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ASYMPTOTIC BEHAVIORS OF SOME ARITHMETIC FUNCTION ASSOCIATED WITH THE VON MANGOLDT FUNCTION

Kouji Yamamuro (Gifu, Japan)

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Abstract. Some function associated with the von Mangoldt function is investigated. It is related to the logarithm of the Riemann zeta function. By means of probability theory, we show that this function is bounded above and below by a certain function. It is possible that the result extends to Dirichlet series.

1. Introduction and main result

Let Λ denote the von Mangoldt function which is defined as follows:

$$\Lambda(n) \quad = \quad \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

for every integer $n \geq 1$. The Riemann zeta function $\zeta(s)$ is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

for $s = \sigma + it$ with $\sigma > 1$. As stated in (1.1.9) of [12], its logarithm has the expression

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n}. \tag{1.2}$$

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Then we have

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n}(\cos(t \log n) - i \sin(t \log n)).$$

As a related study, Gonek [3] investigated behaviors of

$$\sum_{n=2}^{x^2} \frac{\Lambda_x(n) \sin(t \log n)}{n^{\frac{1}{2}} \log n}$$

for $2 \le x \le t^2$, where

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{if } n \le x, \\ \Lambda(n)(2 - \frac{\log n}{\log x}) & \text{if } x < n \le x^2, \\ 0 & \text{if } x > x^2. \end{cases}$$

We note that

$$\log |\zeta(s)| = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma \log n} \cos(t \log n).$$

Akatsuka [1] treated asymptotic behaviors of

$$\sum_{n=2}^{x} \frac{\Lambda(n)}{n^{s_0} \log n}$$

for $s_0 = \sigma_0 + it$ with $\sigma_0 \in [2^{-1}, 1)$. In this paper we investigate asymptotic behaviors of the arithmetic function

$$\Pi_0(x) := \sum_{n > x} \frac{\Lambda(n)}{n^\sigma \log n}$$

for $\sigma > 1$. We focus on the Riemann zeta distribution and evaluate Π_0 by means of probability theory. Let $\sigma > 1$. The Riemann zeta distribution Ψ is an infinitely divisible distribution whose characteristic function is represented as

$$\begin{aligned} \hat{\Psi}(t) &= \int_{\mathbb{R}} e^{itu} \Psi(du) = \\ &= \exp\left[\sum_{n=2}^{\infty} (e^{it \log n} - 1) \frac{\Lambda(n)}{n^\sigma \log n}\right] = \\ &= \exp\left[\int_0^{\infty} (e^{itu} - 1) \Pi(du)\right], \end{aligned}$$

where the Lévy measure Π is as follows:

$$\Pi(du) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{\sigma} \log n} \delta_{\log n}(du).$$

Here δ_a represents the probability measure concentrated at a, that is,

$$\delta_a(B) = \begin{cases} 1 & \text{if } a \in B, \\ 0 & \text{otherwise} \end{cases}$$

for any Borel set B. We see from (1.2) that

$$\hat{\Psi}(t) = \frac{\zeta(\sigma - it)}{\zeta(\sigma)}, \quad t \in \mathbb{R}.$$

In [5], Gnedenko and Kolmogorov have shown that $\frac{\zeta(\sigma-it)}{\zeta(\sigma)}$ is a characteristic function of an infinitely divisible distribution. Recently, Lin and Hu [6] considered the Riemann zeta distribution and Nakamura [7] focused on it. Now we set

$$\lambda = \Pi((0,\infty)) = \log \zeta(\sigma) \quad \text{and} \quad \rho(du) = \lambda^{-1}\Pi(du).$$

We notice that ρ is a probability measure on $(0,\infty)$. Then it follows that

$$\Pi_0(x) = \Pi(\{u : u > \log x\}) = \lambda\rho(\{u : u > \log x\}).$$

For any finite measure H, we denote by $\overline{H}$ the tail of H, that is,

$$\overline{H}(x) = H(\{u : u > x\}).$$

Hence we have

$$\Pi_0(x) = \lambda\overline{\rho}(\log x).$$

Theorem 1.1. *Let $\sigma > 1$. There is a positive constant C_0 such that*

$$C_0 \sum_{n>x} \frac{1}{n^{\sigma}} \le \Pi_0(x) = \sum_{n>x} \frac{\Lambda(n)}{n^{\sigma} \log n} \le \sum_{n>x} \frac{1}{n^{\sigma}}.$$

for all positive integers x.

We show that Theorem 1.1 is extended to Dirichlet series in the last section. There are a lot of studies on the tails of infinitely divisible distributions. For example, see [8], [9], [11], [13] and [14]. Furthermore, see Sato's book [10] for infinitely divisible distributions. Terminology follows Sato's book [10].

2. Proof of Theorem 1.1

We make preparations for proving Theorem 1.1.

Lemma 2.1. *For any positive integer x,*

$$\Pi_0(x) \le \sum_{n>x} \frac{1}{n^\sigma}.$$

Proof. We see from Theorem 296 of [4] that

$$\Lambda(n) \le \sum_{d|n} \Lambda(d) = \log n.$$

Hence the lemma holds. ■

We use the following convolution for two finite measures H_1 and H_2 on $(0, \infty)$.

$$H_1 * H_2(B) = \int_{(0,\infty)} H_1(B - x) H_2(dx)$$

for any Borel set B in $(0, \infty)$. Let $A > 0$ and define a finite measure R_λ by

$$R_\lambda(du) = A^{-1} e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \rho^{n*}(du) \tag{2.1}$$

on $(0, \infty)$. Here ρ^{n*} denotes the n-fold convolution of ρ.

Lemma 2.2. *For $x \ge 0$, we have*

$$A\overline{R_\lambda}(x) \quad = \quad \overline{\Psi}(x). \tag{2.2}$$

Proof. Now Ψ has a representation as follows:

$$\Psi(du) \quad = \quad e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \rho^{n*}(du),$$

where ρ^{0*} is understood to be δ_0. The equation (2.1) tells us that (2.2) holds. The lemma has been proved. ■

Take λ_0 such that $\lambda_0 > \max\{\lambda, \ \log 2\}$ and take $A > 0$ such that $Ae^{\lambda_0} < 1$. We define a new finite measure R_{λ_0} by

$$R_{\lambda_0}(du) = A^{-1} e^{-\lambda_0} \sum_{n=1}^{\infty} \frac{\lambda_0^n}{n!} \rho^{n*}(du) \tag{2.3}$$

on $(0, \infty)$. Here we note that $R_{\lambda_0}(du) \ge e^{-\lambda_0+\lambda} R_\lambda(du)$.

Lemma 2.3. *For $k \geq 1$ and $l \geq 1$, we have*

$$\overline{R_{\lambda_0}^{(k+l)*}}(x) \geq (A^{-1}(1-e^{-\lambda_0}))^l \cdot \overline{R_{\lambda_0}^{k*}}(x) \tag{2.4}$$

and

$$\overline{R_{\lambda_0}^{k*}}(x) \geq \left(A^{-1}(1-e^{-\lambda_0})\right)^{k-1} \cdot \overline{R_{\lambda_0}}(x). \tag{2.5}$$

Proof. We have

$$\begin{aligned}
\overline{R_{\lambda_0}^{(k+l)*}}(x) &= \overline{R_{\lambda_0}^{(k+l-1)*} * R_{\lambda_0}}(x) = \\
&= \overline{R_{\lambda_0}^{(k+l-1)*}}(x) R_{\lambda_0}((0,\infty)) + \int_0^x \overline{R_{\lambda_0}}(x-y) R_{\lambda_0}^{(k+l-1)*}(dy) \geq \\
&\geq \overline{R_{\lambda_0}^{(k+l-1)*}}(x) R_{\lambda_0}((0,\infty)) = \overline{R_{\lambda_0}^{(k+l-1)*}}(x) \cdot A^{-1}(1-e^{-\lambda_0}).
\end{aligned}$$

If we repeat this operation l times, the first inequality comes out. A similar operation yields the second inequality. ■

Lemma 2.4. *Let $m \geq 2$. For $x \geq 0$, we have*

$$\lambda_0 \overline{\rho}(x) = \sum_{k=1}^{m} \frac{(Ae^{\lambda_0})^{2k-1}}{2k-1} \overline{R_{\lambda_0}^{(2k-1)*}}(x) + \sum_{k=1}^{\infty} \frac{(Ae^{\lambda_0})^{2k}}{2(m+k)-1} S(k,x),$$

where

$$S(k,x) = (Ae^{\lambda_0})^{2m-1} \overline{R_{\lambda_0}^{(2(m+k)-1)*}}(x) - \frac{2(m+k)-1}{2k} \overline{R_{\lambda_0}^{(2k)*}}(x).$$

Proof. Denote the Laplace transforms of R_{λ_0} and ρ by $\check{R}_{\lambda_0}(s)$ and $\check{\rho}(s)$, respectively. That is,

$$\begin{aligned}
\check{R}_{\lambda_0}(s) &= \int_0^\infty e^{-su} R_{\lambda_0}(du), \\
\check{\rho}(s) &= \int_0^\infty e^{-su} \rho(du)
\end{aligned}$$

for $s \geq 0$. From (2.3), it follows that

$$A\check{R}_{\lambda_0}(s) = e^{-\lambda_0} \sum_{n=1}^{\infty} \frac{\lambda_0^n}{n!} \check{\rho}(s)^n = e^{-\lambda_0} \exp[\lambda_0 \check{\rho}(s)] - e^{-\lambda_0}.$$

This implies

$$\lambda_0\check{\rho}(s) = \log(1 + Ae^{\lambda_0}\check{R}_{\lambda_0}(s)).$$

Since $|Ae^{\lambda_0}\check{R}_{\lambda_0}(s)| < 1$, we obtain that

$$\begin{aligned}
\lambda_0\check{\rho}(s) &= -\sum_{l=1}^{\infty}\frac{(-Ae^{\lambda_0})^l}{l}\check{R}_{\lambda_0}(s)^l = \\
&= -\sum_{l=1}^{\infty}\frac{(-Ae^{\lambda_0})^l}{l}\int_{(0,\infty)} e^{-su}R_{\lambda_0}^{l*}(du) = \\
&= \int_{(0,\infty)} e^{-su}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}(Ae^{\lambda_0})^l}{l}R_{\lambda_0}^{l*}(du).
\end{aligned}$$

This leads to

$$\lambda_0\rho(du) = \sum_{l=1}^{\infty}\frac{(-1)^{l+1}(Ae^{\lambda_0})^l}{l}R_{\lambda_0}^{l*}(du).$$

Hence we obtain that

$$\begin{aligned}
\lambda_0\overline{\rho}(x) &= \sum_{k=1}^{m}\frac{(Ae^{\lambda_0})^{2k-1}}{2k-1}\overline{R_{\lambda_0}^{(2k-1)*}}(x) + \\
&\quad + \sum_{k=1}^{\infty}\frac{(Ae^{\lambda_0})^{2(m+k)-1}}{2(m+k)-1}\overline{R_{\lambda_0}^{(2(m+k)-1)*}}(x) - \sum_{k=1}^{\infty}\frac{(Ae^{\lambda_0})^{2k}}{2k}\overline{R_{\lambda_0}^{(2k)*}}(x) = \\
&= \sum_{k=1}^{m}\frac{(Ae^{\lambda_0})^{2k-1}}{2k-1}\overline{R_{\lambda_0}^{(2k-1)*}}(x) + \sum_{k=1}^{\infty}\frac{(Ae^{\lambda_0})^{2k}}{2(m+k)-1}S(k,x).
\end{aligned}$$

The lemma has been proved. ∎

Now we prove Theorem 1.1.

Proof of Theorem 1.1. We see from Lemma 2.1 that the second inequality holds. We prove the first inequality below. Lemma 2.4 tells us that

$$\lambda_0\overline{\rho}(x) = \sum_{k=1}^{m}\frac{(Ae^{\lambda_0})^{2k-1}}{2k-1}\overline{R_{\lambda_0}^{(2k-1)*}}(x) + \sum_{k=1}^{\infty}\frac{(Ae^{\lambda_0})^{2k}}{2(m+k)-1}S(k,x).$$

Recall that $\lambda_0 > \log 2$ and take $m \geq 2$ such that

$$e^{\lambda_0} - 1 > \left(m + \frac{1}{2}\right)^{\frac{1}{2m-1}}.$$

Applying (2.4) in Lemma 2.3, we obtain that

$$\begin{aligned}
&S(k,x)\geq\\
&\geq (Ae^{\lambda_0})^{2m-1}\left(A^{-1}(1-e^{-\lambda_0})\right)^{2m-1}\overline{R^{(2k)*}_{\lambda_0}}(x)-\left(1+\frac{2m-1}{2k}\right)\overline{R^{(2k)*}_{\lambda_0}}(x)\geq\\
&\geq\left[\left(e^{\lambda_0}-1\right)^{2m-1}-\left(m+\frac{1}{2}\right)\right]\overline{R^{(2k)*}_{\lambda_0}}(x).
\end{aligned}$$

In addition, applying (2.5) in Lemma 2.3, we have

$$\begin{aligned}
\lambda_0\overline{\rho}(x) &\geq \sum_{k=1}^{m}\frac{(Ae^{\lambda_0})^{2k-1}}{2k-1}\overline{R^{(2k-1)*}_{\lambda_0}}(x)+\\
&\quad+\sum_{k=1}^{\infty}\frac{(Ae^{\lambda_0})^{2k}}{2(m+k)-1}\left[\left(e^{\lambda_0}-1\right)^{2m-1}-\left(m+\frac{1}{2}\right)\right]\overline{R^{(2k)*}_{\lambda_0}}(x)\geq\\
&\geq \sum_{k=1}^{m}\frac{(Ae^{\lambda_0})^{2k-1}}{2k-1}\left(A^{-1}(1-e^{-\lambda_0})\right)^{2k-2}\cdot\overline{R_{\lambda_0}}(x)+\\
&\quad+\left[\left(e^{\lambda_0}-1\right)^{2m-1}-\left(m+\frac{1}{2}\right)\right]\times\\
&\quad\times\sum_{k=1}^{\infty}\frac{(Ae^{\lambda_0})^{2k}}{2(m+k)-1}\left(A^{-1}(1-e^{-\lambda_0})\right)^{2k-1}\cdot\overline{R_{\lambda_0}}(x).
\end{aligned}$$

Hence, using Lemma 2.2, we obtain that for $x\geq 1$,

$$\begin{aligned}
\frac{\Pi_0(x)}{\overline{\Psi}(\log x)} &= \frac{\lambda\overline{\rho}(\log x)}{A\overline{R_\lambda}(\log x)}=\\
&= \lambda(\lambda_0 A)^{-1}\frac{\lambda_0\overline{\rho}(\log x)}{\overline{R_{\lambda_0}}(\log x)}\cdot\frac{\overline{R_{\lambda_0}}(\log x)}{\overline{R_\lambda}(\log x)}\geq\\
&\geq \lambda(\lambda_0 A)^{-1}\frac{\lambda_0\overline{\rho}(\log x)}{\overline{R_{\lambda_0}}(\log x)}\cdot e^{-\lambda_0+\lambda}=\\
&= \lambda(\lambda_0 A)^{-1}e^{-\lambda_0+\lambda}\sum_{k=1}^{m}\frac{(Ae^{\lambda_0})^{2k-1}}{2k-1}\left(A^{-1}(1-e^{-\lambda_0})\right)^{2k-2}+\\
&\quad+\lambda(\lambda_0 A)^{-1}e^{-\lambda_0+\lambda}\left[\left(e^{\lambda_0}-1\right)^{2m-1}-\left(m+\frac{1}{2}\right)\right]\times\\
&\quad\times\sum_{k=1}^{\infty}\frac{(Ae^{\lambda_0})^{2k}}{2(m+k)-1}\left(A^{-1}(1-e^{-\lambda_0})\right)^{2k-1}>0.
\end{aligned}$$

Since we have
$$\overline{\Psi}(\log x) = \frac{1}{\zeta(\sigma)} \sum_{n>x} \frac{1}{n^\sigma},$$
the theorem holds. ■

3. Dirichlet series

Let $f(n)$ be an arithmetic function being completely multiplicative. In what follows, we assume that $f(n) \geq 0$ for all positive integers n and $f(n) > 0$ for some $n \geq 2$. In this section, we consider a Dirichlet series denoted by

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{3.1}$$

for $s = \sigma + it$, where $F(s)$ converges absolutely for $\sigma > \sigma_a$. In the case of $\zeta(s)$, $f(n)$ is equal to 1 for any positive integer n. We extend Theorem 1.1 to the Dirichlet series. By virtue of Theorem 11.14 of [2], $F(s)$ has a representation as follows:

$$F(s) = \exp\left[\sum_{n=2}^{\infty} \frac{(f' \circ f^{-1})(n)}{n^s \log n}\right],$$

where f^{-1} is the Dirichlet inverse of f and $f'(n) = f(n)\log n$. The symbol $f' \circ f^{-1}$ denotes the Dirichlet convolution of f' and f^{-1}, that is,

$$(f' \circ f^{-1})(n) = \sum_{d|n} f'(d) f^{-1}\left(\frac{n}{d}\right).$$

Now we set

$$\begin{aligned}\hat{\Phi}(t) &:= \frac{F(\sigma - it)}{F(\sigma)} = \\ &= \exp\left[\sum_{n=2}^{\infty} (e^{it\log n} - 1) \frac{(f' \circ f^{-1})(n)}{n^\sigma \log n}\right].\end{aligned}$$

Similarly to the Riemann zeta distribution, $\hat{\Phi}(t)$ becomes a characteristic function of an infinitely divisible distribution. Such a fact has been already pointed out in Lin and Hu [6]. Hence $\hat{\Phi}(t)$ has the following representation:

$$\hat{\Phi}(t) = \exp\left[\int_0^{\infty} (e^{itu} - 1) G(du)\right]$$

and its Lévy masure is

$$G(du) = \sum_{n=2}^{\infty} \frac{(f' \circ f^{-1})(n)}{n^{\sigma} \log n} \delta_{\log n}(du).$$

We set

$$\gamma = G((0,\infty)) = \log F(\sigma) \quad \text{and} \quad \nu(du) = \gamma^{-1} G(du).$$

Then we have

$$\Phi(du) = e^{-\gamma} \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \nu^{n*}(du), \tag{3.2}$$

where ν^{0*} is understood to be δ_0. On the other hand, we have another representation by (3.1).

$$\Phi(du) = \frac{1}{F(\sigma)} \sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma}} \delta_{\log n}(du).$$

Let $A > 0$ and define a finite measure U_γ by

$$U_\gamma(du) = A^{-1} e^{-\gamma} \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \nu^{n*}(du) \tag{3.3}$$

on $(0, \infty)$.

Lemma 3.1. *Assume that $f(n) \geq 0$ for all positive integers n and $f(n) > 0$ for some $n \geq 2$. If f is completely multiplicative, then*

$$\frac{(f' \circ f^{-1})(n)}{\log n} \leq f(n)$$

for $n \geq 2$.

Remark 3.1. The proof requires the Möbius function μ which is defined as follows: $\mu(1) = 1$. For $n > 1$, we write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ denote prime numbers. Then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Lemma 3.1. Since f is completely multiplicative, then

$$f^{-1}(n) = \mu(n) f(n)$$

by Theorem 2.17 of [2]. We see from Theorem 295 of [4] that

$$\begin{aligned}(f' \circ f^{-1})(n) &= \sum_{d|n} f(d)(\log d)\mu\left(\frac{n}{d}\right) f\left(\frac{n}{d}\right) = \\ &= f(n)\sum_{d|n} \mu\left(\frac{n}{d}\right)\log d = \\ &= f(n)\Lambda(n).\end{aligned}$$

By Theorem 296 of [4], we have

$$\Lambda(n) \leq \sum_{d|n} \Lambda(d) = \log n.$$

Hence the lemma holds. ∎

Lemma 3.2. *For $x \geq 0$, we have*

$$A\overline{U_\gamma}(x) = \overline{\Phi}(x). \tag{3.4}$$

Proof. The equations (3.2) and (3.3) tell us that (3.4) holds. ∎

Take γ_0 such that $\gamma_0 > \max\{\gamma, \log 2\}$ and take $A > 0$ such that $Ae^{\gamma_0} < 1$. We define a new finite measure U_{γ_0} by

$$U_{\gamma_0}(du) = A^{-1}e^{-\gamma_0}\sum_{n=1}^{\infty}\frac{\gamma_0^n}{n!}\nu^{n*}(du) \tag{3.5}$$

on $(0, \infty)$. Here we note that $U_{\gamma_0}(du) \geq e^{-\gamma_0+\gamma}U_\gamma(du)$.

Lemma 3.3. *For $k \geq 1$ and $l \geq 1$, we have*

$$\overline{U_{\gamma_0}^{(k+l)*}}(x) \geq (A^{-1}(1-e^{-\gamma_0}))^l \cdot \overline{U_{\gamma_0}^{k*}}(x)$$

and

$$\overline{U_{\gamma_0}^{k*}}(x) \geq \left(A^{-1}(1-e^{-\gamma_0})\right)^{k-1} \cdot \overline{U_{\gamma_0}}(x).$$

Proof. The proof is the same as the proof of Lemma 2.3. ∎

Lemma 3.4. *Let $m \geq 2$. For $x \geq 0$, we have*

$$\gamma_0\overline{\nu}(x) = \sum_{k=1}^{m}\frac{(Ae^{\gamma_0})^{2k-1}}{2k-1}\overline{U_{\gamma_0}^{(2k-1)*}}(x) + \sum_{k=1}^{\infty}\frac{(Ae^{\gamma_0})^{2k}}{2(m+k)-1}T(k,x),$$

where

$$T(k,x) = (Ae^{\gamma_0})^{2m-1}\overline{U_{\gamma_0}^{(2(m+k)-1)*}}(x) - \frac{2(m+k)-1}{2k}\overline{U_{\gamma_0}^{(2k)*}}(x).$$

Proof. Denote the Laplace transform of U_{γ_0} and ν by $\check{U}_{\gamma_0}(s)$ and $\check{\nu}(s)$, respectively. From (3.5), it follows that

$$A\check{U}_{\gamma_0}(s) = e^{-\gamma_0}\sum_{n=1}^{\infty}\frac{\gamma_0^n}{n!}\check{\nu}(s)^n = e^{-\gamma_0}\exp[\gamma_0\check{\nu}(s)] - e^{-\gamma_0}.$$

This implies

$$\gamma_0\check{\nu}(s) = \log(1 + Ae^{\gamma_0}\check{U}_{\gamma_0}(s)).$$

Since $|Ae^{\gamma_0}\check{U}_{\gamma_0}(s)| < 1$, we obtain that

$$\gamma_0\check{\nu}(s) \quad = \quad \int\limits_{(0,\infty)} e^{-su}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}(Ae^{\gamma_0})^l}{l}U_{\gamma_0}^{l*}(du)$$

in the same way as the proof of Lemma 2.4. This leads to

$$\gamma_0\nu(du) = \sum_{l=1}^{\infty}\frac{(-1)^{l+1}(Ae^{\gamma_0})^l}{l}U_{\gamma_0}^{l*}(du).$$

The rest of the proof is the same as Lemma 2.4. ■

Now we show the following theorem.

Theorem 3.1. *Let $\sigma > \sigma_0$ and assume that $f(n) \geq 0$ for all positive integers n and $f(n) > 0$ for some $n \geq 2$. If f is completely multiplicative, then there is a positive constant C_1 such that*

$$C_1\sum_{n>x}\frac{f(n)}{n^\sigma} \leq \sum_{n>x}^{\infty}\frac{(f' \circ f^{-1})(n)}{n^\sigma \log n} \leq \sum_{n>x}\frac{f(n)}{n^\sigma}$$

for all positive integers x.

Proof. It follows from Lemma 3.1 that

$$\sum_{n>x}^{\infty}\frac{(f' \circ f^{-1})(n)}{n^\sigma \log n} \leq \sum_{n>x}^{\infty}\frac{f(n)}{n^\sigma}.$$

The first inequality can be proved in the same way as the proof of Theorem 1.1, because we have Lemmas 3.2, 3.3 and 3.4. ■

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K. Yamamuro
Faculty of Engineering
Gifu University
Gifu 501-1193
Japan
yamamuro.kouji.f3@f.gifu-u.ac.jp