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MEAN-VALUES ASSOCIATED WITH A GENERALIZED SCHEMMEL FUNCTION

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Abstract. We prove mean-value results for Schemmel's function (generalizing Euler's totient function). This function was originally defined to count the number of sets of m consecutive integers each being $\leq n$ and coprime to n. We generalize this to all integers m and consider its various mean-values.

1. Introduction and motivation

For a positive integer m, Victor Schemmel [15] introduced the arithmetic function

$$n \mapsto \varphi_m(n) := n \prod_{p|n} \left(1 - \frac{m}{p}\right), \tag{1.1}$$

which generalizes Euler's totient function $\varphi = \varphi_1$. He observed that for m smaller than all prime factors of $n > 1$, there exist $\varphi_m(n)$ many sets of m consecutive integers in the interval $[1, n)$ each of which being relatively coprime to n.

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Moreover, for $n \geq 2$, we have $\varphi_m(n) = 0$ if, and only if, m is a prime divisor of n. Since φ_m is multiplicative, by using the inclusion–exclusion principle, we also have

$$\varphi_m(n) = n \sum_{d|n} \frac{\mu(d)}{d} m^{\omega(d)}, \tag{1.2}$$

where $\omega(d)$ counts the number of distinct prime divisors of d and μ is the Möbius μ-function, i.e., $\mu(d) = (-1)^{\omega(d)}$ for square-free n, and $\mu(d) = 0$ otherwise.

It seems that not much is known about Schemmel's function and its inventor. Victor Schemmel was a German mathematician of the 19th century (1840-1897); he received his doctorate in Breslau (now Wrocław in Poland) in 1863. His function was used by Derrick Norman Lehmer [9] in the context of magic squares. More recently, Schemmel's function appeared in formulas enumerating cliques in direct product graphs [2] by Colin Defant. He also studied the behavior of the iterates of Schemmel's function [3]. Further references on $\varphi_m(n)$ can be found in the Handbook of Number Theory, vol. II [13].

In this note we prove certain asymptotic formulae for Schemmel's function $\varphi_m(n)$ with respect to n and m. It turns out that our reasoning also holds for negative values of m. Indeed, writing δm with $\delta = \pm 1$ and $m \in \mathbb{N}$ in place of m above, we find

$$\varphi_{\delta m}(n) = n \prod_{p|n} \left(1 - \delta \frac{m}{p}\right) = n \sum_{d|n} \frac{\mu(d)}{d} (\delta m)^{\omega(d)}, \tag{1.3}$$

which generalizes (1.1) and (1.2). Disregarding the original definition we shall also call φ_m with a non-positive m Schemmel's function. Without much effort, quite a few of our investigations can be extended to the case where m is not necessarily an integer.

And here come our main results.

Theorem 1.1. *For arbitrary $m \in \mathbb{Z}\setminus\{0\}$, as $N \to \infty$,*

$$\sum_{1 \leq n \leq N} \varphi_m(n) = \tfrac{1}{2} \prod_p \left(1 - \frac{m}{p^2}\right) \cdot N^2 + O\left(N \frac{(\log N)^{|m|}}{|m|!}\right).$$

The error term is uniform for all m from a compact interval contained in $[-R, R] \setminus \{0, 2^2, 3^2, 5^2, 7^2, 11^2, \ldots\}$, for any $R > 0$.

For $m = 0$ the exact trivial formula

$$\sum_{1 \leq n \leq N} \varphi_0(n) = \sum_{1 \leq n \leq N} n = \tfrac{1}{2} N(N+1).$$

is valid; the right hand side matches the asymptotic formula of the theorem.

For a square $m = p^2$ of a prime the main term vanishes and the asymptotic formula reduces to an error estimate. This phenomenon is illustrated in the figure below, where the logarithms of $\sum_{1 \le n \le N} \varphi_m(n)$ for $m = 1, 3^2, 2 \cdot 5$ are printed.

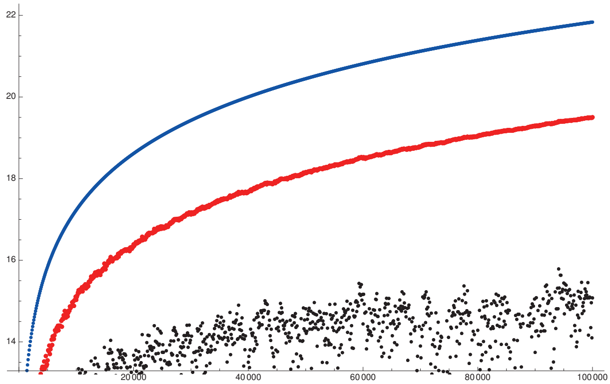

Figure 1. The upper curve is for the classical case of Euler's totient function; the one in the middle belongs to $m = 10$, and the "cloud" below is related to $m = 9$.

Theorem 1.2. *For fixed $n \in \mathbb{N}$, let η be the squarefree kernel of n (or radical of n). Then we have*

$$\sum_{1 \le m < M} \varphi_m(n) = nP(M)$$

with a polynomial $P \in \mathbb{Q}[X]$ of degree $\omega(\eta) + 1$ given by

$$P(X) = -1 + X + \sum_{0 \le k \le \omega(\eta)} c_k X^{k+1},$$

where

$$c_k = \sum_{\substack{d \mid n \\ \omega(d) \ge \max\{1,k\}}} \frac{\mu(d)}{d} \binom{\omega(d)}{\omega(d) - k} \frac{B_{\omega(d)-k}}{k+1},$$

where the Bernoulli numbers B_j are implicitly defined by

$$\frac{x}{\exp(x) - 1} = \sum_{j \ge 0} B_j \frac{x^j}{j!} = 1 - \frac{1}{2}x + \frac{1}{6}x^2 - \frac{1}{30}x^4 \pm \dots .$$

For

$$\sum_{-M < m \le -1} \varphi_m(n) = nQ(M),$$

the coefficients of the polynomial $Q \in \mathbb{Q}[X]$ can be written in a similar way to those of $P(X)$ by replacing $\mu(d)$ with $(-1)^{\omega(d)}\mu(d)$.

Combining the above and $\varphi_0(n) = n$,

$$\sum_{-M<m<M} \varphi_m(n) = n\left(-1 + 2M + 2\sum_{0\leq k\leq \omega(\eta)} d_k M^{k+1}\right),$$

where

$$d_k = \sum_{\substack{d|n,\\ \omega(d)\geq \max\{1,k\},\\ \omega(d):\ even}} \frac{\mu(d)}{d}\binom{\omega(d)}{\omega(d)-k}\frac{B_{\omega(d)-k}}{k+1}.$$

We give the proofs of these theorems in the next two sections. The fourth section deals with mean-value results for the reciprocal of $\varphi_m(n)$. We also mention a few details about the Dirichlet series associated with $\varphi_m(n)$ before concluding our paper.

2. The average with respect to n (Proof of Theorem 1.1)

We follow the reasoning for Euler's totient function (the case $m = 1$) that can be found in many textbooks (e.g. [1]). Using (1.2), we find

$$\begin{aligned}\sum_{n\leq N}\varphi_m(n) &= \sum_{bd\leq N} b\mu(d)m^{\omega(d)} = \sum_{d\leq N}\mu(d)m^{\omega(d)}\sum_{b\leq N/d} b = \\ &= \frac{1}{2}N^2\sum_{d\leq N}\frac{\mu(d)}{d^2}m^{\omega(d)} + O\left(N\sum_{d\leq N}\frac{|m|^{\omega(d)}}{d}\right).\end{aligned} \tag{2.1}$$

In view of $2^{\omega(z)} \leq d(z)$ for $z \in \mathbb{N}$, resp. $m^{\omega(z)} \leq d(z)^{\log m/\log 2} \ll z^{\epsilon}$, where $z \mapsto d(z) = \sum_{d|z} 1$ denotes the divisor function; the error term here is of size $N^{1+\epsilon}$. However, with a little more effort, we can achieve a slightly better error term.

For this aim we use a result of Atle Selberg [14], namely that

$$A_z(x) := \sum_{d\leq x} z^{\omega(d)} = c(z)x(\log x)^{z-1} + O\left(x(\log x)^{\mathrm{Re}\, z-2}\right), \tag{2.2}$$

where z is a complex number and $c(z)$ is the Euler product

$$c(z) = \frac{1}{\Gamma(z)}\prod_p\left(1+\frac{z}{p-1}\right)\left(1-\frac{1}{p}\right)^z.$$

The error term in (2.2) is uniform for $|z| \le R$ with any $R > 0$ and the Euler product (taken over the primes) is absolutely convergent; see also the paper [4] by Hubert Delange. For the Euler product we observe that $c(1) = 1$ and for $z = |m| \in \mathbb{N}$ in general

$$\frac{c(|m|+1)}{c(|m|)} = \frac{\Gamma(|m|)}{\Gamma(|m|+1)} \prod_p \frac{p+|m|}{p+|m|-1} \frac{p-1}{p} \le \frac{1}{m},$$

hence

$$c(|m|) \le \frac{1}{(|m|-1)!}.$$

By partial summation we thus get

$$\sum_{d \le N} \frac{|m|^{\omega(d)}}{d} = A_{|m|}(N)N^{-1} + \int_1^N A_{|m|}(x) \frac{\mathrm{dx}}{x^2} \ll \frac{(\log N)^{|m|}}{(|m|)!}.$$

We continue by noting that

$$\sum_{d \le N} \frac{\mu(d)}{d^2} m^{\omega(d)} = \sum_{d \ge 1} \frac{\mu(d)}{d^2} m^{\omega(d)} - \sum_{d > N} \frac{\mu(d)}{d^2} m^{\omega(d)}.$$

Applying once more partial summation in combination with (2.2) yields in a similar fashion

$$\begin{aligned} \sum_{N < d \le M} \frac{|m|^{\omega(d)}}{d^2} &\ll N^{-1} \frac{(\log N)^{|m|-1}}{(|m|-1)!} + \int_N^M \frac{(\log x)^{|m|-1}}{(|m|-1)!} \frac{\mathrm{dx}}{x^2} \ll \\ &\ll N^{-1} \frac{(\log N)^{|m|-1}}{(|m|-1)!}. \end{aligned}$$

By letting $M \to \infty$, we get

$$\sum_{d \le N} \frac{\mu(d)}{d^2} m^{\omega(d)} = \prod_p \left(1 - \frac{m}{p^2}\right) + O\left(N^{-1} \frac{(\log N)^{|m|-1}}{(|m|-1)!}\right).$$

Substituting this in (2.1) implies the statement of Theorem 1.1. We remark that the main term vanishes when m is a square of a prime, hence this limits the uniformity of our estimate with respect to m.

The best error term for Euler's totient function so far had been given by a method due to Arnold Walfisz [17], using advanced estimates for exponential sums:

$$\sum_{n \le N} \varphi(n) = \frac{3}{\pi^2} N^2 + O(N(\log N)^{\frac{2}{3}} (\log \log N)^{\delta}).$$

Walfisz obtained $\delta = \frac{4}{3}$, and with further refinement of the method, Hongquan Liu [10] obtained $\delta = \frac{1}{3}$. For similar estimates for a more general class of arithmetic functions, we refer to Yuta Suzuki [16]. In the case of $\varphi_m(n)$, an error term $O(N)$ is impossible since $\varphi_m(N) = N - m \sim N$ for prime N.

3. The average with respect to m (Proof of Theorem 1.2)

It is not difficult to compute the mean-value of $\varphi_m(n)$ with respect to m explicitly. For this purpose, we deduce from (1.3) that, for fixed n,

$$\sum_{1\leq m<M} \varphi_m(n) = n \sum_{d|n} \frac{\mu(d)}{d} \sum_{1\leq m<M} m^{\omega(d)}$$

and

$$\sum_{-M<m\leq -1} \varphi_m(n) = n \sum_{d|n} \frac{\mu(d)}{d} \sum_{1\leq m<M} (-m)^{\omega(d)}.$$

Next we use a classical result due to Johann Faulhaber (which in some literature is also addressed as (Jacob) Bernoulli's formula), resp. its modern version

$$\sum_{1\leq m<M} m^{\omega} = \sum_{0\leq j\leq \omega} \binom{\omega}{j} \frac{B_j}{\omega+1-j} M^{\omega+1-j}, \tag{3.1}$$

valid only for $\omega \geq 1$; a proof can be found in [6], for historical details we refer to [7]. Hence,

$$\begin{aligned}\sum_{1\leq m<M} \varphi_m(n) &= n \sum_{\substack{d|n\\ \omega(d)\geq 1}} \frac{\mu(d)}{d} \sum_{0\leq j\leq \omega(d)} \binom{\omega(d)}{j} \times \\ &\quad \times \frac{B_j}{\omega(d)+1-j} M^{\omega(d)+1-j} + n(M-1)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\sum_{-M<m\leq -1} \varphi_m(n) &= n \sum_{\substack{d|n\\ \omega(d)\geq 1}} \frac{\mu(d)}{d} \sum_{0\leq j\leq \omega(d)} (-1)^{\omega(d)} \binom{\omega(d)}{j} \times \\ &\quad \times \frac{B_j}{\omega(d)+1-j} M^{\omega(d)+1-j} + n(M-1),\end{aligned}$$

where the last term results from those d satisfying $\omega(d) = 0$.

Denoting by η be the squarefree kernel of n (or radical of n), that is $\eta = \prod_{p|n} p$, we notice that $\omega(\eta) \geq \omega(d)$ for all divisors d of n with $\mu(d) \neq 0$. Thus collecting all terms with constant difference $k = \omega(d) - j$, we obtain

$$\sum_{1\leq m<M} \varphi_m(n) = n \sum_{0\leq k\leq \omega(\eta)} \sum_{\substack{d|n \\ \omega(d)\geq \max\{1,k\}}} \frac{\mu(d)}{d} \binom{\omega(d)}{\omega(d)-k} \frac{B_{\omega(d)-k}}{k+1} M^{k+1} + n(M-1)$$

and similarly for negative m. Adding the two sums and incorporating the case $m = 0$ yields Theorem 1.2. ■

4. Means for the reciprocal

Next we discuss the average of the reciprocal of Schemmel's function with respect to m. We first observe for any prime power p^α that whenever $m > p$, we have

$$\frac{1}{\varphi_m(p^\alpha)} = -\frac{1}{mp^{\alpha-1} - p^\alpha} = -\frac{1}{mp^{\alpha-1}} \cdot \frac{1}{1-\frac{p}{m}} = -\frac{1}{mp^{\alpha-1}} \left(1 + O\left(\frac{1}{m}\right)\right).$$

Since $\varphi_m(n)$ is multiplicative with respect to n, writing $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $p_1 < p_2 < \ldots < p_k$, $\alpha_1, \ldots, \alpha_k \geq 1$, $k \in \mathbb{N}$, it thus follows that

$$\begin{aligned}
\sum_{p_k<m\leq M} \frac{1}{\varphi_m(n)} &= \sum_{p_k<m\leq M} \prod_{i=1}^{k} \frac{1}{\varphi_m(p_i^{\alpha_i})} = \\
&= \sum_{p_k<m\leq M} \prod_{i=1}^{k} \left(\frac{-1}{mp_i^{\alpha_i-1}} + O\left(\frac{1}{m^2}\right)\right) = \\
&= \sum_{p_k<m\leq M} \left(\prod_{i=1}^{k} \frac{-p_i}{mp_i^{\alpha_i}} + O\left(\frac{1}{m^3}\right)\right) = \\
&= \sum_{p_k<m\leq M} \frac{(-1)^k p_1\cdots p_k}{m^k n} + O\left(\frac{1}{p_k^2}\right).
\end{aligned}$$

Hence for $k = 1$, that is when n has only one prime factor p_1,

$$\sum_{p_1<m\leq M} \frac{1}{\varphi_m(n)} \sim -\frac{p_1}{n} \log M.$$

For $k \geq 2$, however, the mean-value cannot be presented in such a simple form.

For the average with respect to n we again follow Landau's [8] result for the case of Euler's totient function. Unlike the classical case, however, $\varphi_m(n)$ vanishes if, and only if, m is a prime factor of n. Thus for a prime number m, we consider

$$\sum_{\substack{n\le N\\ m\nmid n}}\frac{1}{\varphi_m(n)}.$$

Meanwhile, for any non-prime m, we can consider

$$\sum_{n\le N}\frac{1}{\varphi_m(n)}.$$

Here for $m>0$, we consider the sum in a more general context, that is

$$\sum_{\substack{n\le N\\ \gcd(n,m)=1}}\frac{1}{\varphi_m(n)}$$

for any $m\in\mathbb{N}$, as $N\to\infty$.

Taking into account the multiplicativity of $\varphi_m(n)$ with respect to n, we find

$$\frac{1}{\varphi_m(n)}=\frac{1}{n}\prod_{p|n}\frac{p}{p-m}=\frac{1}{n}\prod_{p|n}\left(1+\frac{m}{p-m}\right)=\frac{1}{n}\sum_{d|n}\frac{\mu(d)^2}{\varphi_m(d)}m^{\omega(d)}. \tag{4.1}$$

For coprime n and m,

$$\begin{aligned}\sum_{\substack{n\le N\\ \gcd(n,m)=1}}\frac{1}{\varphi_m(n)} &= \sum_{\substack{n\le N\\ \gcd(n,m)=1}}\frac{1}{n}\sum_{d|n}\frac{\mu(d)^2}{\varphi_m(d)}m^{\omega(d)}=\\ &= \sum_{\substack{d\le N\\ \gcd(d,m)=1}}\frac{\mu(d)^2}{d}\frac{m^{\omega(d)}}{\varphi_m(d)}\sum_{\substack{b\le N/d\\ \gcd(b,m)=1}}\frac{1}{b}.\end{aligned}$$

For the inner sum, we have

$$\sum_{\substack{n\le X\\ \gcd(n,m)=1}}\frac{1}{n}=\sum_{\substack{a \bmod m\\ \gcd(a,m)=1}}\sum_{\substack{n\le X\\ n\equiv a \bmod m}}\frac{1}{n}\sim\frac{\log X}{m}\sum_{\substack{b \bmod m\\ \gcd(b,m)=1}}1=\frac{\log X}{m}\varphi(m),$$

where we have used the well known asymptotic formula

$$\sum_{n\le x}\frac{1}{n}=\log x+O(1).$$

This leads to

$$\sum_{\substack{n\le N\\ \gcd(n,m)=1}} \frac{1}{\varphi_m(n)} \sim \frac{\varphi(m)}{m}(\log N) \sum_{\substack{d\le N\\ \gcd(d,m)=1}} \frac{\mu(d)^2}{d}\,\frac{m^{\omega(d)}}{\varphi_m(d)} \sim$$

$$\sim \prod_{p|m}\left(1-\frac{1}{p}\right)\prod_{p\nmid m}\left(1+\frac{m}{p(p-m)}\right)\cdot \log N$$

This yields

Theorem 4.1. *For fixed $m\in\mathbb{N}$, as $N\to\infty$,*

$$\sum_{\substack{n\le N\\ \gcd(n,m)=1}} \frac{1}{\varphi_m(n)} \sim \Pi \log N,$$

where Π is a convergent Euler product

$$\Pi = \prod_{p|m}\left(1-\frac{1}{p}\right)\prod_{p\nmid m}\left(1+\frac{m}{p(p-m)}\right).$$

Remark. This result is contained in a more general theorem due to Kent Wooldridge [18].

When $m\le 0$, again using (4.1),

$$\sum_{n\le N}\frac{1}{\varphi_m(n)} = \sum_{n\le N}\frac{1}{n}\sum_{d|n}\frac{\mu(d)^2}{\varphi_m(d)}m^{\omega(d)} = \sum_{d\le N}\frac{1}{d}\frac{\mu(d)^2}{\varphi_m(d)}m^{\omega(d)}\sum_{b\le N/d}\frac{1}{b} =$$

$$= \sum_{d\le N}\frac{1}{d}\,\frac{\mu(d)^2}{\varphi_m(d)}m^{\omega(d)}\left(\log\frac{N}{d}+O(1)\right).$$

We continue by extending the sum of the main term to an infinite series and using multiplicativity; this gives

$$\sum_{n\le N}\frac{1}{\varphi_m(n)} = \left(\sum_{d=1}^{\infty}\frac{1}{d}\,\frac{\mu(d)^2}{\varphi_m(d)}\,m^{\omega(d)}\right)\log N + O\left(\sum_{d>N}\frac{1}{d}\,\frac{|m|^{\omega(d)}}{|\varphi_m(d)|}\log N\right) +$$

$$+ O\left(\sum_{d\le N}\frac{\log d}{d}\,\frac{|m|^{\omega(d)}}{|\varphi_m(d)|}\right) \sim$$

$$\sim \prod_{p}\left(1+\frac{m}{p(p-m)}\right)\cdot \log N.$$

5. Concluding remarks

We begin with a brief discussion of the Dirichlet series associated with Schemmel's function. In the case of Euler's totient function $\varphi(n)$, we have

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^s}=\zeta(s-1)\prod_p\left(1-\frac{1}{p^s}\right)=\frac{\zeta(s-1)}{\zeta(s)}.$$

for $\mathrm{Re}(s)>2$.

For $m\in\mathbb{Z}$ in general, by multiplicativity, one can easily show

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\varphi_m(n)}{n^s} &= \prod_p\left(1+\sum_{\alpha=1}^{\infty}\varphi_m(p^\alpha)p^{-\alpha s}\right)=\\ &= \prod_p\left(1+\sum_{\alpha=1}^{\infty}\left(1-\frac{m}{p}\right)p^{\alpha(1-s)}\right)=\\ &= \zeta(s-1)\prod_p\left(1-\frac{m}{p^s}\right),\end{aligned}\tag{5.1}$$

which again holds if $\mathrm{Re}\,(s)>2$. Note that in this region, the product

$$\prod_p\left(1-\frac{m}{p^s}\right)$$

is absolutely convergent for any fixed m.

Further we note that $s=2$ gives rise to the pole of $\zeta(s-1)$ and applying Perron's formula, the residue of this term would yield the main term in Theorem 1.1. This may explain why we have the prime squares dominating the estimate in Theorem 1.1 as the main term of our asymptotic formula.

We continue our observation in the region $\mathrm{Re}(s)>2$. When $m=0$, we have

$$\sum_{n=1}^{\infty}\frac{\varphi_0(n)}{n^s}=\zeta(s-1),$$

which is a single Riemann zeta function shifted by one to the right. However this is trivial since $\varphi_0(n)=n$.

When $m=-1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\varphi_{-1}(n)}{n^s} &= \zeta(s-1)\prod_p\left(1+\frac{1}{p^s}\right)=\zeta(s-1)\,\frac{\zeta(s)}{\zeta(2s)}=\\ &= \zeta(s-1)\left(\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}\right)^{-1},\end{aligned}$$

where $\lambda(n)$ is the Liouville function.

Next we consider the size of $\varphi_m(n)$. First of all, for positive m,

$$|\varphi_m(n)| = n\prod_{p|n}\left|1-\frac{m}{p}\right| \le n\prod_{\substack{p|n\\ p<m}}\frac{m}{p} \le n\cdot m^{\pi(m)},$$

where $\pi(m)$ counts the number of primes $p\le m$; in view of well-known facts about the distribution of primes we deduce

$$|\varphi_m(n)| \le n\cdot\exp((1+\epsilon)m)$$

for every positive ϵ. If m is negative, however, then

$$\varphi_m(n) = n\prod_{p|n}\left(1+\frac{|m|}{p}\right) \le n\left(1+\frac{|m||}{2}\right)^{\omega(n)}.$$

From [11, Theorem 2.10], we have

$$1\le\omega(n)\le\frac{\log n}{\log\log n}\left(1+O\left(\frac{1}{\log\log n}\right)\right), \qquad n\ge 3.$$

We note that $\omega(1)=0$ and $\omega(2)=1$. Thus we arrive at

$$|\varphi_m(n)| \le n^{1+O\left(\frac{\log\left(1+\frac{|m|}{2}\right)}{\log\log n}\right)}.$$

The above bounds for $|\varphi_m(n)|$ show that the error term we obtained is probably close to optimal.

Finally, we remark that another generalization of the Euler's totient function $\varphi(n)$ has been considered by Pieter Moree et al. [12] by taking the k-th power of n and p with $m=1$ in the definition (1.1), that is

$$n^k\prod_{p|n}\left(1-\frac{1}{p^k}\right).$$

This quantitiy counts the number of k-tuples chosen from a complete residue system modulo n such that the greatest common divisor of each set is coprime to n. It is usually called the k-th Jordan totient function $J_k(n)$. Some of their results are pretty close to ours (Theorem 1.1); in particular in those cases where the Selberg–Delange method is in the background.

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