THE h-FOURIER COSINE-LAPLACE GENERALIZED CONVOLUTION WITH A WEIGHT FUNCTION

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Abstract. We construct h-Fourier cosine-Laplace discrete generalized convolution with a weight function on time scale $\mathbb{T}_h^0$ and study its properties. We obtain some inequalities for this generalized convolution such as Young's type inequalities, Saitoh's type inequalities. In the application, we apply this generalized convolution to solve some linear equations of generalized convolution type.

1. Introduction

Let $h > 0$ be a fixed number and $\mathbb{T}_h^0 = h\mathbb{N}_0$, here $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. The definition of h-Laplace transform is given by M. Bohner and G.Sh. Guseinov in [2, pp. 78]: For $x : \mathbb{T}_h^0 \to \mathbb{C}$, the h-Laplace transform of x, denoted by $\mathcal{L}\{x\}$, is represented as follows:

$$\mathcal{L}\{x\}(u) = h\sum_{n=0}^{\infty} \frac{x(nh)}{(1+hu)^{n+1}}, \quad u \in \mathbb{C} \setminus \left\{\frac{-1}{h}\right\}. \tag{1.1}$$

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Using the h-Fourier transform in [4, pp. 914], the h-Fourier cosine transform is defined as follows: For $x : \mathbb{T}_h^0 \to \mathbb{C}$ such that $\sum\limits_{n=0}^{\infty} |x(nh)| < \infty$, the h-Fourier cosine transform of x, denoted by $\mathcal{F}_c\{x\}$, is given by

$$\mathcal{F}_c\{x\}(u) = hx(0) + 2h \sum_{n=1}^{\infty} x(nh)\cos(unh), \quad u \in \mathbb{R}. \tag{1.2}$$

Recently, the generalized convolutions related to h-Laplace transform have been studied in [8, 12]. However, as far as we know, there have not been any published research results about generalized convolutions related to h-Laplace transform with weight functions.

For the Fourier convolution

$$\big(u \underset{\mathcal{F}}{*} v\big)(x) = \int_{-\infty}^{\infty} u(x-y)v(y)dy, \quad x \in \mathbb{R},$$

Young's theorem states that (see [1])

$$\Big| \int_{-\infty}^{\infty} \big(u \underset{\mathcal{F}}{*} v\big)(x)w(x)dx \Big| \leq \|u\|_p \|v\|_q \|w\|_r,$$

for all $u \in L_p(\mathbb{R}),\ v \in L_q(\mathbb{R}),\ w \in L_r(\mathbb{R}),\ p, q, r \geq 1,\ p^{-1} + q^{-1} + r^{-1} = 2.$

In a recent paper [11], the authors established some Young's type inequalities for a Fourier cosine and sine polyconvolution and a generalized convolution.

The Saitoh's inequality for the Fourier convolution was introduced in [5] as follows:

For two nonvanishing continuous functions $\rho_j(x),\ (j = 1, 2)$ in $L_1(\mathbb{R})$, and for $p > 1$, we have

$$\Big\| \big((F_1\rho_1) \underset{\mathcal{F}}{*} (F_2\rho_2)\big)\big(\rho_1 \underset{\mathcal{F}}{*} \rho_2\big)^{\frac{1}{p}-1} \Big\|_p \leq \|F_1\|_{L_p(\mathbb{R}, |\rho_1|)} \|F_2\|_{L_p(\mathbb{R}, |\rho_2|)},$$

where $F_j \in L_p(\mathbb{R}, |\rho_j|)$.

The reverse Saitoh's inequality for Fourier convolution can be found in [6].

In this article, we are interested in generalized convolution with a weight function for the h-Fourier cosine and h-Laplace transforms. We will investigate some operator properties, generalized convolution type inequalities and its applications.

The structure of this article is as follows. In Section 2, we present some fundamental notations and lemmas used in this article. In Section 3, we give formula for the h-Fourier cosine-Laplace generalized convolution with weight function and study some of its properties such as factorization identity, relation with the h-Fourier cosine convolution, Titchmarsh's type Theorem. In Section 4, we investigate the existence of this generalized convolution on some function spaces and obtain some inequalities such as Young's type inequalities, Saitoh's type inequalities and reverse Saitoh's type inequalities. In the final section, we apply the new generalized convolution to solve some linear equations of generalized convolution type.

2. Preliminaries

Let $1 \leq p < \infty$. We consider the following vector spaces and norms:

$$\ell_p(\mathbb{T}_h^0) = \big\{x : \mathbb{T}_h^0 \to \mathbb{C} \;\big|\; \sum_{n=0}^{\infty} |x(nh)|^p < \infty\big\},$$

$$\|x\|_p = h\Big(\sum_{n=0}^{\infty} |x(nh)|^p\Big)^{\frac{1}{p}}, \quad \|x\|_p^{(1)} = h\Big(|x(0)|^p + 2^p \sum_{n=1}^{\infty} |x(nh)|^p\Big)^{\frac{1}{p}}, \tag{2.1}$$

$$\ell_\infty(\mathbb{T}_h^0) = \big\{x : \mathbb{T}_h^0 \to \mathbb{C} \;\big|\; \sup_{n\geq 0}|x(nh)| < \infty\big\}, \quad \|x\|_\infty = h \sup_{n\geq 0}|x(nh)|.$$

For $x : \mathbb{T}_h^0 \to \mathbb{C}$ we define $H_1\{x\} : \mathbb{T}_h^0 \to \mathbb{C}$ as follows:

$$\big(H_1\{x\}\big)(0) := \frac{x(0)}{2}, \quad (H_1\{x\})(nh) := x(nh), \quad n \in \mathbb{N}. \tag{2.2}$$

From (2.1), it is easily proven that if $x \in \ell_p(\mathbb{T}_h^0)$, $1 \leq p < \infty$ then $H_1\{x\} \in$ $\in \ell_p(\mathbb{T}_h^0)$ and

$$\sum_{n=0}^{\infty} \big|H_1\{x\}(nh)\big|^p = \Big[\frac{\|x\|_p^{(1)}}{2h}\Big]^p. \tag{2.3}$$

Definition 2.1. [9] The h-Fourier cosine convolution on time scale of two functions $x, y \in \ell_1(\mathbb{T}_h^0)$ is defined as

$$(x \underset{\mathcal{F}_c}{*} y)(kh) = h\Big\{\sum_{n=1}^{\infty} x(nh)\Big[y(|kh - nh|) + y(kh + nh)\Big] + x(0)y(kh)\Big\}, \tag{2.4}$$

for $k \in \mathbb{N}_0$.

Lemma 2.1. [9] *Let $x, y \in \ell_1(\mathbb{T}_h^0)$ then $x \underset{\mathcal{F}_c}{*} y \in \ell_1(\mathbb{T}_h^0)$,*

$$\|x \underset{\mathcal{F}_c}{*} y\|_1^{(1)} \leq \|x\|_1^{(1)} \|y\|_1^{(1)}$$

and we have the factorization identity

$$\mathcal{F}_c\{x \underset{\mathcal{F}_c}{*} y\}(u) = \mathcal{F}_c\{x\}(u)\mathcal{F}_c\{y\}(u), \quad u \in \left[0, \frac{\pi}{h}\right]. \tag{2.5}$$

Lemma 2.2. (Wiener-Lévy type Theorem for Fourier cosine series) *Let $x \in \ell_1(\mathbb{T}_h^0)$ and $\Phi(z)$ be an analytic function whose domain contains the range of $\mathcal{F}_c\{x\}(u)$ and satisfies $\Phi(0) = 0$. Then $\Phi(\mathcal{F}_c\{x\}(u))$ equals to h-Fourier cosine transform of a function in $\ell_1(\mathbb{T}_h^0)$.*

3. Generalized convolution with a weight function for *h*-Fourier cosine and *h*-Laplace transforms

In this article, let $\mu \in \mathbb{N}$ be a fixed natural number. We define a weight function $\gamma : \left[0, \frac{\pi}{h}\right] \to \mathbb{R}^+$ by

$$\gamma(u) = (1 + hu)^{-\mu}, \quad u \in \left[0, \frac{\pi}{h}\right]. \tag{3.1}$$

Definition 3.1. The generalized convolution with the weight function (3.1) of two functions $x, y : \mathbb{T}_h^0 \to \mathbb{C}$ with respect to the h-Fourier cosine and h-Laplace transforms on time scale $\mathbb{T}_h^0$ is defined as

$$\begin{aligned}(x*y)(kh) =& \frac{h}{2\pi} x(0) \sum_{m=0}^{\infty} y(mh)\theta(k, 0, m + \mu)+ \\ &+ \frac{h}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} x(nh)y(mh)\theta(k, n, m + \mu),\end{aligned} \tag{3.2}$$

for $k \in \mathbb{N}_0$, in here

$$\theta(k, n, m) = I(n + k, m) + I(|n - k|, m), \quad k, n, m \in \mathbb{N}_0, \tag{3.3}$$

$$I(n, m) = \int_0^{\pi} \frac{\cos(nu)}{(1 + u)^{m+1}} du, \quad n, m \in \mathbb{N}_0, \tag{3.4}$$

assuming that the right hand side of (3.2) converges for all $k \in \mathbb{N}_0$.

We denote $x_1 := H_1\{x\}$, where $H_1\{x\}$ is defined from x by (2.2). The formula (3.2) can be written in the form

$$(x*y)(kh) = \frac{h}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_1(nh)y(mh)\theta(k, n, m + \mu), \quad k \in \mathbb{N}_0. \tag{3.5}$$

Theorem 3.1. *If $x, y \in \ell_1(\mathbb{T}_h^0)$, then $x{*}y \in \ell_1(\mathbb{T}_h^0)$ and*

$$\big\|x{*}y\big\|_1 \le \Big[1 + \frac{1}{\mu\pi}\Big]\|x\|_1^{(1)}\|y\|_1. \tag{3.6}$$

The inequality (3.6) becomes an equality if and only if $x \equiv 0$ or $y \equiv 0$. Moreover, the following factorization identity holds:

$$\mathcal{F}_c\{x{*}y\}(u) = \gamma(u)\mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u), \quad u \in \Big[0, \frac{\pi}{h}\Big]. \tag{3.7}$$

Proof. We define $x_1 := H_1\{x\}$, where $H_1\{x\}$ is given by (2.2). From the result in [8, pp. 26], we have $I(n,m) > 0$ for $m, n \in \mathbb{N}_0$. Hence, $\theta(k,n,m{+}\mu) = I(n+k, m+\mu) + I\big(|n-k|, m+\mu\big) > 0$ for $k, n, m \in \mathbb{N}_0$. We use formula (3.5) to obtain

$$\sum_{k=0}^{\infty}\big|\big(x{*}y\big)(kh)\big| \le \frac{h}{\pi}\sum_{n=0}^{\infty}|x_1(nh)|\sum_{m=0}^{\infty}|y(mh)|\sum_{k=0}^{\infty}\theta(k,n,m+\mu). \tag{3.8}$$

For $m, n \in \mathbb{N}_0$, from (3.3), Lemma 3.1 and Lemma 3.4 in [8], we have

$$\sum_{k=0}^{\infty}\theta(k,n,m+\mu) = I(n,m+\mu) + I(0,m+\mu) + 2\sum_{j=1}^{\infty}I(j,m+\mu) < \\ < \frac{2}{m+\mu} + 2\pi \le 2\Big[\pi + \frac{1}{\mu}\Big]. \tag{3.9}$$

Plugging (3.9) into (3.8) yields

$$\sum_{k=0}^{\infty}\big|\big(x{*}y\big)(kh)\big| \le \frac{2h}{\pi}\Big[\pi + \frac{1}{\mu}\Big]\sum_{n=0}^{\infty}|x_1(nh)|\sum_{m=0}^{\infty}|y(mh)| = \\ = 2h\Big[1 + \frac{1}{\mu\pi}\Big]\frac{\|x\|_1^{(1)}}{2h}\frac{\|y\|_1}{h} < \infty. \tag{3.10}$$

Consequently $x{*}y \in \ell_1(\mathbb{T}_h^0)$. Multiplying (3.10) by h gives (3.6). The equality holds if and only if $x \equiv 0$ or $y \equiv 0$.
For $k \in \mathbb{N}_0$, from (3.5), it follows that

$$\big(x{*}y\big)(kh) = \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}x_1(nh)y(mh)\theta(k,n,m+\mu) = \\ = \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}x_1(nh)y(mh)\Big[I(n+k,m+\mu) + I(|n-k|,m+\mu)\Big] = \\ = \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}x_1(nh)y(mh)\int_0^{\pi}\frac{\cos(n+k)u + \cos(n-k)u}{(1+u)^{m+\mu+1}}du = \tag{3.11}$$

$$= \frac{h^2}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_1(nh)y(mh) \int_0^{\frac{\pi}{h}} \frac{\cos(n+k)uh + \cos(n-k)uh}{(1+hu)^{m+\mu+1}} du =$$

$$= \frac{2h^2}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_1(nh)y(mh) \int_0^{\frac{\pi}{h}} \frac{\cos(nuh)\cos(kuh)}{(1+hu)^{m+\mu+1}} du =$$

$$= \frac{1}{\pi} \int_0^{\frac{\pi}{h}} 2h^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\cos(unh)}{(1+hu)^{m+\mu+1}} x_1(nh)y(mh)\cos(kuh)du.$$

By using formulas (1.2) and (1.1), the product of $\gamma(u)$, h-Fourier cosine of x and h-Laplace transform of y can be written as follows:

$$\gamma(u)\mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u) = 2h^2 \sum_{n=0}^{\infty} x_1(nh)\cos(unh) \sum_{m=0}^{\infty} \frac{y(mh)}{(1+hu)^{m+\mu+1}}$$

$$= 2h^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\cos(unh)}{(1+hu)^{m+\mu+1}} x_1(nh)y(mh). \tag{3.12}$$

Substituting (3.12) into (3.11) yields

$$\big(x*y\big)(kh) = \frac{1}{\pi} \int_0^{\frac{\pi}{h}} \gamma(u)\mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u)\cos(kuh)du, \ \forall k \in \mathbb{N}_0. \tag{3.13}$$

Moreover, we use inverse h-Fourier cosine transform to obtain

$$\big(x*y\big)(kh) = \frac{1}{\pi} \int_0^{\frac{\pi}{h}} \mathcal{F}_c\{x \overset{\gamma}{\underset{\mathcal{F}_c\mathcal{L}}{*}} y\}(u)\cos(kuh)du, \ \forall k \in \mathbb{N}_0. \tag{3.14}$$

Combining (3.13) and (3.14), we derive the factorization identity (3.7). The theorem is proved. ∎

For $m \in \mathbb{N}_0$ we define a function $J_m : \mathbb{T}_h^0 \to \mathbb{C}$ by

$$J_m(nh) := I(n,m), \quad n \in \mathbb{N}_0, \tag{3.15}$$

where $I(n,m)$ is defined in (3.4).
From Lemma 3.4 in [8] we have $J_m \in \ell_1(\mathbb{T}_h^0)$.

Lemma 3.1. *Let x, y be any two functions in $\ell_1(\mathbb{T}_h^0)$ then the following identity holds*

$$(x{*}y)(kh) = \frac{1}{\pi}\sum_{m=0}^{\infty} y(mh)\big[x \underset{\mathcal{F}_c}{*} J_{m+\mu}\big](kh), \quad \forall k \in \mathbb{N}_0. \tag{3.16}$$

Proof. We denote $x_1 := H_1\{x\}$, where $H_1\{x\}$ is defined from x by (2.2). Using (3.15), the definition of generalized convolution and convolution in (3.2) and (2.4), we can transform the left hand side of (3.16) as follows:

$$\begin{aligned}
(x{*}y)(kh) &= \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_1(nh)y(mh)\theta(k,n,m+\mu),\ k \in \mathbb{N}_0 = \\
&= \frac{h}{\pi}\sum_{m=0}^{\infty} y(mh)\sum_{n=0}^{\infty} x_1(nh)\big[I(k+n,m+\mu) + I(|k-n|,m+\mu)\big] = \\
&= \frac{h}{\pi}\sum_{m=0}^{\infty} y(mh)\sum_{n=0}^{\infty} x_1(nh)\big[J_{m+\mu}(kh+nh) + J_{m+\mu}(|kh-nh|)\big] = \\
&= \frac{h}{\pi}\sum_{m=0}^{\infty} y(mh)\frac{\big[x \underset{\mathcal{F}_c}{*} J_{m+\mu}\big](kh)}{h} = \\
&= \frac{1}{\pi}\sum_{m=0}^{\infty} y(mh)\big[x \underset{\mathcal{F}_c}{*} J_{m+\mu}\big](kh).
\end{aligned}$$

■

Theorem 3.2. (Titchmarsh's type Theorem.) *Let $x,\ y \in \ell_1(\mathbb{T}_h^0)$. If $x{*}y \equiv$ $\equiv 0$ then $x \equiv 0$ or $y \equiv 0$.*

Proof. Since $x{*}y \equiv 0$ we obtain

$$(1+hu)^{\mu}\mathcal{F}_c\{x{*}y\}(u) = 0, \quad \text{for all } u \in \Big[0, \frac{\pi}{h}\Big]. \tag{3.17}$$

Combining (3.17) with (3.7) yields

$$\mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u) \equiv 0, \quad \text{for all } u \in \Big[0, \frac{\pi}{h}\Big]. \tag{3.18}$$

Recalling that $\mathcal{L}\{y\}(u)$ is an analytic function in the complex region $|1+hu| > 1$ (see [12]). We consider the following two cases:

- Case 1: $\mathcal{L}\{y\}(u) \equiv 0$ on $\Big(0, \dfrac{\pi}{h}\Big)$. Since $\mathcal{L}\{y\}(u)$ is analytic in the complex region $|1+hu| > 1$, we have $\mathcal{L}\{y\}(u) = 0$ for $|1+hu| > 1$. From Theorem 4.8 in [2] we deduce that $y \equiv 0$.

- Case 2: $\mathcal{L}\{y\}(u) \not\equiv 0$ on $\left(0, \dfrac{\pi}{h}\right)$. From $x \in \ell_1(\mathbb{T}_h^0)$, the series

$$hx(0) + 2h\sum_{n=1}^{\infty} x(nh)\cos(unh)$$

converges uniformly on $\mathbb{R}$. Therefore, $\mathcal{F}_c\{x\}(u)$ is continuous on $\mathbb{R}$.

Let u_0 be an arbitrary point in $\left(0, \dfrac{\pi}{h}\right)$, we will prove that $\mathcal{F}_c\{x\}(u_0) = 0$.

Assume that $\mathcal{F}_c\{x\}(u_0) \neq 0$. By continuity property of $\mathcal{F}_c\{x\}(u)$, there exists a neighbourhood of u_0 such that for u in that neighbourhood we have $\mathcal{F}_c\{x\}(u) \neq 0$. From (3.18) we can conclude that $\mathcal{L}\{y\}(u) = 0$ on that neighbourhood of u_0. Additionally, $\mathcal{L}\{y\}(u)$ is an analytic function in the complex region $|1+hu| > 1$. Hence, $\mathcal{L}\{y\}(u) \equiv 0$ on $\left(0, \dfrac{\pi}{h}\right)$, which is a contradiction.

Therefore $\mathcal{F}_c\{x\}(u_0) = 0$ for $u_0 \in \left(0, \dfrac{\pi}{h}\right)$. Using the inverse h-Fourier cosine transform we get $x \equiv 0$.

The proof is completed. ■

4. Some inequalities for the generalized convolution (3.2)

4.1. The generalized convolution on some function spaces

Lemma 4.1. *Suppose that $1 < p < \infty$, $x \in \ell_p(\mathbb{T}_h^0)$, $y \in \ell_1(\mathbb{T}_h^0)$. Then, the discrete generalized convolution $x{*}y$ is well defined and belongs to $\ell_\infty(\mathbb{T}_h^0)$. Furthermore, we have the following inequality:*

$$\|x{*}y\|_\infty \leq C_p \|x\|_p^{(1)} \|y\|_1, \tag{4.1}$$

where $C_p = \left[\dfrac{1}{\mu\pi}\right]^{\frac{1}{p}} \left[1 + \dfrac{1}{\mu\pi}\right]^{1-\frac{1}{p}}$.

Proof. Let $q = \dfrac{p}{p-1}$. We have $1 < q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.
We define $x_1 := H_1\{x\}$, where $H_1\{x\}$ is given by formula (2.2). For $k \in \mathbb{N}_0$, formula (3.5) implies that

(4.2)

$$\begin{aligned}\big|(x{*}y)(kh)\big| &\leq \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |x_1(nh)||y(mh)|\theta(k,n,m+\mu) = \\ &= \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |x_1(nh)|\big(|y(mh)|\theta(k,n,m+\mu)\big)^{\frac{1}{p}}\big(|y(mh)|\theta(k,n,m+\mu)\big)^{\frac{1}{q}}.\end{aligned}$$

From (4.2) and applying Hölder's inequality, we get

$$\big|(x*y)(kh)\big| \le \frac{h}{\pi} A_k^{\frac{1}{p}} B_k^{\frac{1}{q}}, \tag{4.3}$$

where

$$A_k = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |x_1(nh)|^p |y(mh)| \theta(k,n,m+\mu),$$
$$B_k = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |y(mh)| \theta(k,n,m+\mu).$$

For $k,n,m \in \mathbb{N}_0$, since $0 < \theta(k,n,m+\mu) < \dfrac{2}{m+\mu} \le \dfrac{2}{\mu}$ (see [8, pp. 22, pp. 26]), we have

$$A_k \le \frac{2}{\mu}\sum_{n=0}^{\infty}|x_1(nh)|^p \sum_{m=0}^{\infty}|y(mh)| = \frac{2}{\mu}\Big(\frac{\|x\|_p^{(1)}}{2h}\Big)^p \frac{\|y\|_1}{h}. \tag{4.4}$$

Combining (3.9) and $\theta(k,n,m+\mu) = I(n+k,m+\mu) + I(|n-k|,m+\mu) = = \theta(n,k,m+\mu)$, it follows that

$$\sum_{n=0}^{\infty}\theta(k,n,m+\mu) = \sum_{n=0}^{\infty}\theta(n,k,m+\mu) < 2\Big[\pi + \frac{1}{\mu}\Big], \quad m,k \in \mathbb{N}_0. \tag{4.5}$$

From (4.5), we obtain

$$B_k \le 2\Big[\pi + \frac{1}{\mu}\Big]\sum_{m=0}^{\infty}|y(mh)| = 2\Big[\pi + \frac{1}{\mu}\Big]\frac{\|y\|_1}{h}. \tag{4.6}$$

Substituting (4.4) and (4.6) into (4.3) yields

$$\big|(x*y)(kh)\big| \le \frac{1}{h}\Big[\frac{1}{\mu\pi}\Big]^{\frac{1}{p}}\Big[1 + \frac{1}{\mu\pi}\Big]^{\frac{1}{q}}\|x\|_p^{(1)}\|y\|_1 < \infty.$$

Therefore $x*y \in \ell_\infty(\mathbb{T}_h^0)$ and the norm inequality (4.1) holds. ■

Theorem 4.1. *Suppose that $1 < p < \infty$, $x \in \ell_p(\mathbb{T}_h^0)$, $y \in \ell_1(\mathbb{T}_h^0)$. Then the generalized convolution $x*y$ belongs to $\ell_p(\mathbb{T}_h^0)$. Moreover,*

$$\|x*y\|_p \le \Big[1 + \frac{1}{\mu\pi}\Big]\|x\|_p^{(1)}\|y\|_1. \tag{4.7}$$

The equality holds if and only if $x \equiv 0$ or $y \equiv 0$.

Proof. If $x \equiv 0$ or $y \equiv 0$, the inequality (4.7) evidently holds and the inequality becomes an equality.
Now let $x \not\equiv 0$ and $y \not\equiv 0$, we will prove that

$$\|x{*}y\|_p < \Big[1 + \frac{1}{\mu\pi}\Big] \|x\|_p^{(1)} \|y\|_1.$$

Let $q = \dfrac{p}{p-1}$. Then $1 < q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.
According to Lemma 4.1, $x{*}y$ is well defined and belongs to $\ell_\infty(\mathbb{T}_h^0)$. We denote $x_1 := H_1\{x\}$, where $H_1\{x\}$ is given by (2.2). From (4.3), (4.6), (3.9) and (2.3) we get

(4.8)

$$\begin{aligned} S &= \sum_{k=0}^{\infty} |(x{*}y)(kh)|^p \le \\ &\le \frac{h^p}{\pi^p}\Big[2\pi + \frac{2}{\mu}\Big]^{\frac{p}{q}} \Big[\frac{\|y\|_1}{h}\Big]^{\frac{p}{q}} \sum_{n=0}^{\infty} |x_1(nh)|^p \sum_{m=0}^{\infty} |y(mh)| \sum_{k=0}^{\infty} \theta(k,n,m+\mu) < \\ &< \frac{h^p}{\pi^p}\Big[2\pi + \frac{2}{\mu}\Big]^{1+\frac{p}{q}} \Big[\frac{\|y\|_1}{h}\Big]^{\frac{p}{q}} \sum_{n=0}^{\infty} |x_1(nh)|^p \sum_{m=0}^{\infty} |y(mh)| = \\ &= \frac{h^p}{\pi^p}\Big[2\pi + \frac{2}{\mu}\Big]^{1+\frac{p}{q}} \Big[\frac{\|y\|_1}{h}\Big]^{1+\frac{p}{q}} \Big[\frac{\|x\|_p^{(1)}}{2h}\Big]^{p} < \infty. \end{aligned}$$

From (4.8), we obtain $x{*}y \in \ell_p(\mathbb{T}_h^0)$ and

$$\begin{aligned} \|x{*}y\|_p = hS^{\frac{1}{p}} &< \frac{h^2}{\pi}\Big[2\pi + \frac{2}{\mu}\Big]^{\frac{1}{p}+\frac{1}{q}} \Big[\frac{\|y\|_1}{h}\Big]^{\frac{1}{p}+\frac{1}{q}} \Big[\frac{\|x\|_p^{(1)}}{2h}\Big] = \\ &= \frac{h^2}{\pi}\Big[2\pi + \frac{2}{\mu}\Big] \Big[\frac{\|y\|_1}{h}\Big] \Big[\frac{\|x\|_p^{(1)}}{2h}\Big] = \\ &= \Big[1 + \frac{1}{\mu\pi}\Big] \|x\|_p^{(1)} \|y\|_1. \end{aligned}$$

The theorem is proved. ■

Let $q > 1$ and $\varrho : \mathbb{T}_h^0 \to \mathbb{R}^+$ is a given weight function. We define the following weighted space and norm:

$$\ell_q(\mathbb{T}_h^0, \varrho) = \Big\{x : \mathbb{T}_h^0 \to \mathbb{C} \Big| \sum_{n=0}^{\infty} |x(nh)|^q \varrho(nh) < \infty\Big\},$$

$$\|x\|_{\ell_q(\mathbb{T}_h^0,\varrho)} = h\Big[\sum_{n=0}^{\infty} |x(nh)|^q \varrho(nh)\Big]^{\frac{1}{q}}.$$

Assume that α is a fixed positive number. We consider the weight function $\rho : \mathbb{T}_h^0 \to \mathbb{R}^+$ as follows:

$$\rho(nh) = (1+n)^\alpha, \quad n \in \mathbb{N}_0. \tag{4.9}$$

Lemma 4.2. *Let $p \geq 1, q > 1$ such that $\frac{1}{p} + \frac{1}{q} \geq 1$ and ρ be the weight function defined in (4.9). Then for $x \in \ell_p(\mathbb{T}_h^0)$ and $y \in \ell_q(\mathbb{T}_h^0, \rho)$ the discrete generalized convolution $x{*}y$ is well defined in $\ell_\infty(\mathbb{T}_h^0)$ and we have the following estimate*

$$\|x{*}y\|_\infty \leq C(q)\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\rho)}, \tag{4.10}$$

where the constant $C(q) > 0$ is given by

$$C_0(q) = \sum_{m=0}^{\infty} \frac{1}{(m+\mu)(m+1)^{\frac{\alpha}{q-1}}},$$

$$C(q) = \frac{1}{\pi}\big[C_0(q)\big]^{1-\frac{1}{q}}\Big[\pi + \frac{1}{\mu}\Big]^{\frac{1}{q}}. \tag{4.11}$$

Proof. Let $\beta = \dfrac{\alpha}{q-1} > 0$, $r = \dfrac{q}{q-1}$. Using (3.3) and Lemma 3.1 in [8], for $k, n, m \in \mathbb{N}_0$ we have

$$\theta(k,n,m+\mu) = I(n+k, m+\mu) + I\big(|n-k|, m+\mu\big) < \frac{2}{m+\mu}.$$

Hence

$$\sum_{m=0}^{\infty} \frac{\theta(k,n,m+\mu)}{(m+1)^\beta} < \sum_{m=0}^{\infty} \frac{2}{(m+\mu)(m+1)^\beta} = 2C_0(q). \tag{4.12}$$

From (4.12), (4.5), using Hölder's inequality we can prove (4.10). ■

Lemma 4.3. *Let $x, y : \mathbb{T}_h^0 \to \mathbb{R}$ be two functions in $\ell_1(\mathbb{T}_h^0)$ such that $\forall k \in \mathbb{N}_0$ we have $x(kh) \geq 0$, $y(kh) \geq 0$. Then $x{*}y \in \ell_1(\mathbb{T}_h^0)$ and the following estimate holds*

$$\|x{*}y\|_1 \geq \Big[\frac{1}{4} - \frac{1}{2\mu\pi(1+\pi)^\mu}\Big]\|x\|_1^{(1)}\|y\|_1. \tag{4.13}$$

The equality in (4.13) is attained if and only if $x \equiv 0$ or $y \equiv 0$.

Proof. If $x \equiv 0$ or $y \equiv 0$ then we can see that (4.13) becomes an equality. Suppose that x, y are non-zero, non-negative functions, we will prove that

$$\|x{*}y\|_1 > \Big[\frac{1}{4} - \frac{1}{2\mu\pi(1+\pi)^\mu}\Big]\|x\|_1^{(1)}\|y\|_1.$$

Set $x_1 := H_1\{x\} \in \ell_1(\mathbb{T}_h^0)$, where H_1 is given by formula (2.2).

For $m, n \in \mathbb{N}_0$ we have

$$\sum_{k=0}^{\infty} \theta(k, n, m+\mu) > \frac{\pi}{2} - \frac{1}{(m+\mu)(1+\pi)^{m+\mu}} \geq \frac{\pi}{2} - \frac{1}{\mu(1+\pi)^{\mu}}. \tag{4.14}$$

Since x, y are non-negative functions, from (3.5) and (4.14) we have

$$\|x*y\|_1 = h\sum_{k=0}^{\infty} \big|(x*y)(kh)\big| = \frac{h^2}{\pi}\sum_{n=0}^{\infty} x_1(nh) \sum_{m=0}^{\infty} y(mh) \sum_{k=0}^{\infty} \theta(k, n, m+\mu) >$$
$$> \frac{h^2}{\pi}\Big[\frac{\pi}{2} - \frac{1}{\mu(1+\pi)^{\mu}}\Big] \sum_{n=0}^{\infty} x_1(nh) \sum_{m=0}^{\infty} y(mh) =$$
$$= \frac{h^2}{\pi}\Big[\frac{\pi}{2} - \frac{1}{\mu(1+\pi)^{\mu}}\Big] \frac{\|x\|_1^{(1)}}{2h}\frac{\|y\|_1}{h} = \Big[\frac{1}{4} - \frac{1}{2\mu\pi(1+\pi)^{\mu}}\Big]\|x\|_1^{(1)}\|y\|_1.$$

The proof is completed. ■

4.2. Young's type and Saitoh's type inequalities

Theorem 4.2. (A Young's type theorem.) *Let $p,\ q,\ r > 1$ satisfy the condition*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$$

and ρ be the weight function defined in (4.9). Then for $x \in \ell_p(\mathbb{T}_h^0)$, $y \in \ell_q(\mathbb{T}_h^0, \rho)$, $z \in \ell_r(\mathbb{T}_h^0)$ the generalized convolution $x*y$ is well defined in $\ell_\infty(\mathbb{T}_h^0)$ and the following inequality holds

$$\Big|\sum_{k=0}^{\infty}(x*y)(kh)z(kh)\Big| \leq \frac{C(q)}{h^2}\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\rho)}\|z\|_r, \tag{4.15}$$

where the constant $C(q) > 0$ is given by (4.11).

The inequality in (4.15) becomes an equality if and only if $x \equiv 0$ or $y \equiv 0$ or $z \equiv 0$.

Proof. Using the inequalities (4.5), (4.12), (3.9) and performing some analogous arguments to the proofs of other Young's type theorems for other generalized convolutions in the literature, we get (4.15). The inequality in (4.15) becomes an equality if and only if $x \equiv 0$ or $y \equiv 0$ or $z \equiv 0$. ■

From Theorem 4.2, we have the following corollary:

Corollary 4.1. (A Young's type inequality). *Let $\alpha > 0$ and $p, q, r > 1$ satisfy the condition $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and ρ be the weight function defined in (4.9). Let $x \in \ell_p(\mathbb{T}_h^0)$, $y \in \ell_q(\mathbb{T}_h^0, \rho)$ then $x * y \in \ell_r(\mathbb{T}_h^0)$ and*

$$\|x*y\|_r \leq C(q)\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\rho)}, \tag{4.16}$$

where the constant $C(q) > 0$ is given by (4.11).

The inequality (4.16) becomes an equality if and only if $x \equiv 0$ or $y \equiv 0$.

Theorem 4.3. (A Saitoh's type inequality.) *Let $p > 1$ and $\rho_j \in \ell_1(\mathbb{T}_h^0)$, $(j = 1, 2)$ be two functions such that $\rho_j(nh) > 0$, $\forall n \in \mathbb{N}_0$. Then for any $F_j \in \ell_p(\mathbb{T}_h^0, \rho_j)$, $(j = 1, 2)$ we have*

$$F_1\rho_1, F_2\rho_2 \in \ell_1(\mathbb{T}_h^0), \quad \big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1} \in \ell_p(\mathbb{T}_h^0)$$

and the following $\ell_p(\mathbb{T}_h^0)$-weighted inequality for the h-Fourier cosine-Laplace generalized convolution holds

$$\Big\|\big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1}\Big\|_p \leq C\|F_1\|_{\ell_p(\mathbb{T}_h^0,\rho_1)}\|F_2\|_{\ell_p(\mathbb{T}_h^0,\rho_2)}, \tag{4.17}$$

where

$$C = h^{\frac{1-p}{p}}\left(2 + \frac{2}{\mu\pi}\right)^{\frac{1}{p}}.$$

The equality holds if and only if $F_1 \equiv 0$ or $F_2 \equiv 0$.

Proof. Using the inequality (3.9) and Hölder's inequality, by performing some analogous arguments to the proofs of other Saitoh's type inequalities for other generalized convolutions in the literature, we get (4.17). The equality holds if and only if $F_1 \equiv 0$ or $F_2 \equiv 0$. ■

The Specht's ratio was defined by [7, 3, 10]

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e\log\left(t^{\frac{1}{t-1}}\right)} \tag{4.18}$$

for $t > 0$, $t \neq 1$, where the log function is the natural logarithm function and $S(1) = 1$.

Theorem 4.4. (A reverse Saitoh's type inequality.) *Let $p > 1$ and $\rho_j \in \ell_1(\mathbb{T}_h^0)$, $(j = 1, 2)$ be two functions such that $\rho_j(nh) > 0$, $\forall n \in \mathbb{N}_0$. Let F_1 and F_2 be positive functions on $\mathbb{T}_h^0$ satisfying*

$$0 < M_1^{\frac{1}{p}} \leq F_1(nh) \leq M_2^{\frac{1}{p}}, \; 0 < M_3^{\frac{1}{p}} \leq F_2(nh) \leq M_4^{\frac{1}{p}}, \quad \forall n \in \mathbb{N}_0.$$

Then $F_j\rho_j \in \ell_1(\mathbb{T}_h^0),\ F_j \in \ell_p(\mathbb{T}_h^0, \rho_j),\ (j=1,2),$

$$\big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1} \in \ell_p(\mathbb{T}_h^0).$$

Moreover, the following inequality holds

$$\big\|\big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1}\big\|_p > C\|F_1\|_{\ell_p(\mathbb{T}_h^0,\rho_1)}\|F_2\|_{\ell_p(\mathbb{T}_h^0,\rho_2)}, \tag{4.19}$$

where

$$C = \Big(\frac{1}{4} - \frac{1}{2\mu\pi(1+\pi)^\mu}\Big)^{\frac{1}{p}}\Big[S\Big(\frac{M_1M_3}{M_2M_4}\Big)\Big]^{-1} h^{\frac{1-p}{p}},$$

S *is given by* (4.18).

Proof. Using (4.14) and reverse inequality for Hölder's inequality, by doing some similar arguments to the proofs of other reverse Saitoh's type inequalities for other generalized convolutions in the literature, we can prove (4.19).

5. Applications

5.1. Two linear equations

Consider the following two linear equations

$$x(0)M(kh,0) + \sum_{j=1}^{\infty} x(jh)M(kh,jh) = w(kh), \ \ k \in \mathbb{N}_0, \tag{5.1}$$

$$x(kh) + x(0)M(kh,0) + \sum_{j=1}^{\infty} x(jh)M(kh,jh) = w(kh), \tag{5.2}$$

here, for $k \in \mathbb{N}_0,\ j \in \mathbb{N}$

$$\begin{aligned} M(kh,0) =& \frac{h^2}{\pi} y(0)\sum_{m=0}^{\infty} v(mh)I(k,m+\mu) + \\ &+ \frac{h^2}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} y(nh)v(mh)\theta(k,n,m+\mu), \end{aligned}$$

$$\begin{aligned} M(kh,jh) =& \frac{2h^2}{\pi} y(jh)\sum_{m=0}^{\infty} v(mh)I(k,m+\mu) + \\ &+ \frac{h^2}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \big[y(|nh-jh|) + y(nh+jh)\big]v(mh)\theta(k,n,m+\mu), \end{aligned}$$

where y, v, w are given functions in $\ell_1(\mathbb{T}_h^0)$, μ is fixed in $\mathbb{N}$, x is an unknown function in $\ell_1(\mathbb{T}_h^0)$.

Denote

$$\mathcal{A}_c := \left\{\mathcal{F}_c\{x\}(u), u \in \left[0, \frac{\pi}{h}\right] \middle| x \in \ell_1(\mathbb{T}_h^0)\right\}. \tag{5.3}$$

In this subsection, we will use the weight function γ given by (3.1) and apply the generalized convolution in (3.2) to handle our problems.

Theorem 5.1. *Let $y, v \in \ell_1(\mathbb{T}_h^0)$ and $\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u) \neq 0$ on $\left[0, \frac{\pi}{h}\right]$. Then, the necessary and sufficient condition for the existence of the unique solution of equation (5.1) in $\ell_1(\mathbb{T}_h^0)$ is $\dfrac{(1+hu)^\mu \mathcal{F}_c\{w\}(u)}{\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)} \in \mathcal{A}_c$, where $\mathcal{A}_c$ is defined in (5.3). The solution of equation (5.1) can be written in the form*

$$x(nh) = \frac{1}{\pi}\int_0^{\frac{\pi}{h}} \frac{(1+hu)^\mu \mathcal{F}_c\{w\}(u)}{\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)} \cos(unh)du, \; n \in \mathbb{N}_0. \tag{5.4}$$

Furthermore, we have the following inequality for all $p \geq 1$:

$$\|x \underset{\mathcal{F}_c}{*} y\|_p^{(1)} \geq \frac{\mu\pi\|w\|_p}{(1+\mu\pi)\|v\|_1}. \tag{5.5}$$

Proof. From (2.4) and (3.2), equation (5.1) can be rewritten in the form

$$\left[\left(x \underset{\mathcal{F}_c}{*} y\right) * v\right](kh) = w(kh), \quad k \in \mathbb{N}_0. \tag{5.6}$$

- The necessary condition. Applying the h-Fourier cosine transform to equation (5.6) and using factorization identities (2.5) and (3.7), for $u \in$ $\in \left[0, \frac{\pi}{h}\right]$ we have

  $$\mathcal{F}_c\{w\}(u) = \gamma(u)\mathcal{F}_c\{x \underset{\mathcal{F}_c}{*} y\}(u)\mathcal{L}\{v\}(u) = \gamma(u)\mathcal{F}_c\{x\}(u)\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u).$$

  Hence,

  $$\mathcal{F}_c\{x\}(u) = \frac{(1+hu)^\mu \mathcal{F}_c\{w\}(u)}{\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)}.$$

  Thus, $\dfrac{(1+hu)^\mu \mathcal{F}_c\{w\}(u)}{\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)} \in \mathcal{A}_c$ and the solution is given by (5.4).
  We use (3.6) and (4.7) to obtain

  $$\left\|\left(x \underset{\mathcal{F}_c}{*} y\right) * v\right\|_p \leq \left[1 + \frac{1}{\mu\pi}\right] \|x \underset{\mathcal{F}_c}{*} y\|_p^{(1)} \|v\|_1. \tag{5.7}$$

  From (5.6) and (5.7) we get (5.5).

- The sufficient condition. Assume that $\dfrac{(1+hu)^{\mu}\mathcal{F}_c\{w\}(u)}{\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)} \in \mathcal{A}_c$. There exists $x \in \ell_1(\mathbb{T}_h^0)$ such that

$$\mathcal{F}_c\{x\}(u) = \frac{(1+hu)^{\mu}\mathcal{F}_c\{w\}(u)}{\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)}, \quad u \in \left[0, \frac{\pi}{h}\right].$$

Consequently,

$$\begin{aligned}\mathcal{F}_c\{w\}(u) &= \gamma(u)\mathcal{F}_c\{x\}(u)\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u) = \gamma(u)\mathcal{F}_c\{x \underset{\mathcal{F}_c}{*} y\}(u)\mathcal{L}\{v\}(u)\\ &= \mathcal{F}_c\{(x \underset{\mathcal{F}_c}{*} y)*v\}(u), \quad u \in \left[0, \frac{\pi}{h}\right].\end{aligned}$$

Taking the inverse h-Fourier cosine transform of the above we get (5.6).

The proof is completed. ■

Remark 5.1. In Theorem 5.1, if we make additional assumption that $v \in$ $\in \ell_q(\mathbb{T}_h^0, \rho)$, where $p, q, r > 1$ satisfy $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \dfrac{1}{r}$ and ρ be the weight function defined in (4.9), then by using Young's type inequality (4.16) and the same arguments we obtain

$$\|x \underset{\mathcal{F}_c}{*} y\|_p^{(1)} \geq \frac{\|w\|_r}{C(q)\|v\|_{\ell_q(\mathbb{T}_h^0,\rho)}},$$

where the constant $C(q) > 0$ is given by (4.11).

Theorem 5.2. *The necessary and sufficient condition for the equation (5.2) to have a unique solution in $\ell_1(\mathbb{T}_h^0)$, for all right hand side $w \in \ell_1(\mathbb{T}_h^0)$ is*

$$1 + (1+hu)^{-\mu}\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u) \neq 0, \quad u \in \left[0, \frac{\pi}{h}\right]. \tag{5.8}$$

Moreover, the solution of (5.2) can be presented in closed form as follows:

$$x(nh) = w(nh) - (w \underset{\mathcal{F}_c}{*} \psi)(nh), \ n \in \mathbb{N}_0, \tag{5.9}$$

where ψ is defined by

$$\mathcal{F}_c\{\psi\}(u) = \frac{(1+hu)^{-\mu}\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)}{1 + (1+hu)^{-\mu}\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)}, \quad u \in \left[0, \frac{\pi}{h}\right].$$

Proof. *Necessity.* Equation (5.2) can be written as

$$x(kh) + \left[(x \underset{\mathcal{F}_c}{*} y)*v\right](kh) = w(kh), \ \ k \in \mathbb{N}_0. \tag{5.10}$$

Applying the h-Fourier cosine transform to both sides of (5.10) and using factorization identities (2.5) and (3.7), we have

$$\begin{aligned}\mathcal{F}_c\{w\}(u) &= \mathcal{F}_c\{x\}(u) + \gamma(u)\mathcal{F}_c\{x \underset{\mathcal{F}_c}{*} y\}(u)\mathcal{L}\{v\}(u)\\ &= \mathcal{F}_c\{x\}(u)\Big[1 + \gamma(u)\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)\Big], \quad u \in \Big[0, \frac{\pi}{h}\Big].\end{aligned} \tag{5.11}$$

It shows that (5.8) is the necessary condition for the equation (5.2) to have a unique solution in $\ell_1(\mathbb{T}_h^0)$, for all right hand side $w \in \ell_1(\mathbb{T}_h^0)$.

Sufficiency. Assume that (5.8) holds. We use (5.11), (5.8) and (3.7) to obtain

$$\mathcal{F}_c\{x\}(u) = \frac{\mathcal{F}_c\{w\}(u)}{1 + \gamma(u)\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)} = \frac{\mathcal{F}_c\{w\}(u)}{1 + \mathcal{F}_c\{y*v\}(u)}, \quad u \in \Big[0, \frac{\pi}{h}\Big]. \tag{5.12}$$

Since $y*v \in \ell_1(\mathbb{T}_h^0)$, due to Wiener-Lévy type Theorem, there exists a function $\psi \in \ell_1(\mathbb{T}_h^0)$ satisfying

$$\mathcal{F}_c\{\psi\}(u) = \frac{\mathcal{F}_c\{y*v\}(u)}{1 + \mathcal{F}_c\{y*v\}(u)} = \frac{(1+hu)^{-\mu}\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)}{1 + (1+hu)^{-\mu}\mathcal{F}_c\{y\}(u)\mathcal{L}\{v\}(u)}, \; u \in \Big[0, \frac{\pi}{h}\Big].$$

Using factorization identity (2.5), equation (5.12) can be written as

$$\begin{aligned}\mathcal{F}_c\{x\}(u) &= \mathcal{F}_c\{w\}(u) - \mathcal{F}_c\{w\}(u)\mathcal{F}_c\{\psi\}(u)\\ &= \mathcal{F}_c\{w\}(u) - \mathcal{F}_c\{w \underset{\mathcal{F}_c}{*} \psi\}(u), \quad u \in \Big[0, \frac{\pi}{h}\Big].\end{aligned} \tag{5.13}$$

Taking the inverse h-Fourier cosine transform of (5.13) we get (5.9). ■

Remark 5.2. In Theorem 5.2, we consider the case when the function y satisfies $y(0) = \dfrac{1}{h}$, $y(nh) = 0$, $\forall n \in \mathbb{N}$. We have $\mathcal{F}_c\{y\}(u) \equiv 1$, $u \in \mathbb{R}$. Hence, $x \underset{\mathcal{F}_c}{*} y \equiv x$, $\forall x \in \ell_1(\mathbb{T}_h^0)$.

Let $p \geq 1$ be a given number. From (5.10), we obtain the following estimate for the solution x of equation (5.2):

$$\|w\|_p \leq \|x\|_p + \|x * v\|_p \tag{5.14}$$

Combining (3.6) and (4.7) with (5.14) yields

$$\begin{aligned}\|w\|_p &\leq \|x\|_p + \Big[1 + \frac{1}{\mu\pi}\Big]\|x\|_p^{(1)}\|v\|_1 \leq\\ &\leq \|x\|_p + 2\Big[1 + \frac{1}{\mu\pi}\Big]\|x\|_p\|v\|_1.\end{aligned}$$

Therefore,

$$\|x\|_p \geq \Big\{1 + 2\Big[1 + \frac{1}{\mu\pi}\Big]\|v\|_1\Big\}^{-1}\|w\|_p.$$

5.2. A class of linear equations related to operator K

Assume that $p>1$, $a_0=1$, $a_k\in\mathbb{C}$, $k=1,2,\dots,m$ are given numbers, ϱ is a given function in $\ell_1(\mathbb{T}_h^0)$ such that $\varrho(nh)>0$, $\forall n\in\mathbb{N}_0$. We define operator $K:\ell_1(\mathbb{T}_h^0)\to\ell_1(\mathbb{T}_h^0)$ by

$$\big(K(x)\big)(0) := \frac{-1}{h^2}\Big[\frac{\pi^2}{3}x(0)+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}x(nh)\Big],$$
$$\big(K(x)\big)(jh) := \frac{-1}{h^2}\Big[\frac{2(-1)^j}{j^2}x(0)+A\{x\}(jh)\Big],\quad j\in\mathbb{N},$$

where

$$A\{x\}(jh)=\Big(\frac{\pi^2}{3}+\frac{1}{2j^2}\Big)x(jh)+\sum_{n\in\mathbb{N}\setminus\{j\}}(-1)^{j+n}\frac{4(j^2+n^2)}{(j^2-n^2)^2}x(nh),\quad j\in\mathbb{N}.$$

We consider the following linear equation related to the operator K

$$\Big(\sum_{k=0}^{m}(-1)^k a_k K^k\Big)x = y\varrho. \tag{5.15}$$

Here, $y\in\ell_p(\mathbb{T}_h^0,\varrho)$ is a given function and x is an unknown function in $\ell_1(\mathbb{T}_h^0)$.

Lemma 5.1. *Suppose $\eta\in\ell_1(\mathbb{T}_h^0)$ is a given function such that $\eta(nh)>0$, $\forall n\in\mathbb{N}_0$. Assume that there exists $Q\in\ell_p(\mathbb{T}_h^0,\eta)$ with the following property:*

$$\mathcal{L}\{Q\eta\}(u)=\frac{(1+hu)^{\mu}}{\sum\limits_{k=0}^{m}a_k u^{2k}},\quad \forall u\in\Big[0,\frac{\pi}{h}\Big]. \tag{5.16}$$

Then the equation (5.15) has a unique solution in $\ell_1(\mathbb{T}_h^0)$. Moreover, the solution can be written in the form $x=(y\varrho)*(Q\eta)$ and we have the following inequality

$$\big\|x(\varrho*\eta)^{\frac{1}{p}-1}\big\|_p\le h^{\frac{1-p}{p}}\Big(2+\frac{2}{\mu\pi}\Big)^{\frac{1}{p}}\|y\|_{\ell_p(\mathbb{T}_h^0,\varrho)}\|Q\|_{\ell_p(\mathbb{T}_h^0,\eta)}. \tag{5.17}$$

The equality holds if and only if $y\equiv 0$.

Proof. The following formula is valid:

$$\mathcal{F}_c\big\{K(x)\big\}(u)=-u^2\mathcal{F}_c\{x\}(u),\quad u\in\Big[0,\frac{\pi}{h}\Big]. \tag{5.18}$$

Applying the h-Fourier cosine to equation (5.15) and using formula (5.18), we have

$$\Big(\sum_{k=0}^{m} a_k u^{2k}\Big)\mathcal{F}_c\{x\}(u) = \mathcal{F}_c\{y\varrho\}(u), \quad u \in \Big[0, \frac{\pi}{h}\Big]. \tag{5.19}$$

Combining (5.16) and (5.19), it follows that

$$\mathcal{F}_c\{x\}(u) = (1+hu)^{-\mu}\mathcal{F}_c\{y\varrho\}(u)\mathcal{L}\{Q\eta\}(u),\ u \in \Big[0, \frac{\pi}{h}\Big]. \tag{5.20}$$

From factorization identity (3.7) and (5.20) we deduce that $x = (y\varrho)*(Q\eta)$. We then use Theorem 4.3 to obtain (5.17). ■

Remark 5.3. In Lemma 5.1, if the functions y and Q satisfy

$$0 < M_1^{\frac{1}{p}} \le y(nh) \le M_2^{\frac{1}{p}}, \qquad 0 < M_3^{\frac{1}{p}} \le Q(nh) \le M_4^{\frac{1}{p}}, \qquad \forall n \in \mathbb{N}_0,$$

then from Theorem 4.4, we obtain the following inequality

$$\frac{\big\|x(\varrho*\eta)^{\frac{1}{p}-1}\big\|_p}{\|Q\|_{\ell_p(\mathbb{T}_h^0,\eta)}} > \Big(\frac{1}{4} - \frac{1}{2\mu\pi(1+\pi)^{\mu}}\Big)^{\frac{1}{p}} \Big[S\Big(\frac{M_1M_3}{M_2M_4}\Big)\Big]^{-1} h^{\frac{1-p}{p}} \|y\|_{\ell_p(\mathbb{T}_h^0,\varrho)}.$$

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