb{N} \to \mathcal{U}$ be a function for which

$$\frac{1}{x} \sum_{n \leq x} a(p_1 n)\overline{a}(p_2 n) \to 0 \quad \textit{as} \quad x \to \infty$$

for every $p_1 \neq p_2$, $p_1, p_2 \in \mathcal{P}_1$.

Then

$$\frac{1}{x} \sum_{n \leq x} f(n)a(n) \to 0 \quad \textit{as} \quad x \to \infty.$$

By this method the theorem of Daboussi can be generalized in different direction [15].

2.2. In [18] I considered the function

$$\Delta(\alpha, x) := \frac{1}{\pi_2(x)} \max_{X_{p_1}, X_{p_2} \in \mathcal{U}} \Big| \sum_{\substack{p_1 p_2 < x \\ p_1 < p_2}} X_{p_1} X_{p_2} e(\alpha p_1 p_2) \Big| \tag{2.3}$$

and proved that $\Delta(\alpha, x) \to 0$ for almost all irrational α and formulated the conjecture that it holds for every irrational α. This is proved by G. Harman [9].

In a joint paper written with K.-H. Indlekofer [10] we studied the sum

$$S(x|\alpha; Y_m, X_p) := \sum_{\substack{m_j \in \mathcal{M}_x \\ m_j p \le x}} Y_{m_j} X_p e(\alpha m_j p),$$

where

$$\mathcal{M}_x = \{m_1 < \cdots < m_t\} \subseteq \mathbb{N} \quad \text{and} \quad Y_m, X_p \in \mathcal{U}.$$

We proved: Let $m_\ell < x^{\delta_x}$, $\delta_x \to 0$, and that

$$\mu_x := \sum_{j=1}^{t} \frac{1}{m_j} \to \infty \quad \text{as} \quad x \to \infty.$$

Then

$$\max_{Y_{m_1} X_p \in \mathcal{U}} S(x|\alpha; Y_{m_1}, X_p) = o(1) \sum_{j=1}^{t} \pi\Big(\frac{x}{m_j}\Big) \quad \text{as} \quad x \to \infty \tag{2.4}$$

for almost all α.

We formulated the conjecture that (2.4) is true for every irrational α. G. Harman disproved this conjecture, and proved my next conjecture:

Let

$$\Delta_k(\alpha, x) = \frac{1}{\pi_k(x)} \max_{X_p \in \mathcal{U}} \Big| \sum_{p_1 \cdots p_k \le x} X_{p_1} \cdots X_{p_k} e(\alpha p_1 \cdots p_k) \Big|,$$

where

$$\pi_k(x) = \sum_{\substack{p_1 \cdots p_k \le x \\ p_1 < \cdots < p_k}} 1.$$

Then $\Delta_k(\alpha, x) \to 0$ $(x \to \infty)$ for every irrational α.

G. Harman proved my more strict conjecture.

Theorem. (G. Harman) *Let α be irrational, $k \geq 3$. Let*

$$\Delta_k(\alpha, x) = \frac{1}{\pi_k(x)} \max_{X_p^{(j)} \in \mathcal{U}} \Big| \sum_{p_1 \cdots p_k \leq x} X_{p_1}^{(1)} \cdots X_{p_k}^{(k)} e(\alpha p_1 \cdots p_k) \Big|.$$

Then

$$\Delta_k(\alpha, x) \to 0 \quad \textit{as} \quad x \to \infty.$$

3. On some question in the probabilistic number theory

3.1. In a paper written with J.-M. De Koninck [6] we investigated the function $\mathcal{U}_\lambda(n)$, where it is the number of those prime divisors p of n, for which in the interval $(p, p^{\frac{1}{\lambda}})$ there no exist prime divisor of n. Here $0 < \lambda < 1$.

We proved that, if $\epsilon > 0$ is am arbitrary fixed number, then

$$\lim_{x \to \infty} \frac{1}{x} \sharp \Big\{ n \leq x : \quad \Big| \frac{\mathcal{U}_\lambda(n)}{\omega(n)} - \lambda \Big| > \epsilon \Big\} \to 0$$

and that

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sharp \Big\{ n \leq x : \quad \Big| \frac{\mathcal{U}_\lambda(p+1)}{\omega(p+1)} - \lambda \Big| > \epsilon \Big\} \to 0.$$

Let $f_\lambda(n) = \mathcal{U}_\lambda(n) - \lambda\omega(n)$. Our conjecture is the following.

Conjecture 1. *We have that for every $u \in \mathbb{R}$*

$$\lim_{x \to \infty} \frac{1}{x} \sharp \Big\{ n \leq x : \ \frac{f_\lambda(n)}{c(\lambda)\sqrt{\log\log n}} < u \Big\} \to \Phi(u),$$

$c(\lambda)$ is a suitable positive constant.

The first step to prove it would be to prove that

$$\frac{1}{x} \sum_{n \leq x} f_\lambda^2(n) = (1 + o_x(1)) c(\lambda) \log\log x.$$

Highly probable our conjecture is true for

$$\frac{f_\lambda(p+1)}{c(\lambda)\sqrt{\log\log(p+1)}}.$$

We remark that Conjecture 1 is proved by A. Sofos in arxiv: 2106.00298v3

3.2. In [7] we investigated the following question.

Let $a(n) = n(n+1)$ $(n \in \mathbb{N}, n \geq 2)$. Let $p_1 \leq p_2 \leq \cdots \leq p_k$ be the complete list of the prime divisors of $a(n)$. Let $s_n : \{p_1, \cdots, p_k\} \to \{0, 1\}$.

We write

$$s_n(p_j) = \begin{cases} 0 \text{ if } & p_j | n \\ 1 \text{ if } & p_j | n+1 \end{cases}$$

Then consider the binary sequence:

$$h(n) := s_n(p_1) \cdots s_n(p_k).$$

We proved that

$$\xi = 0, h(2)h(3)\ldots$$

is a binary normal number.

To prove it we considered

$$K(n|\delta_1 \ldots \delta_\ell) := \sharp\{j \in \{1, \ldots, k-\ell\} \mid s_n(p_{j+r}) = \delta_r, r = 1, \ldots \ell\}.$$

Here $k = \Omega(n) + \Omega(n+1)$.

We proved that, for every fixed $\delta_1, \ldots, \delta_\ell$

$$\lim_{x\to\infty} \frac{1}{x} \sharp\Big\{n \le x \ : \ \Big| \frac{K(n|\delta_1 \ldots \delta_\ell)}{2 \log\log n} - \frac{1}{2^\lambda}\Big| > \epsilon\Big\} = 0$$

holds for every fixed $\epsilon > 0$.

Conjecture 2. *We have that*

(A) $$\frac{1}{x} \sum_{n \le x} (2^\ell K(n|\delta_1 \ldots \delta_\ell) - 2 \log\log x)^2 = c(1 + o_x(1)) \log\log x,$$

moreover that

(B) $$\Theta(n) := \frac{2^\ell K(n|\delta_1 \cdots \delta_\ell) - 2 \log\log x}{d_\ell \sqrt{\log\log n}}$$

is distributed according to the normal law. Here d_ℓ is a suitable positive constant.

Let $b(q) = (q-1)(q+1)$ $(q \in \mathcal{P})$. Let

$$b(q) = 2^\alpha p_1 \cdots p_k, \quad 2 < p_1 \le \cdots \le p_k$$

and

$$\overline{s}_n(p_j) = \begin{cases} 0 \text{ if } & p_j | q-1 \\ 1 \text{ if } & p_j | q+1. \end{cases}$$

Let

$$\overline{h}(n) := \overline{s}_n(p_1) \cdots \overline{s}_n(p_k),$$

$$\overline{K}(q|\delta_1 \ldots \delta_\ell) := \sharp\{s_q(p_{j+r}) = \delta_r, \quad r = 1, \ldots \ell\},$$
$$k = \Omega(b(q)) - \sum_{2^q | b(q)} 1.$$

By our method we can prove that

$$\eta = 0, \overline{h}(3)\overline{h}(5) \ldots \overline{h}(q) \ldots$$

is a binary normal number.

We can prove that

$$\text{(3.1)} \qquad \frac{1}{li\, x} \sum_{q \le x} \left(\overline{K}(q|\delta_1 \cdots \delta_\ell) - \frac{2 \log\log x}{2^k} \right)^2 = o\left(x(\log\log x)^2 \right).$$

Conjecture 3. *We guess that*

$$\text{(C)} \qquad \frac{1}{li\, x} \sum_{q \le x} \left(\overline{K}(q|\delta_1 \cdots \delta_\ell) - \frac{2 \log\log x}{2^k} \right)^2 = c(1 + o_x(1)) \log\log x$$

with a suitable constant $c > 0$, and that

$$\text{(D)} \qquad \frac{1}{\pi(x)} \sharp\{q \le x \mid \frac{\overline{K}(q|\delta_1 \ldots \delta_\ell) - \frac{2\log\log x}{2^k}}{d_1 \sqrt{\log\log x}} u\} = \Phi(u).$$

Here d_1 is a suitable positive constant.

3.3. Let $\mathcal{R}_{A,B} = \{n \in \mathbb{N} | \Omega(n) = A, \Omega(n+1) = B\}$, $k = A + B$. Let $\mathcal{T}_{A,B}$ be the set of those $\delta_1, \cdots, \delta_k \in \{0,1\}^k$ sequences in which 0 occur exactly A-times (and then 1 occur B-times).

Conjecture 4. *Let $\Theta_1, \Theta_2 \in \mathcal{T}_{A,B}, \Theta_1 \ne \Theta_2$. Then, under the condition*

$$\max\{|A - \log\log x|, |B - \log\log x|\} < c\sqrt{\log\log x},$$

we have

$$\sup_{\Theta_1, \Theta_2 \in \mathcal{R}_{A,B}} \left| \frac{\sharp\{n \le x \mid n \in \mathcal{R}_{A,B}, h(n) = \Theta_1\}}{\sharp\{n \le x \mid n \in \mathcal{R}_{A,B}, h(n) = \Theta_2\}} - 1 \right| \to 0 \quad \textit{as} \quad x \to \infty.$$

3.4. Let $u_q(n) = \Omega(n) \pmod q, q \in \mathbb{N}, q \ge 2$

Open problem. Let $(2 \le) q_1 < q_2 < \cdots$ be an infinite sequence of pairwise coprime integers. How can we construct an infinite sequence of integers $\mathcal{B} = \{q_1 < q_2 < \cdots\}$ such that

$$\xi_j = 0, u_{q_j}(a_1) u_{q_j}(a_2) \ldots,$$

are q_j-ary normal numbers for every $j = 1, 2, \cdots$, and more over that

$$x_m = \begin{cases} [\xi_1, q_1^m] \\ \dots \\ [\xi_k, q_k^m] \end{cases} \quad (m = 1, 2 \dots)$$

is uniformly distributed in $[0, 1]^k$.

I do not know how we can construct $\xi_1, \xi_2 \dots$ with these properties.

4. The distribution of prime numbers in short intervals and some consequences

4.1. K. Ramachandra [26] proved that

$$\pi(x+h) - \pi(x) = \frac{h}{\log x} + O\left(\frac{h}{\log^2 x}\right) \tag{4.1}$$

if

$$x^{\frac{7}{12}+\epsilon} \le h \le x, \quad \epsilon \text{ arbitrary constant.} \tag{4.2}$$

His main observation was to use a complicated contour to estimate

$$\int \frac{(x+h)^s - x^s}{s} \cdot \frac{\xi'(s)}{\xi(s)} ds,$$

the so called modified Hooley–Huxley contour. The contour depends on the estimation of $N(\sigma, T)$, that is the number of roots of $\xi(s)$ in the domain $\mathrm{Re}\, s >$ $> \sigma$, $|\mathrm{Im} s| \le T$.

By this he improved an older results of Huxley, namely that (4.1) holds under the condition

$$x^{5/8} \le h \le x. \tag{4.3}$$

By using the method of Ramachandra I proved [19]:

Let
$$\mu_k(x) = \frac{(\log\log x)^{k-1}}{(k-1)!\log x}, \quad R_x = \log\log x + c_x\sqrt{\log\log x}$$

$c_x \to \infty$ appropriately slowly.

Theorem 2. *Under the condition* (4.1) *we have*

$$\frac{1}{k}\sharp\{n \in [x, x+h] | \omega(n) = k\} = (1 + o_x(1))\, \mu_k(x) \tag{4.4}$$

uniformly as $1 \le k \le R_x$. *Consequently*

$$\max_{u\in\mathbb{R}} \max_{h under\ (4.2)} \left|\frac{1}{h}\sharp\Big\{n \in [x, x+h] \mid \frac{\omega(n) - \log\log x}{\sqrt{\log\log x}} < u\Big\} - \Phi(u)\right| \to 0 \tag{4.5}$$

as $x \to \infty$.

Repeating the procedure of Ramachandra one can obtain that

$$\pi(x+h, q, a) - \pi(x, q, a) = \frac{1}{\varphi(q)} \cdot \frac{h}{\log x} + O\Big(\frac{h}{\log^2 x}\Big) \tag{4.6}$$

for every fixed $q \ge 3$, and $(a, q) = 1$.

4.2. Let $(2 \le)\, A$ be a constant,

$$(\log x)^A \le Y(= T(x)) \le (\log x)^{A+1}.$$

Let

$$S(X, Y) = \{n \le X \mid p(n) > Y\}, \tag{4.7}$$

$$N(X, Y) = \sharp S(X, Y). \tag{4.8}$$

By using the classical sieve method, we obtain that

$$N(X, Y) = (1 + o_x(1))\, \frac{c_1 X}{\log Y}, \quad c_1 = e^{-\gamma}. \tag{4.9}$$

The following remark quite obvious:

$$\sharp\{n \in S(X, Y) \mid \mu(n) = 0\} \le \frac{c_2 X}{\log Y}. \tag{4.10}$$

Let

$$S_k(X, Y) = \{n \in S(X, Y) \mid \omega(n) = k\}. \tag{4.11}$$

One can prove that

$$N_k(X, Y) = \sharp S_k(X, Y) = (1 + o_x(1)) N(X, Y) \rho_k(X, Y) \tag{4.12}$$

uniformly as $1 \le k \le R_x$, (R_x is defined earlier), where

$$\rho_k(X, Y) = \frac{(\log\log x - \log\log Y)^{k-1}}{(k-1)!\,\log x}. \tag{4.13}$$

Let

$$f(p) = \begin{cases} \dfrac{1}{\log p} & \text{if} \quad p \in [Y, X] \\ 0 \quad \text{if} & p \notin [Y, X]. \end{cases}$$

Let f be additive. Let

$$\sum\nolimits_k = \sum_{\substack{n \le x \\ n \in S_k(X,Y)}} f(n). \tag{4.14}$$

We can prove that

$$\sum\nolimits_k < \frac{cN_k(X,Y)}{\log x} \tag{4.15}$$

uniformly as $k \le R_x$.

It is clear that (4.15) is true for $k = 1$. Let $k \ge 2$. We have

$$\sum\nolimits_k \le \sum_{Y < p < \sqrt{x}} \frac{1}{\log p} N_{k-1}\left(\frac{X}{p}, Y\right) + \sum_{Y < p < \sqrt{x}} \frac{1}{\log p} N\left(\frac{X}{p^2}, Y\right) + \\ + \sum_{\sqrt{x} < p < x} \frac{1}{\log p} N_{k-1}\left(\sqrt{X}, Y\right).$$

Since

$$\max_{\substack{\sqrt{x} \le u \le x \\ k \in R_x}} \frac{N_{k-1}(u,Y)}{N_k(u,Y)} = O(1),$$

and

$$\sum_{Y < p < X} \frac{1}{p \log p} \ll \frac{1}{\log Y},$$

therefore (4.15) is true.

4.3. Let $q \ge 3$. $\mathcal{A}_q = \{\ell_1, \cdots, \ell_\varphi(q)\}$ be the set of reduced residue classes (mod q).Let $\kappa(p) = a$ if $p \equiv \ell_a \pmod q$. For some n, coprime to q, $|\mu(n)| = 1$ let $n = p_1 \cdots p_r$, $p_1 < \cdots < p_r$. Write $\kappa(n) = \kappa(p_1) \cdots \kappa(p_r)$.

Let $\mathcal{H}_q$ be the set of words composed from $\{1, \cdots, \varphi(n)\}$. For some $\alpha \in \mathcal{H}_q$ let $\lambda(\alpha)$ be the length of α.

Let

$$S_k(X, Y, \alpha) = \{n \in S_k(X, Y, \alpha) | \quad |\mu(n)| = 1, \kappa(n) = \alpha\}$$

and

$$N_k(X, Y, \alpha) = \sharp S_k(X, Y, \alpha).$$

Theorem 3. *We have*

$$\varphi(q)^k N_k(X,Y,\alpha) = (1+o_x(1))N_k(X,Y)$$

uniformly as $\alpha \in \mathcal{H}_q$, $\lambda(\alpha)=k$, $k \le R_x$. *In the other words*

$$\sup_{1\le k\le R_x} \max_{\substack{\alpha,\beta\in\mathcal{H}_q\\ \lambda(\alpha)=\lambda(\beta)=k}} \left|\frac{N_k(X,Y,\alpha)}{N_k(X,Y,\beta)} - 1\right| \to 0 \quad \textit{as} \quad x\to\infty.$$

Proof. Let $N_0 = Y, \mathcal{F}_\ell = [N_\ell, N_{\ell+1}]$, where $N_{\ell+1} = N_\ell + N_\ell^{\frac{5}{8}}$ $(\ell = 0,\dots,T)$ and T is defined by $N_T \le x \le N_{T+1}$. (Note: we shall use (4.1) instead of (4.4))

Let us consider those $n \in S_k(X,Y,\alpha)$ for which in every interval $\mathcal{F}_\ell$ $(\ell = = 0,\dots,T)$ no more than one prime divisors of n exists.

Let $s_1 < \cdots < s_k$ $(\le T)$, $\underline{s} = \{s_1,\dots,s_k)$ and $\mathcal{M}(s)$ be the set of those $n = p_1\dots p_k$ for which $p_j \in \mathcal{F}_{s_j}(j=1,\dots k)$, and $\mathcal{M}(s,\alpha)$ be the set of those n for which additionally $\kappa(n) = \alpha$ holds.

Let $M(\underline{s}) = \sharp\mathcal{M}(\underline{s}), M(\underline{s},\alpha) = \sharp\mathcal{M}(\underline{s},\alpha)$. Let

$$U_{\underline{s}} = \prod_{j=1}^k N_{s_j}, \quad V_{\underline{s}} = \prod_{j=1}^k N_{s_j+1}.$$

If there exists $n \in \mathcal{M}(\underline{s})$ for which $n \le x$, then $U_{\underline{s}} \le x$, and if there some $n > x$ in $\mathcal{M}(\underline{s})$, then $V_{\underline{s}} > x$. Let $\underline{s} \in \mathcal{A}$ if $V_{\underline{s}} \le x$, and $\underline{s} \in \mathcal{B}$ if $U_{\underline{s}} \le x < V_{\underline{s}}$.

If $n \in \bigcup_{s\in\mathcal{B}} \mathcal{M}(\underline{s})$, then

$$n \le x\prod_{j=1}^k (1 + N_{s_j}^{-3/8}) \le x(1+Y^{-3/8})^k \le$$

$$\le x\exp\left(2(\log\log x)(\log x)^{-3/8A}\right) \le x + O\left(\frac{x}{\log x}\right)$$

and

$$n \ge x\prod_{j=1}^k (1 - N_{s_j+1}^{-3/8}) \ge x - O\left(\frac{x}{\log x}\right).$$

Consequently

$$\sum_{\underline{s}\in\mathcal{B}} M(\underline{s}) = O\left(\frac{x}{\log x}\right). \tag{4.16}$$

Let now $\underline{s}$ be such a vector for which $U_{\underline{s}} \le x$. From (4.1) we obtain that

$$\frac{M(\underline{s}|\alpha)}{M(\underline{s})} = \frac{1}{\varphi(q)^k}\prod_{j=1}^k \left(1 + O\left(\frac{1}{\log N_{s_j}}\right)\right). \tag{4.17}$$

The product of the right hand side is bounded since $R_x/\log N_0 = O(1)$. Hence

$$\sum_{\underline{s}\in\mathcal{B}} M(\underline{s}|\alpha) < \frac{1}{\varphi(q)^k}\frac{x}{\log x} \tag{4.18}$$

follows.

Let

$$\nu(\underline{s}) = \sum_{j=1}^{k} \frac{1}{\log N_{s_j}}.$$

Then (4.17) can be rewrite as

$$\left|\varphi(q)^k M(\underline{s}|\alpha) - M(\underline{s})\right| \le c\nu(\underline{s})M(\underline{s}). \tag{4.19}$$

Observe that

$$f(n) = \sum_{p|n}\frac{1}{\log p} = \nu(\underline{s}) + O\left(\frac{1}{\sqrt{n}}\right)$$

if $n \in \mathcal{M}(\underline{s})$. From (4.14), (4.15) we obtain that

$$\varphi(q)^k \sum_{\underline{s}\in\mathcal{A}} M(\underline{s}|\alpha) - \sum_{\underline{s}\in\mathcal{A}} M(\underline{s}) = O\Big(\sum_{n\le x} f(n)\Big) = O\Big(\frac{N_k(X,Y)}{\log Y}\Big).$$

It remains to estimate the contribution of those n for which there are at least two prime divisors in one of $\mathcal{F}_\ell$ $(\ell = 0\cdots, T)$. We can use the same method to prove that contribution of these n is small. We omit the proof. ∎

5. On interval filling sequences

Assume that $(0<)\lambda_n$ tends to zero monotonically. Let $L_n = \lambda_{n+1} + L_{n+2} + + \cdots$. Assume that $L_0 < \infty$. Let

$$S(\{\lambda_n\}) = \Big\{x = \sum \epsilon_n\lambda_n \mid \epsilon_n \in \{0,1\}\Big\}.$$

We say that $\{\lambda_n\}$ is an interval filling sequence if $S(\{\lambda_n\})$ is an interval. Since $0, L_0 \in S(\{\lambda_n\})$, therefore it means that $S(\{\lambda_n\}) = [0, L_0]$. According to a theorem of S. Kakeya [19] a sequence $\lambda_n \downarrow 0$ is an interval filling sequence if and only if $\lambda_n \le L_{n+1}$ $(n \in \mathbb{N}_0)$.

We say that F is an additive function with respect to the interval filling sequence $\{\lambda_n\}$, if

$$F(x) = \sum_{n=1}^{\infty} \epsilon_n F(\lambda_n)$$

for every $x \in [0, L_0]$, where $x = \sum_{n=1}^{\infty} \epsilon_n \lambda_n$ is the regular expansion of x. The regular expansion means that $\epsilon_1 = 1$ iff $x \geq \lambda_1$ and in general

$$\epsilon_N = 1 \iff \sum_{j=1}^{N-1} \epsilon_j \lambda_j + \lambda_N \leq x.$$

We wrote some papers with Z. Daróczy and A. Járai on such additive functions several years ago ([1], [3], [4]).

The next question seems to be hard.

Let $K = \{0 = K_0, K_1, \ldots, K_t\} \subset \mathbb{C}$. Let W_j $(j = 1, 2, \ldots)$ be a sequence of complex numbers such that $|W_j| \neq 0$ $(j \in \mathbb{N})$, furthermore that $\sum_{j=1}^{\infty} |W_j| <$ $< \infty$.

Let

$$S(\{W_j\}|K) = \{z = \sum_{j=1}^{\infty} \epsilon_j W_j | \epsilon_n \in K\}.$$

Open problem: *Give necessary and sufficient condition for $S(\{W_j\}|K)$ to be a connected domain the 0 of which is an interior point.*

Let $\{\lambda_n\}$ be an interval filling sequence, $z_1 = e(\theta) = U + iV$, $0 < \theta < \frac{1}{2}$. Let

$$K = \{0, 1, z_1, 1 + z_1\}.$$

Then

$$S(\{W_j\}|K) = \Big\{ u + v z_1 | u, v \in S(\{\lambda_n\}|\{0, 1\}) \Big\}$$

is a paralelogramma with endpoints $(0, 0), (0, L_0), L_0 U, L_0 V), (L_0 U + L_0, L_0 V)$.

Let us choose $z_1 = \omega = e(\frac{1}{3}), z_2 = \overline{\omega}$, $K = \{0, -1, 1, \omega, -\omega, \overline{\omega}, -\overline{\omega}\}$. Then $S(\{\lambda_n\}|K)$ is a hexagon with the endpoints $\pm L_0, \pm\omega L_0, \pm\overline{\omega} L_0$.

This is very special case. More than ten years ago Prof. M. Laczkovich proved my conjecture: *Let ℓ_1 be a continuous curve connecting 0 and $A, A \neq 0$. Let ℓ_2 be another curve connection 0 and $B, B \notin \ell_2$. Then there exists an interior point in $\{z_1 + z_2 | z_1 \in \ell_1, z_2 \in \ell_2\}$.*

6. Mean values of *q*-multiplicative function over the set of primes

Let $q \geq 2, q \in \mathbb{N}, \mathcal{A}_q = \{0, \cdots, q - 1\}$. We say that $g : \mathbb{N}_0 \to \mathbb{C}$ is a q-multiplicative function if $g(n) = \sum_{j=0}^{k} g(\epsilon_j q^j)$, if $n = \sum_{j=0}^{k} \epsilon_j q^j$ $(\epsilon_j \in \mathcal{A}_q)$. Let $\mathcal{M}_q$ be the set of q-multiplicative functions, and $\mathcal{M}_q^{(1)}$ be those for which additionally $|g(m)| \leq 1$ holds.

Assume that $g \in \mathcal{M}_q^{(1)}$, and

$$\sum_{j=0}^{\infty}\sum_{a=0}^{q-1}(g(aq^j)-1) \tag{6.1}$$

is convergent.

Let

$$P(x) := \sum_{p\leq x} g(p). \tag{6.2}$$

In [17] we proved that in this case

$$M_q := \lim_{x\to\infty}\frac{P(x)}{\pi(x)} \tag{6.3}$$

exists, furthermore

$$M_q = \prod_{j=0}^{\infty} K_j,$$

where

$$K_0 = \frac{1}{\varphi(q)}\sum_{(a,q)=1} g(a) \quad \text{and} \quad K_j = \frac{1}{q}\sum_{a=0}^{q-1} g(aq^j) \quad (j\geq 1).$$

Consequently, if $K_j \neq 0 \ \ (j\in\mathbb{N}_0)$, then $M_q \neq 0$.

Conjecture 5. *Let $g \in \mathcal{M}_q^{(1)}$. Assume that (6.3) exists, and $M_q \neq 0$. Then $K_j \neq 0 \ \ (j \in \mathbb{N}_0)$ and (6.1) holds true.*

Let $g \in \mathcal{M}_q^{(1)}$,

$$S(x|\alpha) = \sum_{\substack{\ell\leq x\\ (\ell,q)=1}} g(\ell)e(\alpha\ell). \tag{6.4}$$

We would like to know under which condition holds

$$\frac{P(x)}{\pi(x)} \to 0 \ \ (x\to\infty). \tag{6.5}$$

Conjecture 6. (6.5) *holds if and only if*

$$\frac{S(x|r)}{r} \to 0 \qquad \text{as} \ \ x\to\infty \tag{6.6}$$

for every $r\in\mathbb{R}$.

In [17] we proved that (6.5) implies (6.6).

Let $Y(x)$ be monotonically increasing, $Y(x) \to \infty$, $\frac{\log Y(x)}{\log x} \to 0$ as $x \to \to \infty$,

$$\mathcal{N}_x := \{n \le x \mid p(n) > Y_x\}, \quad N(x) = \sharp \mathcal{N}_x.$$

Let

$$L(p) := \begin{cases} \dfrac{1}{p-2} & \text{if} \quad p > 2, p \nmid q \\ 0 & \text{otherwise.} \end{cases}$$

Let L be strongly multiplicative.

Let $g \in \mathcal{M}_q^{(1)}$,

$$U(x) = \sum_{\substack{n \le x \\ n \in \mathcal{N}_x}} g(n). \tag{6.7}$$

In [19] we proved the following theorem.

Theorem 4. *Under the above conditions, we have*

$$\left|\frac{U(x)}{N(x)}\right|^2 \le \sum_{d<\infty} \frac{L(d)}{d} \sum_{a=0}^{d-1} \left| q^{-M} S\left(q^M | \frac{a}{d}\right)\right|^2 + \frac{c_1}{D} + o_x(1), \tag{6.8}$$

where M is an integer satisfying

$$q^{-1} x^{\frac{1}{4}} \le q^M \le q x^{\frac{1}{4}},$$

where c_1 is a positive constant, which may depend only on q, $o_x(1)$ depends on Y_x, $D \ge 1$ is an arbitrary integer.

If (6.6) holds, then $\dfrac{U(x)}{N(x)} \to 0$ *as* $x \to \infty$.

7. On a functional equation with polynomials

In our paper written together with Z. Daróczy [4] we investigated the equation

$$Q(S(x)) = cQ(x)Q(x+1), \tag{7.1}$$

where S, Q are polynomials in $\mathbb{C}[x]$, $\deg S = 2$.

Let $\mathcal{A} := \{\beta_1, \cdots, \beta_n\}$ be the roots of Q. Q is a solution of (7.1) with some $S(x) = Ax^2 + E$, where $AE \ne 0$, if

$$\mathcal{A} = \{1-\beta_1, \cdots, 1-\beta_n\} = \{S(\beta_1), \cdots, S(\beta_n)\}. \tag{7.2}$$

We could determine the solutions of (7.2) if Q is a polynomial with real coefficients, or if Q has a real root.

Open question. *Let $S(x) = Ax^2 + E, AE \neq 0,$ and let $\mathcal{A} = \{\beta_1, \cdots, \beta_n\}$ be a set of n complex numbers, satisfying the conditions*

$$\mathcal{A} = \{1-\beta_1, \cdots, 1-\beta_n\} = \{S(\beta_1), \cdots, S(\beta_n)\}.$$

How can we characterize the set $\mathcal{A}$?

Kovács A. [25] proved an interesting partial result concerning this question.

Naturally, the solution of the equation

$$Q(S(x)) = c\prod_{j=1}^{k} Q(x+\lambda_j), \quad \deg S = k$$

seems to be much harder.

8. On some uniformly summable functions on the set of primes

In our paper [12] we investigated the sum

$$\sum_{(p+1)g(p+1)\leq x} 1,$$

where g is a positive multiplicative function with light condition on primes.

We mentioned that we are unable to give the asymptotic of

$$\sum_{(p+1)\tau(p+1)\leq x} 1 \quad \text{or} \quad \sum_{(p+1)2^{\omega(p+1)}\leq x} 1.$$

The problem is almost the same as to give the asymptotic of

$$\sharp\{p \leq x \mid \omega(p+1) = k\}.$$

Let $\leq Y(x) < x, \ \dfrac{\log Y(x)}{\log x} \to 0 \quad \text{as} \quad Y(x) \to \infty.$ Let

$$S(X,Y) = \{n \leq x \mid p(n) > Y\} \quad \text{and} \quad N(X,Y) = \sharp S(X,Y).$$

As we known,

$$N(X,Y) = (1+o_x(1))\,\frac{e^{-\gamma}X}{\log Y} \quad \text{as} \quad X \to \infty.$$

Highly probable by using the Selberg method we can determine the asymptotic of

$$\frac{\sharp\{n \le x, p(n) > Y, \omega(n+1) = k\}}{N(X,Y)}$$

uniformly as $1 \le k \le R_x$, and hence we can give the asymptotic of

$$\sum_{\substack{(n+1)2^{\omega(n+1)} \le x \\ n \in S(X,Y)}} 1.$$

9.

Let α, β be positive real numbers such that $\frac{\alpha}{\beta} \notin \mathbb{Q}$. In joint papers written with B. M. Phong ([20]–[24]) we formulated the following conjecture:

Conjecture 7. *If $f \in \mathcal{M}, f(n) \in \mathcal{U}$, and there exists some C for which either*

(a) $$f([\beta n]) - Cf([\alpha n] \to 0 \quad \textit{as} \quad n \to \infty$$

or

(b) $$\frac{1}{\log x} \sum_{n \le x} \frac{f([\beta n]) - Cf([\alpha n])}{n} \to 0,$$

then $f(n) = n^{i\tau}$.

We could prove this conjecture in the special case, when $\alpha = 1, \beta = \sqrt{2}$. Try to prove it for $\alpha = 1, \beta = \sqrt{3}$.

References

[1] **Daboussi, H. and H. Delange,** Quelques proprié tés des fonctions multiplicatives de module au plus égal á 1. (French), *C. R. Acad. Sci. Paris Sér. A*, **278** (1974), 657–660.

[2] **Daróczy, Z., A. Járai and I. Kátai,** Interval filling sequences, *Annales Univ. Sci. Budapest., Sect. Comp.*, **6** (1985), 53–63.

[3] **Daróczy, Z. and I. Kátai,** Interval filling sequences and additive functions, *Acta Sci. Math. (Szeged)*, **52(3–4)** (1988), 337–347.

[4] **Daróczy, Z. and I. Kátai,** On a functional equation with polynomials, *Acta Math. Hungar.*, **52(3–4)** (1988), 305–320.

[5] **De Koninck, J.-M. and I. Kátai,** Consecutive neighbour spacings between the prime divisors of an integer, *Colloq. Math.*, **170(2)** (2022), 289–305.

[6] **De Koninck, J.-M. and I. Kátai,** On the consecutive prime divisors of an integer, *Funct. Approx. Comment. Math.*, **66(1)** (2022), 7–33.

[7] **De Koninck, J.-M. and I. Kátai,** Creating normal numbers using the prime divisors of consecutive integers, *Distr. Theory*, (accepted)

[8] **De Koninck, J.-M. and I. Kátai,** Exponential sums running over particular sets of positive integers, *Annales Univ. Sci. Budapest., Sect. Comp.*, **51** (2020), 51–57.

[9] **Harman, G.,** On exponential sums studied by Indlekofer and Kátai, *Acta Math. Hungar.*, **124(3)** (2009), 289–298.

[10] **Indlekofer, K.-H. and I. Kátai,** Some remarks on trigonometric sums, *Acta Math. Hungar.*, **118(4)** (2008), 313–318.

[11] **Indlekofer, K.-H., I. Kátai, O.I. Klesov and B.M. Phong,** A remark on uniformly distributed functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 167–172.

[12] **Indlekofer, K.-H., I. Kátai and B.M. Phong,** On some uniformly distributed functions on the set of shifted primes, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 173–181.

[13] **Kakeya, S.,** On the set of partial sums of an infinite series, *Tohoku Sic. Rep.*, **3** (1914), 159–164.

[14] **Kátai, I.,** A remark on a theorem of H. Daboussi, *Acta Math. Hungar.*, **47(1–2)** (1986), 223–225.

[15] **Kátai, I.,** On the Turán–Kubilius inequality, *Number Theory, Analysis, and Combinatorics, Proc. in Math.*, 177–182.

[16] **Kátai, I.,** Distribution of q-additive functions, Kluwer.

[17] **Kátai, I.,** Some remarks and problems on q-additive functions, *Arithmetical functions*, 57–70, Leaflets in Math.

[18] **Kátai, I.,** A remark on trigonometric sums, *Acta Math. Hungar.*, **112(3)** (2006), 221–225.

[19] **Kátai, I.,** A remark on a paper of Ramachandra, in: *Number Theory, Proc. Ootacamund*, K. Alladi (Ed.), Lecture Notes in Math. 1122, Springer (1984), pp. 147–152.

[20] **Kátai, I. and B.M. Phong,** On the multiplicative group generated by $\{\frac{[\sqrt{2}n]}{n} | n \in \mathbb{N}\}$, *Acta Math. Hungar.*, **145(1)** (2015), 80–87.

[21] **Kátai, I. and B.M. Phong,** On the multiplicative group generated by $\{\frac{[\sqrt{2}n]}{n} | n \in \mathbb{N}\}$ II., *Acta Sci. Math.* (*Szeged*), **81(3–4)** (2015), 431–436.

[22] **Kátai, I. and B.M. Phong,** On the multiplicative group generated by $\{\frac{[\sqrt{2}n]}{n} | n \in \mathbb{N}\}$ III., *Acta Math.Hung.*, **147** (2015), 247–254.

[23] **Kátai, I. and B.M. Phong,** On the multiplicative group generated by $\{\frac{[\sqrt{2}n]}{n} | n \in \mathbb{N}\}$ IV., *Panonica*, Pécs, **25/1** 2014–2015, 105–112.

[24] **Kátai, I. and B.M. Phong,** On the multiplicative group generated by $\{\frac{[\sqrt{2}n]}{n} | n \in \mathbb{N}\}$ V., *Notes on Number Theory and Discrete Mathematics*, **29(2)** (2023), 348–353.

[25] **Kovács, A.,** Sets of complex numbers generated from a polynomial functional equation, *Annales Univ. Sci. Budapest., Sect. Comp.*, **18** (1999), 115-–124.

[26] **Ramachandra, K.,** Some problems of analytic number theory, *Acta Arith.*, **31** (1976), 313–324.

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