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CONVOLUTION OPERATORS ON THE DISK

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Abstract. In this work, we introduce hyperbolic convolution operators using a subgroup of the Blaschke group. The subgroup in question can be interpreted as the group of direction preserving translations on the complex unit disk. We introduce a group norm and the Haar integral which are invariant with respect to this hyperbolic translation. Using these tools, we propose the notions of hyperbolic convolution and approximate identity. Finally, we extend the investigated group of direction preserving hyperbolic translations with a dilation operation and investigate some properties of the resulting field.

1. Introduction

Neural networks employing convolution layers have revolutionized 2D and 3D image processing applications in recent years (see [14]). One of the main contributions of this paper is the introduction of a new convolution concept which remains inside a certain region (such as the complex unit disk). This property could be beneficial especially for applications. Starting from the class of Blaschke functions $\mathfrak{B}$ we identify a (transformation) group, in which the group operation is the function composition operation $\circ$ [18, 20]. In this way we can also construct the concept of hyperbolic wavelets which have applications in for example control theory [4, 17, 18]. Other important function systems, such as the Zernike system commonly used in optics, or the discrete Laguerre system used in control theory can also be derived from the representations of the

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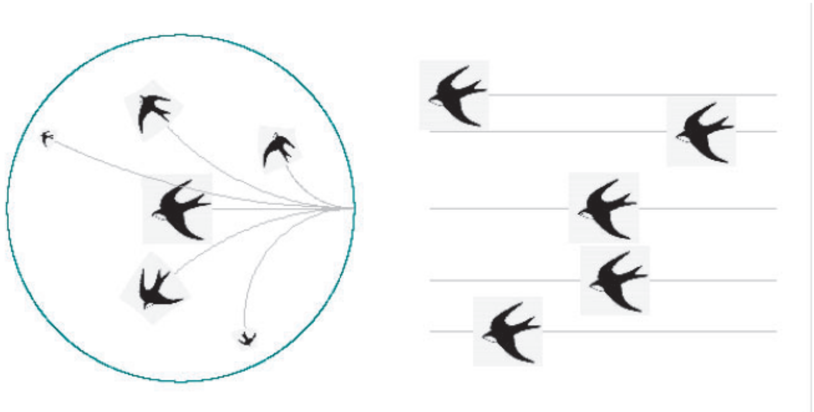
Blaschke group [2, 3, 7, 12, 19, 20]. Blaschke introduced the functions named after him in [1], where he used them to describe so-called inner functions of the space H^p $(0 < p \le \infty)$. Functions belonging to Hardy spaces, whose values on the unit circle are 1 a.e. are referred to as *inner functions*. It turned out, that such functions can be written as products of Blaschke functions. In addition to contributing greatly to the theory of analytic functions, such factorization theorems also help to mathematically formalize important concepts in control theory [12]. The Blaschke group also has important geometrical interpretations. Namely, Blaschke functions describe the congruence transformations of the Bólyai–Lobachevsky (hyperbolic) geometry in the Poincaré disk model [5]. Hyperbolic translation operators can also be introduced easily with Blaschke functions, for example the translation of the trigonometric system leads to the discrete Laguerre system. In this paper, we use this notion of translation to introduce hyperbolic convolution operators. Using Blaschke products, Malmquist and Takenaka have independently introduced the rational orthonormal Malmquist–Takenaka (MT) systems [15, 21]. The MT systems contain as a special case, several other important orthogonal systems (such as the discrete Laguerre and Kautz systems). Today, MT expansions have become an important tool for the identification of dynamic systems. In the last decade, MT expansions were successfully applied for the identification, compression and classification of signals [4, 9, 13, 14, 15]. Functions belonging to the Blaschke group can be parameterized by $\mathfrak{B} := \{B_{\mathfrak{b}} = \epsilon B_b : \mathfrak{b} := (\epsilon, b) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}\}$, where $B_b(z) := (z-b)/(1-\bar{b}z)$. The Blaschke group is isomorphic to the group $\mathbb{SU}(1,1)$ used in representation theory [23]. Describing this group by Blaschke functions has several benefits. Fundamental concepts in harmonic analysis (i.e. representation, convolution etc.) can be directly related to MT expansions in this model. *Unger et. al.* [8, 22] introduced the so-called *gyro-operation* and in connection to this, a complicated algebraic structure to describe relative velocity. Using operations of the Blaschke group introduced here, these structures can be significantly simplified. Another advantage of the above mentioned parameterezation of the Blaschke functions is that using the group $\mathbb{B}$ we can describe the congruence transformations in the Poincaré disk model of the Bólyai-Lobachevsky geometry [5]. This connection makes it possible to geometrically describe the introduced concepts and algorithms as well as allowing us to introduce a new *hyperbolic wavelet* concept [18] akin to the classical *affine* wavelets [6, 10, 11, 16]. In this work, instead of the group $\mathfrak{B}$ we consider its subgroup, which maps the point $1 \in \mathbb{T}$ onto itself: $\mathfrak{B}^* := \{B \in \mathfrak{B} : B(1) = 1\}$. This subgroup allows us to parameterize $\mathfrak{B}^*$ using only the points of $\mathbb{D}$ and introduce a structure on the disk that is isomorphic to the group $(\mathfrak{B}^*, \circ)$. We will refer to members of this subgroup as *direction preserving hyperbolic motions*.

The rest of this paper is organized as follows. In section 2 we discuss the geometric interpretation of the group $(\mathfrak{B}^*, \circ)$ and introduce a group norm as well as a translation invariant integral. In section 3, we consider convolution operator on the $\mathbb{D}$ disk and prove the analogue of the theorem about the convergence of integral means. In section 4, we prove an analogous theorem to the theorem on approximate identity. This section can be interpreted as a special case of the following. Classical wavelets and related transformations are based on affine mappings of the real line. Taking the Blaschke group instead of the affine group a new, hyperbolic wavelet concept can be introduced, for which we previously proved some fundamental formulas. In this paper, we provide the tools to introduce a new type of hyperbolic wavelet concept, which uses the analogues of classical translation and dilation operators in the hyperbolic case. Section 4 can be interpreted as an example on how to use this approach to represent signals and images on the disk.



Direction preserving motions on the hyperbolic and Euclidean planes.

2. Group of direction preserving motions

In this section we study subgroups of the Blaschke group which map a point of the torus onto itself. Henceforth, we assume that this special point is $1 \in \mathbb{T}$. Blaschke functions are defined as

$$B_b(z) := \frac{z-b}{1-\bar{b}z} \quad (b \in \mathbb{D}, z \in \overline{\mathbb{D}}). \tag{2.1}$$

From the identity

$$1 - |B_b(z)|^2 = \frac{(1-|b|^2)(1-|z|^2)}{|1-\bar{b}z|^2} \quad (b \in \mathbb{D}, z \in \overline{\mathbb{D}}), \tag{2.2}$$

we get that Blaschke functions map the torus and the disk onto themselves. Since by

$$B_a(B_b(z)) = \epsilon B_c(z),\ c = B_{-b}(a),\ \epsilon = B_{-a\bar{b}}(1) \quad (z \in \overline{\mathbb{D}}, a, b \in \mathbb{D}), \tag{2.3}$$

the composition of two Blaschke functions is not necessarily of form (2.1), it is useful to generalize the class of Blaschke functions by introducing the set

$$\mathfrak{B} := \{B_{\mathfrak{b}} := \epsilon B_b : \mathfrak{b} = (b, \epsilon) \in \mathbb{B} := \mathbb{T} \times \mathbb{D}\}. \tag{2.4}$$

The class of functions (2.4) also includes the inverse mapping $B_{\mathfrak{b}}^{-1} = B_{\mathfrak{b}^-}$ of any element $B_{\mathfrak{b}}$, where

$$\mathfrak{b}^- := (b^-, \epsilon^-),\ b^- := -\epsilon b, \epsilon^- := \overline{\epsilon} \quad (\mathfrak{b} = (b, \epsilon) \in \mathbb{B}). \tag{2.5}$$

The composition of two functions from $\mathfrak{B}$ also belongs to $\mathfrak{B}$:

$$\begin{gathered} B_{\mathfrak{a}} \circ B_{\mathfrak{b}} = B_{\mathfrak{c}},\ \mathfrak{c} = (B_{\mathfrak{b}^-}(a), \epsilon_c),\ \epsilon_c = \epsilon_a B_{-a\bar{b}}(\epsilon_b) \\ \mathfrak{a} = (a, \epsilon_a), \mathfrak{b} = (b, \epsilon_b) \in \mathbb{B}). \end{gathered} \tag{2.6}$$

By the above, the ordered pair $(\mathfrak{B}, \circ)$ defines a *transformation group* on $\overline{\mathbb{D}}$, whose identity element is the $B_{\mathfrak{e}}$ $(\mathfrak{e} = (0, 1))$ identity map. We call the group $(\mathfrak{B}, \circ)$ *Blaschke group*. By (2.6), the bijection $\mathbb{B} \ni \mathfrak{b} \to B_b \in \mathfrak{B}$ induces a group structure on the set $\mathbb{B}$. We denote this group by $(\mathbb{B}, \circ)$ and note that it is isomorphic to the Blaschke group.

The class of Blaschke functions which leave in place $1 \in \mathbb{T}$ is given as

$$\mathfrak{B}^* := \{B \in \mathfrak{B} : B(1) = 1\}. \tag{2.7}$$

Elements of (2.7) form a subgroup of the Blaschke group. Obviously, for any $(\mathfrak{b} = (b, \epsilon_b)) \in \mathbb{B}$, $B_{\mathfrak{b}} \in \mathfrak{B}^*$ if and only if $\epsilon_b = \overline{B}_b(1)$. We also introduce the index set $\mathbb{B}^* := \{\mathfrak{b} := (b, \overline{B}_b(1)) : b \in \mathbb{D}\}$ corresponding to $\mathfrak{B}^*$. Then, the map $(b, \overline{B}_b(1)) \to b$ is a bijection between $\mathbb{B}^*$ and $\mathbb{D}$. This bijection induces a group operation on the disk. We also denote this operation with the symbol $\circ$, and note that it can be easily expressed using Blaschke functions:

$$a \circ b^- := B_{\mathfrak{b}}(a) = \overline{B}_b(1) B_b(a),\ b^- = -\overline{B}_b(1) b \quad (a, b \in \mathbb{D}). \tag{2.8}$$

The number 0 is the identity element of the group $(\mathbb{B}^*, \circ)$, furthermore one can prove based on the definition that the above mentioned structure fulfills the criteria for groups.

The mapping

$$\rho(a, b) := |B_a(b)| \quad (a, b \in \mathbb{D}) \tag{2.9}$$

is referred to as *pseudo hyperbolic metric* on $\mathbb{D}$, for which

$$\rho(a,b) \le \frac{|a|+|b|}{1+|a||b|} \le |a|+|b| \ (a,b\in\mathbb{D}) \tag{2.10}$$

holds. Indeed, using the polar coordinates $a = r_a e^{i\alpha}$, $b = r_b e^{i\beta}$, $\theta := \alpha - \beta$ the property (2.10) can be written as

$$f(\theta) := \frac{r_a^2 + r_b^2 - 2r_a r_b cos(\theta)}{1 + r_a^2 r_b^2 - 2r_a r_b cos(\theta)} \le \frac{(r_a + r_b)^2}{(1 + r_a r_b)^2} \quad (0 \le \theta \le \pi).$$

Since

$$f'(\theta) = \frac{2r_a r_b(1-r_a^2)(1-r_b^2)sin(\theta)}{(1 + r_a^2 r_b^2 - 2r_a r_b cos(\theta))^2} \ge 0 \quad (0 \le \theta \le \pi),$$

therefore $f(\theta) \le f(\pi)$ end so (2.10) holds. From the above it also follows that equality only occurs when $\theta = \pi$, or in other words a and b point in opposite directions. The second part of (2.10) is obvious. Now we proceed to show, that the absolute value function $|\cdot|$ is a *group norm* on the $(\mathbb{D}, \circ)$ group. Indeed,

$$i)\ |a| = 0 \iff a = 0, \quad ii)\ |a^-| = |a| \quad (a \in \mathbb{D}),$$

furthermore by (2.8) and (2.10)

$$iii)\ |a \circ b^-| = |B_b(a)| = \rho(a,b) \le |a| + |b| \quad (a,b\in\mathbb{D}).$$

From the above, it follows that the triangle inequality holds for the metric $\rho(a,b)$:

$$\rho(a,b) = |a\circ b^-| = |(a\circ c^-)\circ(c\circ b^-)| \le |a\circ c^-| + |c^- \circ b^-| = \rho(a,c) + \rho(c,b).$$

We will refer to the mapping

$$a \to \tau_b(a) := a \circ b^- = \overline{B}_b(1)\, B_b(a) \quad (a,b\in\mathbb{D}) \tag{2.11}$$

as *(right side) translation*. The identity

$$\tau_{b_1}(\tau_{b_2}(a)) = (a\circ b_2^-)\circ b_1^- = a\circ(b_2^- \circ b_1^-) = a\circ(b_1\circ b_2)^- = \tau_{b_1\circ b_2}(a)$$

can be interpreted by noting that $b \to \tau_b^r$ is a homomorphism (automorphism).

The metric ρ is invariant with respect to the right sided translation:

$$\rho(a,b) = |a\circ b| = |(a\circ c^-)\circ(b\circ c^-)^-| = \rho(\tau_c(a), \tau_c(b)) \quad (a,b,c\in\overline{\mathbb{D}}). \tag{2.12}$$

We note, that the metric ρ is usually not inverse invariant $\rho(a,b) \ne \rho(a^-, b^-)$.

By (2.8), using the metric $\rho_E(a,b) = |a-b|$ $(a,b \in \mathbb{D})$ the mapping $(a,b) \to a \circ b$ is obviously continuous on the disk $\mathbb{D}$, therefore $(\mathbb{D}, \circ)$ is a topological group. We note that the metrics ρ and ρ_E are not equivalent on $\mathbb{D}$. By definition we have

$$\begin{gathered} \frac{|a-z|}{2} \le \rho(a,z) \le \frac{|a-z|}{1-r^2} \quad (|a|, |z| \le r < 1), \\ \rho(z \circ a, a) = |z|, \ |z \circ a - a| \le 2|z| \quad (a, z, \in \mathbb{D}). \end{gathered} \tag{2.13}$$

From (2.13) it follows that the metrics ρ and ρ_E are equivalent on the closed disc $\mathbb{D}_r := \{z \in \mathbb{D} : |z| \le r\}$ $(r < 1)$, furthermore the sets of continuous function defined using ρ and ρ_E are the same. Denote by $C(\mathbb{D})$ the set of bounded and continuous functions on $\mathbb{D}$. Furthermore, denote the set of continuous functions which disappear on the torus by

$$C_0(\mathbb{D}) := \{f \in C(\mathbb{D}) : \lim_{|z| \to 1} f(z) = 0\} \tag{2.14}$$

and the set of compactly supported functions by

$$C_{00}(\mathbb{D}) := \{f \in C(\mathbb{D}) : \exists 0 < r < 1 : f(z) = 0 \ |z| > r\}. \tag{2.15}$$

Now we can construct a translation invariant integral (i.e. Haar-integral) belonging to the group $(\mathbb{D}, \circ)$. Consider the weight functions

$$\sigma_s(z) := \frac{1}{(1-|z|^2)^s}, \quad \sigma := \sigma_2 \quad (z \in \mathbb{D}).$$

Denote by L^p_σ the set of (more precisely the equivalence classes of) Borel-measurable functions on $\mathbb{D}$, for which

$$\|f\|_{p,\sigma} := \left(\int_{\mathbb{D}} |f(z)|^p \sigma(z) \, dz \right)^{1/p} < \infty \quad (dz = dx\,dy, z = x + iy).$$

We prove that the below translation invariant integral formula holds:

$$\int_{\mathbb{D}} f(z \circ b^-) \sigma(z) \, dz = \int_{\mathbb{D}} f(z) \sigma(z) \, dz \quad (f \in L^1_\sigma, b \in \mathbb{D}). \tag{2.16}$$

This statement, using (2.8) is a direct consequence of the two integral formulas below:

(2.17)

$$\int_{\mathbb{D}} f(\epsilon z) \sigma(z) \, dz = \int_{\mathbb{D}} f(z) \sigma(z) \, dz, \quad \int_{\mathbb{D}} f(B_b(z)) \sigma(z) \, dz = \int_{\mathbb{D}} f(z) \sigma(z) \, dz$$
$$(f \in L^1_\sigma, b \in \mathbb{D}, \epsilon \in \mathbb{T}).$$

The first formula can be easily seen, if we apply integral transformation to $z \to \epsilon z$.

The second statement is also a consequence of the integral transformation formula. By the Cauchy–Riemann equations, the absolute value of the determinant belonging to the Jacobi matrix of $z \to B_b(z)$ is given by $|B_b'(z)|^2$. Using the identity (2.2), we get

$$|B_b'(z)| = \frac{1-|b|^2}{|1-\bar{b}z|^2} = \frac{\sigma_1(z)}{\sigma_1(B_b(z))} \quad (b, z \in \mathbb{D}). \tag{2.18}$$

Applying the formula of integral transformation to the mapping $z \to B_b(z)$ we get the statement

$$\int_{\mathbb{D}} f(z)\sigma_2(z)\, dz = \int_{\mathbb{D}} f(B_b(z))\sigma_2(B_b(z))|B_b'(z)|^2\, dz = \int_{\mathbb{D}} f(B_b(z))\sigma_2(z)\, dz$$

which concludes the proof.

From definition (2.8) and formula (2.16) we get that the proposed integral is invariant with respect to inversion:

$$\int_{\mathbb{D}} f(z^-)\sigma(z)\, dz = \int_{\mathbb{D}} f(z)\sigma(z)\, dz \quad (f \in L^1_\sigma).$$

Using the above statements, we say that the $d\mu(z) = \sigma(z)\, dz$ *Haar-measure which generates the integral is unimodular.*

Using polar coordinates we can write the Haar-integral as

$$\int_{\mathbb{D}} f(z)\, d\mu(z) = \int_0^{2\pi}\int_0^1 f(re^{i\varphi})\frac{r}{(1-r^2)^2}\, dr\, d\varphi.$$

From this, using the substitution $r = th(s)$ $(0 \le s < \infty)$ and

$$\frac{dr}{ds} = \frac{1}{ch^2(s)}, \quad \frac{r\, dr}{(1-r^2)^2} = \frac{th(s)\, ds}{ch^2(s)}\, ch^4(s) = \frac{1}{2} sh(2s)\, ds$$

we get the following formula:

$$\int_{\mathbb{D}} f(z)\, d\mu(z) = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} f(th(s)e^{i\varphi})\, sh(2s)\, ds\, d\varphi. \tag{2.19}$$

3. Convolution operators on the disk

In this section we introduce and describe convolution operators on the $(\mathbb{D}, \circ)$ unimodular group. We begin with a short review of theorems and formulas which hold for every unimodular group [24].

For any $f, g \in L^1_\mu$ $(d\mu(z) = \sigma_2(z)\, dz)$ function and any μ a.e. $a \in \mathbb{D}$ point the integral

$$(3.1) \qquad (f \star g)(a) := \int\limits_{\mathbb{D}} f(z) g(z^- \circ a)\ d\mu(z)$$

exists and $f \star g \in L^1_\mu$. If $f \in L^1_\mu$ and $g \in L^p_\mu$ $(1 \leq p \leq \infty)$, then $f \star g \in L^p_\mu$ and

$$\|f \star g\|_p \leq \|f\|_1 \|g\|_p.$$

Since the group is unimodular, convolution can be given in the following equivalent form:

$$(f \star g)(a) = \int\limits_{\mathbb{D}} f(a \circ z^-) g(z)\, d\mu(z) = \int\limits_{\mathbb{D}} f(a \circ z) g(z^-)\, d\mu(z) =$$
$$= \int\limits_{\mathbb{D}} f(z^-) g(z \circ a)\, d\mu(z).$$

We call the mappings

$$g \to T_f(g) := f \star g \quad (f \in L^1_\mu, g \in L^p_\mu\ (1 \leq p \leq \infty)$$

convolution operators. By the above equation $T_f : L^p_\mu \to L^p_\mu$ is a bounded linear operator and has the norm $\|T_f\|_p \leq \|f\|_1$.

We now prove the analogue of the theorem on approximate identity for the group $(\mathbb{D}, \circ)$.

Theorem 3.1. *Suppose that the sequence $f_n \in L^1_\mu$ $(n \in \mathbb{N}^*)$ satisfies*

$$(3.2) \qquad \begin{aligned} &i) \quad K := sup_{n\in\mathbb{N}^*} \|f_n\|_1 < \infty,\ ii)\ \xi_n := \int\limits_{\mathbb{D}} f_n\ d\mu \to 1 \quad (n \to \infty), \\ &iii)\ \forall 0 < r < 1 : \eta_n(r) := \int\limits_{|z|\geq r} |f_n|\ d\mu \to 0 \quad (n \to \infty). \end{aligned}$$

Then, in every continuity point $a \in \mathbb{D}$ of the function g we have

$$(3.3) \qquad \lim_{n\to\infty} (f_n \star g)(a) = g(a).$$

Proof. In the continuity points $a \in \mathbb{D}$ of the function g we have

$$\begin{aligned} &\omega_r := \sup_{|z|\le r} |g(z\circ a) - g(a)| \to 0 \quad (r \to 0), \\ &\exists r_0 < 1 : \ |g(z\circ a) - g(a)| \le M < \infty \ (|z| \le r_0). \end{aligned} \tag{3.4}$$

Using this and considering $r \le r_0$ we get

$$\begin{aligned} |(f_n \star g)(a) - \xi_n g(a)| &= \left| \int_{\mathbb{D}} f_n(z^-)(g(z\circ a) - g(a))\, d\mu(z) \right| \le \\ &\le \left(\int_{|z|\le r} + \int_{|z|\ge r} \right) |f_n(z^-)||(g(z\circ a) - g(a))|\, d\mu(z) \le \\ &\le \sup_{|z|\le r} |g(z\circ a) - g(a)| \|f_n\|_1 + M\eta_n(r) \le K\omega_r + M\eta_n(r) \end{aligned} \tag{3.5}$$

This proves the theorem. ■

For the function $\chi_r(z) = 1$ $(|z| \le r)$, $\chi_r(z) = 0$ $(|z| > r)$ we have

$$\int_{\mathbb{D}} \chi_r \, d\mu = \pi \int_0^r \frac{s}{(1-s^2)^2} ds = \left(\frac{1}{1-r^2} - 1 \right) \pi = \frac{r^2\pi}{1-r^2},$$

which implies that the sequence

$$f_n := \frac{(n^2-1)}{\pi} \chi_{1/n} \ (n \in \mathbb{N}^*)$$

satisfies the conditions of theorem 3.1:

$$n^2/\pi \int_{|z|\le 1/n} g(z\circ a)\, d\mu(z) \to g(a) \quad (n \to \infty).$$

4. Dilation, approximate identity

On the $L^1(\mathbb{R})$ space, conditions (i) to (iii) of theorem 3.1 are easily satisfied using dilation. Suppose that the function $f \in L^1(\mathbb{R})$ satisfies $\int_{\mathbb{R}} f(x)\, dx = 1$. Then, the function $f_\lambda(x) = \lambda f(\lambda x)$ $(x \in \mathbb{R}, \lambda > 0)$ has the following properties

$$\int_{\mathbb{R}} f_\lambda(x)\, dx = 1, \int_{|x|>\delta} |f_\lambda(x)|\, dx = \int_{|t|>\lambda\delta} |f(t)|\, dt \to 0 \quad (\lambda \to \infty).$$

By the above, for $\lambda = n$ the needed conditions are satisfied, therefore the classical theorem regarding the approximate identity is applicable.

Based on the above, the question of dilation (multiplication) on the group $(\mathbb{D}, \circ)$ arises naturally. Using the bijection $th : \mathbb{R} \to \mathbb{I}$ we can introduce a field structure on the $(\mathbb{D}, \circ)$ group:

$$s \oplus t = th(ath(s) + ath(t)), \ s \odot t = th(ath(s) \cdot ath(t)) \ \ (s, t \in \mathbb{I}). \tag{4.1}$$

The additive structure is compatible with the $(\mathbb{I}, \circ)$ group structure: $s \circ t^- = s \oplus t$ $(s, t \in \mathbb{I})$.

We will now investigate how the dilation $t \to th(\lambda) \odot t$ effects the Haar integral. Consider the analogue of the function f_λ:

$$F_\lambda(z) := \lambda^2 f((th(\lambda) \odot r)e^{i\varphi})) \ \ (z = re^{i\varphi} \in \mathbb{D}, \lambda > 0).$$

Now we will prove an analogous statement to the one in the introduction of section 3.

Lemma 4.1. *Suppose $f \in L^1_\mu$. Then, the limit*

$$\lim_{\lambda \to \infty} \int_{\mathbb{D}} F_\lambda(z) \ d\mu(z) = \kappa \tag{4.2}$$

exists, furthermore for all $0 < \delta < 1$ number

$$\lim_{\lambda \to \infty} \int_{|z| \geq \delta} |F_\lambda(z)| \ d\mu(z) = 0.$$

Proof. We start with the polar coordinate form of the Haar integral:

$$\int_{\mathbb{D}} F_\lambda(z) \ d\mu(z) = \int_0^{2\pi} \int_0^1 F_\lambda(re^{i\varphi}) \frac{r}{(1 - r^2)^2} \ dr \ d\varphi.$$

In the inner integral, consider the substitution $r = th(s)$, identity (4.1), then the substitution $s\lambda = t$ and the notation $\epsilon = e^{i\varphi}$:

$$\begin{aligned}
\int_0^1 F_\lambda(r\epsilon)) \frac{r \ dr}{(1 - r^2)^2} &= \lambda^2 \int_0^1 f((th(\lambda) \circ r)\epsilon) \frac{r \ dr}{(1 - r^2)^2} = \\
&= \lambda^2 \int_0^\infty f((th(\lambda) \circ th(s))\epsilon) \frac{th(s)}{(1 - th^2(s))^2} \frac{ds}{ch^2(s)} = \\
&= \frac{\lambda^2}{2} \int_0^\infty f((th(\lambda \cdot s)\epsilon) \ sh(2s) \ ds = \\
&= \frac{1}{2} \int_0^\infty f(th(t\epsilon)) \ sh(2t) \frac{sh(2t/\lambda)}{sh(2t)/\lambda} \ dt.
\end{aligned} \tag{4.3}$$

On the other hand

$$\int_0^1 f(s\epsilon)\frac{s\,ds}{(1-s^2)^2}\,ds = \int_0^\infty f(th(t)\epsilon)\ th(t)\ ch^2(t)\ dt = \tag{4.4}$$
$$= \frac{1}{2}\int_0^\infty f(th(t)\epsilon)\ sh(2t)\ dt$$

Since

$$\phi_\lambda(t) := \frac{sh(2t/\lambda)}{sh(2t)/\lambda} \quad (\lambda > 1, t > 0)$$

satisfies

$$0 \le \phi_\lambda(t) \le 1,\ lim_{\lambda\to\infty}\phi_\lambda(t) = \frac{2t}{sh(2t)} \quad (t > 0).$$

From here, using the Lebesgue convergence theorem we get the first part of the statement, furthermore

$$\kappa = \int_0^\infty f(e^{i\varphi}th(t))t\ dt.$$

Repeating the above steps we have

$$\int_{r\ge\delta} |F_\lambda(r)|\frac{r\ dr}{(1-r^2)^2} = \frac{\lambda^2}{2}\int_{s\ge\ ath(\delta)} |f(th(\lambda\cdot s)\epsilon)|\ sh(2s)\ ds =$$
$$= \frac{1}{2}\int_{t\ge\lambda\cdot ath(\delta)} |f(th(t\epsilon))|\ sh(2t)\frac{sh(2t/\lambda)}{sh(2t)/\lambda}\ dt \tag{4.5}$$
$$\le \frac{1}{2}\int_{t\ge\lambda\cdot ath(\delta)} |f(th(t\epsilon))|\ sh(2t)\ dt.$$

From here, using the Beppo-Levi theorem the second part of the statement follows. ■

Let us apply theorem 3.1 to the function sequence

$$f_n(z) = F_n(z)/c \quad (n\in\mathbb{N}^*), c = \int_0^{2\pi}\int_0^\infty f(e^{i\varphi}th(t))t\ dt\ d\varphi \ne 0.$$

Then, based on lemma 4.1, the conditions of theorem 3.1 are satisfied. Therefore, on the group $(\mathbb{D},\circ)$ the analogue the theorem holds.

Theorem 4.1. *Let $f \in L^1_\mu$ and*

$$f_n(z) = \frac{n^2}{c} f(th(n) \circ z) \quad (z \in \mathbb{D}, n \in \mathbb{N}^*)$$

Then, for every $g \in C_0(\mathbb{D})$ function

$$\sup_{a \in \mathbb{D}} |(f_n \star g)(a) - g(a)| \to 0 \quad (n \to \infty).$$

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