Annales Univ. Sci. Budapest., Sect. Comp. 54 (2023) 265–280

THE FOURIER GENERALIZED CONVOLUTIONS ON TIME SCALES $h\mathbb{N}^0$ AND THEIR APPLICATIONS

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Communicated by Bui Minh Phong

(Received January 19, 2023; accepted June 3, 2023)

Abstract. In this article, we study and construct Fourier sine and Fourier cosine generalized convolutions on time scale, the existence of these convolutions on some specific function spaces, obtain some properties and application in solving some discrete integral equations of Toeplitz–Hankel type.

1. Introduction

Time scale was first mentioned in S.Hilger's thesis in 1988, which is a nonempty closed subset of $\mathbb{R}$. Until now, many beautiful results related to time scale are known. But the Fourier convolution on time scale is really new. In 1999, in [1], S. Hilger studied the Fourier transform on the time scales $\mathbb{R}$ and $h\mathbb{Z}, h > 0$. According to [1], the Fourier transform of a given function $f : h\mathbb{Z} \longrightarrow \mathbb{C}$ is

$$F(\omega) = h \sum_{n=-\infty}^{\infty} e^{-i\omega nh} f(nh),\ \omega \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$$

and the inverse Fourier transform is

$$f(t) = \frac{1}{2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} e^{i\omega t} F(\omega) d\omega.$$

Key words and phrases: Convolution; h-Fouriersine, h-Fouriercosine; integral transform; discrete integral equations.
2010 Mathematics Subject Classification: 42A38, 65T50.

The convolution of Fourier transform on time scales was mentioned in [5]. As a result in [5], on time scale $h\mathbb{Z}$, we obtain the Fourier convolution of two functions $f, g : h\mathbb{Z} \longrightarrow \mathbb{C}$, which is defined as

$$(f * g)(t) = h \sum_{m=-\infty}^{\infty} f(mh)g(t - mh), \tag{1.1}$$

for $t \in h\mathbb{Z}$ and the right-hand side is convergent. Throughout the article, we denote $\mathbb{N}^0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ and some function spaces

$\ell_p(h\mathbb{N}^0) = \{x : h\mathbb{N}^0 \to \mathbb{C},$ such that $|x(0)|^p + 2 \sum\limits_{n=1}^{\infty} |x(nh)|^p < \infty\},$

$||x||_{\ell_p(h\mathbb{N}^0)} = h \left(|x(0)|^p + 2 \sum\limits_{n=1}^{\infty} |x(nh)|^p\right)^{\frac{1}{p}}, \; p \in \{1, 2\}.$

$\ell_\infty(h\mathbb{N}^0) = \{x : h\mathbb{N}^0 \longrightarrow \mathbb{C},$ such that $\sup\limits_{n\geq 0}|x(nh)| < \infty\}, ||x||_{\ell_\infty(h\mathbb{N}^0)} =$ $= h \sup\limits_{n\geq 0}|x(nh)|.$

$L_2(0, \frac{\pi}{h}) = \left\{x : [0, \frac{\pi}{h}] \longrightarrow \mathbb{C},\right.$ such that $\left.\int\limits_0^{\frac{\pi}{h}} |x(t)|^2 dt < \infty\right\}, \; ||x||_{L_2(0,\frac{\pi}{h})} =$ $= h\left(\int\limits_0^{\frac{\pi}{h}} |x(t)|^2 dt\right)$, where h is a given real positive number.

In this article, we study some generalized convolutions related to Fourier sine and Fourier cosine integral transforms on time scale $h\mathbb{N}^0, h > 0$ which is defined by the formulas (3.1) and (4.1). We show the existence, factorization identity, and estimation of these convolutions (Theorem 3.1, Theorem 4.1), and Parseval's identity (Theorem 3.2, Theorem 4.2). With the help of convolutions, we solve the equations of Toeplitz–Hankel type on time scale (5.3), (5.4) in a closed form and show the boundedness of the solutions.

2. h-Fourier cosine and h-Fourier sine transforms on time scale $h\mathbb{N}^0$

Definition 2.1. The h-Fourier sine transform of a function $f : h\mathbb{N}^0 \to \mathbb{C}$ is of the form

$$F_s(\omega) = F_s\{f\}(\omega) = 2h \sum_{n=1}^{\infty} f(nh) \sin(\omega nh), \quad \omega \in \left[0, \frac{\pi}{h}\right]$$

and its inverse transform is $f(t) = \dfrac{1}{\pi} \int\limits_0^{\frac{\pi}{h}} F_s(\omega) \sin(\omega t) d\omega.$

The h-Fourier cosine transform of a function $f : h\mathbb{N}^0 \to \mathbb{C}$ is of the form

$$F_c(\omega) = F_c\{f\}(\omega) = 2h\sum_{n=1}^{\infty} f(nh)\cos(\omega nh) + hf(0), \quad \omega \in \left[0, \frac{\pi}{h}\right]$$

and its inverse transform is $f(t) = \dfrac{1}{\pi}\displaystyle\int_0^{\frac{\pi}{h}} F_c(\omega)\cos(\omega t)d\omega.$

3. h-Fourier sine generalized convolutions on time scale $h\mathbb{N}^0$

Definition 3.1. The $h-$Fourier sine generalized convolution of two functions $f, g : h\mathbb{N}^0 \to \mathbb{C}$ is defined by

$$(f * g)(t) = h\sum_{m=1}^{\infty} f(mh)\big(g(|t-mh|) - g(t+mh)\big), \quad t \in h\mathbb{N}^0, \tag{3.1}$$

where the right-hand side is convergent.

Theorem 3.1. *Suppose that $f, g \in \ell_1(h\mathbb{N}^0), f(0) = 0$ then the convolution $(f * g)$ belongs to $\ell_1(h\mathbb{N}^0)$, and satisfies the following factorization identity*

$$F_s\{f * g\}(\omega) = F_s\{f\}(\omega)F_c\{g\}(\omega), \omega \in \left[0, \frac{\pi}{h}\right]. \tag{3.2}$$

Moreover, the following estimation holds

$$||f * g||_{\ell_1(h\mathbb{N}^0)} \le ||f||_{\ell_1(h\mathbb{N}^0)}||g||_{\ell_1(h\mathbb{N}^0)}. \tag{3.3}$$

Proof. Firstly, we prove that $(f * g) \in \ell_1(h\mathbb{N}^0)$. Indeed, we have

$$\begin{aligned}
&h\left(|(f*g)(0)| + 2\sum_{n=1}^{\infty}|(f*g)(nh)|\right) = \\
&= h^2|(f*g)(0)| + 2h^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big|f(mh)\big(g(|nh-mh|) - g(nh+mh)\big)\Big| = \\
&= 2h^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big|f(mh)\big(g(|nh-mh|) - g(nh+mh)\big)\Big| \le \\
&\le 2h^2\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|f(mh)|\big(|g(|nh-mh|)| + |g(nh+mh)|\big)\right) \le \\
&\le 2h^2\left(\sum_{m=1}^{\infty}|f(mh)g(mh)| + \sum_{m=1}^{\infty}\sum_{\tau=m+1}^{\infty}|f(mh)g(\tau h)|\right) + \\
&\qquad +2h^2\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{\infty}|f(mh)g(\tau h)| + 2h^2\sum_{m=1}^{\infty}\sum_{\tau=0}^{m-1}|f(mh)g(\tau h)|\right) = \\
&= 2h^2\sum_{n=1}^{\infty}|f(nh)|\left(|g(0)| + 2\sum_{r=1}^{\infty}|g(nh)|\right) = ||f||_{\ell_1(h\mathbb{N}^0)}||g||_{\ell_1(h\mathbb{N}^0)} < \infty.
\end{aligned}$$

Therefore, we get the estimation (3.3) and $(f*g)\in \ell_1(h\mathbb{N}^0)$.
Next, we prove the factorization identity (3.2). According to Definition 2.1, we rewrite the right-hand side of (3.2) as follows

$$
\begin{aligned}
& F_s\{f\}(\omega)F_c\{g\}(\omega) = \\
= \quad & 2h^2\sum_{n=1}^{\infty} f(nh)\sin(\omega nh)\big(g(0)+2\sum_{m=1}^{\infty} g(mh)\sin(\omega mh)\big) = \\
= \quad & 2h^2\Big(g(0)\sum_{n=1}^{\infty} f(nh)\sin(\omega nh)+ \\
& \qquad +2\sum_{n=1}^{\infty} f(nh)\sin(nh\omega)\sum_{n=1}^{\infty} g(mh)\cos(mh\omega)\Big).
\end{aligned}
$$

We set $A = 2\sum_{n=1}^{\infty} f(nh)\sin(nh\omega)\sum_{n=1}^{\infty} g(mh)\cos(mh\omega)$ then

$$
\begin{aligned}
\text{A} \quad &= \sum_{n=1}^{\infty} f(nh)\Big\{\sum_{m=1}^{\infty} g(mh)\big(\sin(\omega h(n+m))+\sin(\omega h(n-m))\big)\Big\} = \\
&= \sum_{n=1}^{\infty} f(nh)\sum_{m=1}^{\infty} g(mh)\sin(\omega h(n+m))+ \\
&\qquad +\sum_{n=1}^{\infty} f(nh)\sum_{m=1}^{\infty} g(mh)\sin(\omega h(n-m)) = \\
&= \sum_{n=1}^{\infty} f(nh)\sum_{\tau=n+1}^{\infty} g(\tau h-nh)\sin(\omega\tau h)+ \\
&\qquad +\sum_{n=1}^{\infty} f(nh)\sum_{\tau=-\infty}^{n-1} g(nh-\tau h)\sin(\omega\tau h) = \\
&= \sum_{n=1}^{\infty} f(nh)\Big(\sum_{\tau=1}^{\infty} g(\tau h-nh)\sin(\omega\tau h)-\sum_{\tau=1}^{n-1} g(\tau h-nh)\sin(\omega\tau h)- \\
&\qquad -g(0)\sin(\omega nh)\Big)+\sum_{n=1}^{\infty} f(nh)\sum_{\tau=-\infty}^{-1} g(nh-\tau h)\sin(\omega\tau h)+ \\
&\qquad +\sum_{n=1}^{\infty} f(nh)\sum_{\tau=1}^{m-1} g(nh-\tau h)\sin(\omega\tau h).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& F_s\{f\}(\omega)F_c\{g\}(\omega) = 2h^2(g(0)\sum_{n=1}^{\infty} f(nh)\sin(\omega nh)+A) = \\
& = 2h^2\sum_{n=1}^{\infty} f(nh)\sum_{m=1}^{\infty} g(|nh-mh|)\sin(\omega nh)- \\
& \qquad -2h^2\sum_{n=1}^{\infty} f(nh)\sum_{m=1}^{\infty} g(mh+nh)\sin(\omega nh) = F_s\{f*g\}(\omega). \blacksquare
\end{aligned}
$$

Theorem 3.2. (Identity of Parseval's type) *Let f, g be functions belonging to $\ell_2(h\mathbb{N}^0)$, then*

$$(3.4)\qquad (f * g)(t) = \frac{1}{\pi}\int_0^{\frac{\pi}{h}} F_s\{f\}(\omega)F_c\{g\}(\omega)\sin(\omega t)d\omega, \quad t \in h\mathbb{N}^0.$$

Proof. Notation $F_o(\omega)$ and $G_e(\omega)$ are respectively the odd component of $F\{f\}(\omega)$ and the even component of $F\{g\}(\omega)$ on $\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$. Using the Fourier transform on $h\mathbb{Z}$, we have

$$\begin{aligned} F_o(\omega) =& F\{f_o(t)\}(\omega) = h\sum_{n=-\infty}^{\infty} f_o(nh)e^{-i\omega nh}, \\ G_e(\omega) =& F\{g_e(t)\}(\omega) = h\sum_{n=-\infty}^{\infty} g_e(nh)e^{-i\omega nh}, \end{aligned}$$

where $f_o(nh) = \operatorname{sign}(n)f(|nh|), f(0) = 0$, $g_e(nh) = g(|nh|)$, for all $n \in \mathbb{Z}$. On the other hand, the Parseval's identity for the Fourier transform on time scale $h\mathbb{Z}$ for F_o and G_e is

$$(3.5)\qquad \begin{aligned} h\sum_{n=-\infty}^{\infty} f_o(nh)g_e(t-nh) &= \frac{1}{2\pi}\int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} F\{f\}(\omega)F\{g\}(\omega)e^{i\omega t}d\omega = \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{h}} F_o(\omega)G_e(\omega)\sin(\omega t)d\omega. \end{aligned}$$

Since $f, g \in \ell_2(h\mathbb{Z})$, the left-hand side series of (3.5) is absolutely convergent, and it can be rewritten as

$$\begin{aligned} h\sum_{n=-\infty}^{\infty} f_o(nh)g_e(t-nh) =& h\sum_{n=-\infty}^{-1} f_o(nh)g_e(t-nh)+ \\ &+ h\sum_{n=1}^{\infty} f_o(nh)g_e(t-nh) = \\ =& h\sum_{n=1}^{\infty} f(nh)\big(g(|t-nh|) - g(t+nh)\big). \end{aligned}$$

The proof is complete. ■

Remark 3.1. If $f, g \in \ell_2(h\mathbb{N}^0)$ then we have

$$||f * g||_{\ell_\infty(h\mathbb{N}^0)} \leq ||f||_{\ell_2(h\mathbb{N}^0)}||g||_{\ell_2(h\mathbb{N}^0)}.$$

Proof. From (3.5) we get

$$\begin{aligned}\big|(f * g)(t)\big| &= \frac{1}{\pi}\bigg|\int\limits_0^{\frac{\pi}{h}} F_s\{f\}(\omega)F_c\{g\}(\omega)\sin(\omega t)d\omega\bigg| \leq \\ &\leq \frac{1}{\pi}\int\limits_0^{\frac{\pi}{h}} \big|F_s\{f\}(\omega)F_c\{g\}(\omega)\big|d\omega \leq \\ &\leq \frac{1}{\pi}||F_s\{f\}||_{L_2(0,\frac{\pi}{h})}||F_c\{g\}||_{L_2(0,\frac{\pi}{h})} = ||f||_{\ell_2(h\mathbb{N}^0)}||g||_{\ell_2(h\mathbb{N}^0)}.\end{aligned}$$

4. The generalized convolution with a weight function

Definition 4.1. The generalized convolution with a weight function $\gamma = \cos(\omega h)$ for the $h-$Fourier cosine and $h-$Fourier sine transforms of two functions f and g is defined by

(4.1)

$$\begin{aligned}(f \underset{\gamma}{*} g)(t) = h\sum_{m=1}^{\infty} f(mh)\big(&g(mh+h+t) + g(|mh-h+t|)sign(mh-h+t)+ \\ &+g(|t-mh+h|)sign(t-mh+h) + g(|t-mh-h|)sign(t-mh-h)\big),\end{aligned}$$

where $t \in h\mathbb{N}^0$, $f, g : h\mathbb{N}^0 \to \mathbb{C}$ and the right hand side is convergent.

Theorem 4.1. *Let $f, g \in \ell_1(h\mathbb{N}^0)$ and $g(0) = 0$ then $f \underset{\gamma}{*} g$ belongs to $\ell_1(h\mathbb{N}^0)$ and the following factorization equality holds*

$$F_s(f \underset{\gamma}{*} g)(\omega) = \cos(\omega h)F_c\{f\}(\omega)F_s\{g\}(\omega), \quad \omega \in \left[0, \frac{\pi}{h}\right]. \tag{4.2}$$

Proof. Firstly, we show that $(f \underset{\gamma}{*} g) \in \ell_1(h\mathbb{N}^0)$. We have

$$\begin{aligned}(f \underset{\gamma}{*} g)(0) = h\sum_{m=1}^{\infty} f(mh)\big(&g(mh+h) + g(mh-h)+ \\ +g(|h-mh|)sign(1-m) &+ g(|-mh-h|)sign(-mh-h)\big) = 0.\end{aligned}$$

Moreover,

$$\begin{aligned}||f \underset{\gamma}{*} g||_{\ell_1(h\mathbb{N}^0)} &\leq h^2 \sum_{m=1}^{\infty} |f(mh)| \sum_{n=1}^{\infty} \big(|g(mh+nh+h)| + |g((m+n-1)h)| + \\ &\quad + |g(|n+1-m|)h)| + |g(|n-m-1|h)|\big) \leq \\ &\leq h^2 \Bigg(\sum_{m=1}^{\infty} |f(mh)| \sum_{\tau=m+1}^{\infty} |g(\tau h)| + \sum_{m=1}^{\infty} |f(mh)| \sum_{\tau=m-1}^{\infty} |g(\tau h)| + \\ &\quad + \sum_{m=1}^{\infty} |f(mh)| \sum_{\tau=1-m}^{\infty} |g(\tau h)| + \sum_{m=1}^{\infty} |f(mh)| \sum_{\tau=-m-1}^{\infty} |g(\tau h)|\Bigg) \leq \\ &\leq 2h^2 \big(f(0) + 2\sum_{m=1}^{\infty} |f(mh)|\big) \sum_{\tau=1}^{\infty} |g(\tau h)| = ||f||_{\ell_1(h\mathbb{N}^0)} ||g||_{\ell_1(h\mathbb{N}^0)} < \infty.\end{aligned}$$

This implies that $(f \underset{\gamma}{*} g) \in \ell_1(h\mathbb{N}^0)$.

Next, we prove the factorization equality (4.2). We have

$$\cos(\omega h) F_c\{f\}(\omega) F_s\{g\}(\omega) =$$
$$4h^2 \cos(\omega h) \sum_{n=1}^{\infty} f(nh) \cos(\omega nh) \sum_{m=1}^{\infty} g(mh) \sin(\omega mh).$$

Using trigonometric transforms, one can easily see that

$$\begin{aligned}4\cos(\omega h)\cos(\omega nh)\sin(\omega nh) &= \sin(\omega h(m+n+1)) + \sin((\omega h(m+n-1)) + \\ &\quad + \sin(\omega h(m-n+1)) + \sin(\omega h(m-n-1)),\end{aligned}$$

the right-hand side of (4.2) is

$$\begin{aligned}&\cos(\omega h) F_c\{f\}(\omega) F_s\{g\}(\omega) = \\ &\quad = \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g(nh) \sin(\omega h(m+n+1)) + \\ &\qquad + \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g(nh) \sin((\omega h(m+n-1)) + \\ &\qquad + \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g(nh) \sin(\omega h(m-n+1)) + \\ &\qquad + \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g(nh) \sin(\omega h(m-n-1)).\end{aligned}$$

We see that

$$F_s\{f \underset{\gamma}{*} g\}(\omega) = 2h^2 \sum_{n=1}^{\infty} \Big\{ \sum_{m=1}^{\infty} f(mh)\big(g((m+n+1)h)+$$

$$+g((m+n-1)h)sign(m+n-1) + g(|n+1-m|h)sign(n+1-m)+ \tag{4.3}$$

$$+g(|n-m-1|h)sign(n-m-1)\big)\Big\}sin(\omega nh).$$

Setting $A = \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g((m+n+1)h)sin(\omega nh)$. Substituting $\tau = m+n+1$ into A, we get

$$A = \sum_{m=1}^{\infty} f(mh) \sum_{\tau=m+2}^{\infty} g(\tau h) \sin(\omega(\tau - m - 1)h) = \tag{4.4}$$

$$= \sum_{m=1}^{\infty} f(mh)\Big(\sum_{\tau=1}^{\infty} g(\tau h) \sin(\omega(\tau - m - 1)h) - \sum_{\tau=1}^{m+1} g(\tau h) \sin(\omega(\tau - m - 1)h)\Big).$$

Setting $B = \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g((m+n-1)h)sin(\omega nh)$. Substituting $\tau = m+n-1$ into B, we get

$$B = \sum_{m=1}^{\infty} f(mh) \sum_{\tau=m}^{\infty} g(\tau h) \sin(\omega(\tau + 1 - m)) = \tag{4.5}$$

$$= \sum_{m=1}^{\infty} f(mh) \sum_{t=1}^{\infty} g(\tau h) \sin(\omega(\tau + 1 - m)) - \sum_{\tau=1}^{m-1} g(\tau h) \sin(\omega(\tau + 1 - m)h).$$

Similarly, letting $C = \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g((m-n+1)h)sign(m-n+1) \sin(\omega nh)$ and replacing τ by $m-n+1$, we have

$$\begin{aligned} C &= \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g(|nh - mh + h|)sign(n-m+1) \sin(\omega nh) = \\ &= \sum_{m=1}^{\infty} f(mh) \sum_{\tau=2-m}^{\infty} g(|\tau| h) \sin(\omega(m-1+\tau)h) = \\ &= \sum_{m=1}^{\infty} f(mh) \sum_{\tau=0}^{\infty} g(\tau h) \sin(\omega(m-1+\tau)h)+ \\ &\quad + \sum_{m=1}^{\infty} f(mh) \sum_{\tau=2-m}^{0} g(|\tau| h)sign(\tau) \sin(\omega(m-1+\tau)h). \end{aligned} \tag{4.6}$$

And $D = \sum_{m=1}^{\infty} f(mh) \sum_{n=1}^{\infty} g(|n-m-1|h) sign(n-m-1)\sin(\omega nh)$.
Letting $t = n-m-1$, we have

$$D = \sum_{m=1}^{\infty} f(mh) \sum_{\tau=-m}^{\infty} g(|\tau|h) sign(\tau)\sin((\tau+m+1)\omega) =$$

$$= -\sum_{m=1}^{\infty} f(mh) \sum_{\tau=0}^{m} g(\tau h)\sin((\tau+m+1)h\omega) + \tag{4.7}$$

$$+ \sum_{m=1}^{\infty} f(mh) \sum_{\tau=1}^{\infty} g(|mh+t+h|)\sin(\omega nh).$$

Combining (4.4), (4.5), (4.6) and (4.7), we deduce (4.2). ■

Theorem 4.2. (Identity of Parseval's type) *Assume that f, g are functions belonging to $\ell_2(h\mathbb{N}^0)$. Then we have the identity of Parseval's type*

$$(f \underset{\gamma}{*} g)(t) = \frac{1}{\pi} \int_0^{\frac{\pi}{h}} F_c\{f\}(\omega) F_s\{g\}(\omega) \cos(\omega h)\sin(\omega t) d\omega, t \in h\mathbb{N}^0.$$

Proof. Let f_1 be the even extension of f from $h\mathbb{N}^0$ to $h\mathbb{Z}$ and g_1 be the odd extension of g from $h\mathbb{N}^0$ to $h\mathbb{Z}$. We have the Parseval's identity for the Fourier transform on time scale $h\mathbb{Z}$

$h \sum_{n=-\infty}^{\infty} f(nh) g(t-nh) = \frac{1}{2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} F\{f\}(\omega) F\{g\}(\omega) e^{i\omega t} d\omega$. The left-hand side of the above identity can be rewritten as follows

$$h \sum_{m=0}^{\infty} f_1(mh) g_1(mh+nh+h) + h \sum_{m=0}^{\infty} f_1(mh) g_1(mh+nh-h) +$$

$$+ h \sum_{m=0}^{\infty} f_1(mh) g_1(-mh+nh+h) + h \sum_{m=0}^{\infty} f_1(mh) g_1(-mh+nh-h) =$$

$$= h \sum_{m=-\infty}^{\infty} f_1(mh) g_1(-mh+nh+h) + h \sum_{m=-\infty}^{\infty} f_1(mh) g_1(-mh+nh-h) =$$

$$= \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} F\{f_1\}(\omega) F\{g_1\}(\omega) e^{i\omega(nh+h)} d\omega +$$

$$+ \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} F\{f_1\}(\omega) F\{g_1\} e^{i\omega(nh-h)} d\omega = \quad \left(\text{here } B = \tfrac{1}{2h}\right)$$

$$= \frac{1}{2\pi} \int\limits_{-2\pi B}^{2\pi B} F\{f_1\}(\omega)F\{g_1\}(\omega)\big(\cos(\omega nh + \omega h) + i\sin(\omega nh + \omega h)\big)d\omega +$$

$$+ \frac{1}{2\pi} \int\limits_{-2\pi B}^{2\pi B} F\{f_1\}(\omega)F\{g_1\}(\omega)\big(\cos(\omega nh - \omega h) + i\sin(\omega nh - \omega h)\big)d\omega.$$

Since $F\{f_1\}(\omega)F\{g_1\}(\omega)\cos(\omega nh+\omega h)$ and $F\{f_1\}(\omega)F\{g_1\}(\omega)\cos(\omega nh-\omega h)$ are odd functions, we have

$$\int\limits_{-2\pi B}^{2\pi B} F\{f_1\}(\omega)F\{g_1\}(\omega)\cos(\omega nh - \omega h) =$$

$$= \int\limits_{-2\pi B}^{2\pi B} F\{f_1\}(\omega)F\{g_1\}(\omega)\cos(\omega nh + \omega h)d\omega = 0.$$

Therefore,

$$\int\limits_{-2\pi B}^{2\pi B} F\{f_1\}(\omega)F\{g_1\}(\omega)\big(\cos(\omega nh + \omega h) + i\sin(\omega nh + \omega h)\big)d\omega +$$

$$+ \int\limits_{-2\pi B}^{2\pi B} F\{f_1\}(\omega)F\{g_1\}(\omega)\big(\cos(\omega nh - \omega h) + i\sin(\omega nh - \omega h)\big)d\omega =$$

$$= i \int\limits_{-2\pi B}^{2\pi B} F\{f_1\}(\omega)F\{g_1\}(\omega)\big(\sin(\omega nh + \omega h) + \sin(\omega nh - \omega h)\big)d\omega = \tag{4.8}$$

$$= -2i^2 \int\limits_{-2\pi B}^{2\pi B} F_c\{f_1\}(\omega)F_s\{g_1\}(\omega)\cos(\omega h)(\omega)\sin(\omega nh)d\omega =$$

$$= 2 \int\limits_{-2\pi B}^{2\pi B} F_c\{f_1\}(\omega)F_s\{g_1\}\cos(\omega h)\sin(\omega nh)d\omega.$$

From (4.8), we obtain (3.2). ∎

Remark 4.1. If f, g belong to $\ell_2(h\mathbb{N}^0)$, then we have

$$||f \underset{\gamma}{*} g||_{\ell_\infty(h\mathbb{N}^0)} \leq ||f||_{\ell_2(h\mathbb{N}^0)}||g||_{\ell_2(h\mathbb{N}^0)}.$$

Proof. According to Theorem 4.2,

$$(f \underset{\gamma}{*} g)(t) = \frac{1}{\pi} \int\limits_0^{\frac{\pi}{h}} F_c\{f\}(\omega) F_s\{g\}(\omega) \cos(\omega h) \sin(\omega t) d\omega.$$

We have

$$\begin{aligned} \left|(f \underset{\gamma}{*} g)(t)\right| &= \left| \frac{1}{\pi} \int\limits_0^{\frac{\pi}{h}} F_c\{f\}(\omega) F_s\{g\}(\omega) \cos(\omega h) \sin(\omega t) d\omega \right| \leq \\ &\leq \frac{1}{\pi} \int\limits_0^{\frac{\pi}{h}} \left|F_c\{f\}(\omega) F_s\{g\}(\omega)\right| d\omega \leq \\ &\leq \frac{1}{\pi} ||F_c\{f\}||_{L_2(0,\frac{\pi}{h})} ||F_s\{g\}||_{L_2(0,\frac{\pi}{h})} = ||f||_{\ell_2(h\mathbb{N}^0)} ||g||_{\ell_2(h\mathbb{N}^0)}. \end{aligned}$$ ■

5. Applications

According to [8, 10], we consider the Toeplitz–Hankel integral equations as of the forms

$$\int\limits_0^\infty \big(k_1(x-y) + k_2(x+y)\big) f(y) dy = g(x), \quad x > 0 \tag{5.1}$$

and

$$f(x) + \int\limits_0^\infty \big(k_1(x-y) + k_2(x+y)\big) f(y) dy = g(x), \quad x > 0, \tag{5.2}$$

where g is the given function, k_1 is the Toeplitz kernel, k_2 is the Hankel kernel, and f is an unknown function. Until now, solving the Toeplitz–Hankel integral equations in general cases of k_1, k_2 as arbitrary kernels is an open problem. However, by various ways, some types of Toeplitz–Hankel were solved. For example, in [7], authors H. M. Srivastava and R. G Buschman obtained the solution in case $k_1(x-y) = (x-y)^\alpha$; $e^{-\alpha(x-y)}$; $sinh(a(x-y))$; $aJ_1(a(x-y))$, where J_1 is the Bessel function. Recently, there has been some results of solving a class of equation (5.1) in a closed form, by the technique of using generalized convolutions related to Fourier, Kontorovich–Lebedev, Hartly integral transforms (see [9, 10, 11]). In this paper, we solve the classes of equations (5.3) and (5.4) in a closed form on time scale $h\mathbb{N}^0$, which are the discretization from the equations (5.1) and (5.2) for kernels $k_1 = k_2 = k$.

$$h\sum_{m=0}^{\infty} x(mh)\big(k(|nh-mh|) - k(nh+mh)\big) = z(nh), \quad n \in \mathbb{N}^0 \tag{5.3}$$

and

$$x(nh) + h\sum_{m=1}^{\infty} x(mh)\big(k(|nh-mh|) - k(nh+mh)\big) = z(nh), \quad n \in \mathbb{N}^0, \tag{5.4}$$

here $h > 0$ is a constant, z is the given function, x is an unknown function.

Based on the obtained results, we find the solutions of equations (5.3), (5.4) in a closed form and study the boundedness of the solutions.

First of all, we recall the Wiener–Levy's theorem for the Fourier transform [6]: "Assume that f is the Fourier transform of a function in $L_1(\mathbb{R})$ and Φ is an analytic function in a neighbourhood of origin, which contains the range of f, i.e. $\{f(y), \forall y \in \mathbb{R}\}$ and $\Phi(0) = 0$. Then $\Phi(f)(y)$ is a Fourier transform of a function in $\ell_1(\mathbb{R})$, for all $y \in \mathbb{R}$. In particular, if $f(y) \neq 0, \forall y \in \mathbb{R}$ then there exists a function $u \in \ell_1(\mathbb{R})$ such that $F\{u\}(y) = \dfrac{1}{f(y)}, \forall y \in \mathbb{R}$." This result still holds true for the Fourier cosine transform on time scale $h\mathbb{N}^0, h > 0$.

Theorem 5.1. *Suppose that k and z are the given functions in $\ell_1(h\mathbb{N}^0)$ and satisfy the condition $F_c\{k\} \neq 0$, $\forall \omega \in \left[0, \frac{\pi}{h}\right]$. Then, the equation (5.3) has a unique solution in $\ell_1(h\mathbb{N}^0)$ and has the form $x(t) = (z{*}u)(t), t \in h\mathbb{N}^0$. Moreover, we have the following estimation*

$$||x||_{\ell_1(h\mathbb{N}^0)} \leq ||u||_{\ell_1(h\mathbb{N}^0)}||z||_{\ell_1(h\mathbb{N}^0)},$$

where $u \in \ell_1(h\mathbb{N}^0)$ and u is determined by $u(t) = \dfrac{1}{\pi}\displaystyle\int_0^{\frac{\pi}{h}} \frac{1}{F_c\{k\}(\omega)}\cos(\omega t)d\omega,$ *generalized convolution $(.{*}.)$ is defined by (3.1).*

Proof. To solve the equation (5.3), we'll apply the h-Fourier sine transform to both sides of (5.3), and use the factorization identity (3.2), we have

$$F_s\{x\}(\omega)F_c\{k\}(\omega) = F_s\{z\}(\omega), \quad \omega \in \left[0, \frac{\pi}{h}\right].$$

Under the condition $F_s\{k\}(\omega) \neq 0$, by the Theorem of Wiener–Levy's type, there exists a unique function $u \in \ell_1(h\mathbb{N}^0)$ that $F_c\{u\}(\omega) = \dfrac{1}{F_c\{k\}(\omega)},$

$\forall \omega \in \left[0, \frac{\pi}{h}\right]$. Function u can be found by the inverse of h-Fourier cosine transform on $h\mathbb{N}^0$

$$u(t) = \frac{1}{\pi} \int\limits_0^{\frac{\pi}{h}} \frac{1}{F_c\{k\}(\omega)} \cos(\omega t) d\omega.$$

Therefore, using factorization identity (3.2), we get

$$F_s\{x\}(\omega) = F_s\{z\}(\omega) F_c\{u\}(\omega) = F_s\{z * u\}(\omega), \quad \omega \in \left[0, \frac{\pi}{h}\right].$$

So $x(t) = (z * u)(t)$, $t \in h\mathbb{N}^0$. The inequality (3.3) yields that $x \in \ell_1(h\mathbb{N}^0)$ and $||x||_{\ell_1(h\mathbb{N}^0)} \leq ||u||_{\ell_1(h\mathbb{N}^0)} ||z||_{\ell_1(h\mathbb{N}^0)}$. ■

Theorem 5.2. *Assume that y, z are given functions belonging to the space $\ell_1(h\mathbb{N}^0)$, $z(0) = 0$ and $1 + F_c\{y\}(\omega) \neq 0$, $\forall \omega \in \left[0, \frac{\pi}{h}\right], h > 0$. Then, the equation (5.4) has a unique solution in $\ell_1(h\mathbb{N}^0)$ and it has the form*

$$x(t) = z(t) - (z * v)(t), t \in h\mathbb{N}^0.$$

Moreover, the following estimation holds

$$||x||_{\ell_1(h\mathbb{N}^0)} \leq ||z||_{\ell_1(h\mathbb{N}^0)} (1 + ||v||_{\ell_1(h\mathbb{N}^0)}), \tag{5.5}$$

where $v \in \ell_1(h\mathbb{N}^0)$ and v is defined by

$$F_c\{v\}(\omega) = \frac{F_c\{y\}(\omega)}{1 + F_c\{y\}(\omega)}, \quad \omega \in \left[0, \frac{\pi}{h}\right],$$

generalized convolution $(. * .)$ is defined by (3.1).

Proof. Applying the h-Fourier sine transform to both sides of (5.4) and using the factorization identity (3.2), we get

$$F_s\{x\}(\omega) + F_c\{y\}(\omega) F_s\{x\}(\omega) = F_s\{z\}(\omega), \quad \omega \in \left[0, \frac{\pi}{h}\right].$$

It leads to

$$F_s\{x\}(\omega) = F_s\{z\}(\omega) \left(1 - \frac{F_c\{y\}(\omega)}{1 + F_c\{y\}(\omega)}\right), \quad \omega \in \left[0, \frac{\pi}{h}\right].$$

By the Theorem of Wiener–Levy's type, there exists a unique function $v \in \in \ell_1(h\mathbb{N}^0)$ such that

$$F_c\{v\}(\omega) = \frac{F_c\{y\}(\omega)}{1 + F_c\{y\}(\omega)}, \quad \omega \in \left[0, \frac{\pi}{h}\right].$$

Clearly, v can be found by the inverse of $h-$ Fourier cosine transform on time scale $h\mathbb{N}^0$, as below

$$v(t) = \frac{1}{\pi}\int_0^{\frac{\pi}{h}} \frac{F_c\{y\}(\omega)}{1+F_c\{y\}(\omega)}\cos(\omega t)d\omega.$$

Therefore,

$$F_s\{x\}(\omega) = F_s\{z\}(\omega) - F_s\{z\}(\omega)F_c\{v\}(\omega) = F_s\{z - z * v\}(\omega), \quad \omega \in \left[0, \frac{\pi}{h}\right].$$

This implies that $x(t) = z(t) - (z * v)(t)$ for all $t \in h\mathbb{N}^0$.

Using (3.3), we can estimate the solution

$$||x||_{\ell_1(h\mathbb{N}^0)} \leq ||z||_{\ell_1(h\mathbb{N}^0)}(1 + ||v||_{\ell_1(h\mathbb{N}^0)}|). \qquad \blacksquare$$

Now, we give an example to illustrate the Theorem 5.2.

Example 5.3. We consider the equation of Toeplitz plus Hankel 5.4 on time scale $\mathbb{N}^0$ with $y, z \in \ell_1(\mathbb{N}^0)$, which are determined as

$$y(0) = \frac{1}{\pi}(1 - \pi - e^{-\pi}), y(n) = \frac{(-1)^n e^{-\pi} + 1}{\pi(n^2+1)}, \quad n \geq 1,$$
$$z(0) = 0, z(n) = \frac{4n(1 - e^{-2\pi}(-1)^{n-1})}{\pi((n-1)^2+4))((n+1)^2+4)}, \quad n \geq 1.$$

The h-Fourier cosine transform of y and h-Fourier sine transform of z are as follows

$$F_c\{y\}(\omega) = \frac{1}{\pi}(1 - \pi - e^{-\pi}) + 2\sum_{n=1}^{\infty} \frac{(-1)^n e^{-\pi} + 1}{\pi(n^2+1)}\cos(n\omega) = e^{-\omega} - 1,$$
$$F_s\{z\}(\omega) = 2\sum_{n=1}^{\infty} z(n)\sin(n\omega) = \sum_{n=1}^{\infty} \frac{8n(1 - (-1)^{n-1}e^{-\pi})}{\pi((n-1)^2+1)((n+1)^2+1)}\sin(n\omega) =$$
$$= e^{-2\omega}\sin\omega, \quad \omega \in \left[0, \frac{\pi}{h}\right].$$

From the proof of Theorem 5.2, we have

$$F_s\{x\}(\omega) = e^{-\omega}\sin\omega, \quad \omega \in [0, \pi].$$

Use the inverse of the Fourier sine transform on time scale $h\mathbb{N}^0$, we get the solution of equation (5.4)

$$x(n) = \frac{4n(1 - (-1)^{n-1}e^{-\pi})}{\pi((n-1)^2+1)((n+1)^2+1)}, \quad n \geq 1 \text{ and } x(0) = 0.$$

It is clear that $x \in \ell_1(h\mathbb{N}^0)$.

Acknowledgements. I would like to express my sincere gratitude to Professor Nguyen Xuan Thao (School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Vietnam) and Professor Trinh Tuan (Department of Mathematics, Electric Power University, Vietnam) for their valuable guidance and exceptional support during the planning and completeness of this paper.

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