Annales Univ. Sci. Budapest., Sect. Comp. 54 (2023) 253–263

RANDOM INHOMOGENEOUS BINARY RECURRENCES

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Communicated by Bui Minh Phong

(Received July 7, 2023; accepted September 5, 2023)

Abstract. The random inhomogeneous binary recurrence $(G_n)_{n=0}^{\infty}$ is defined by the initial values $G_0 = 0$, $G_1 = 1$, and by the recurrence rule $G_n = AG_{n-1} + BG_{n-2} + w_{n-2}$, where A, B are given real numbers, and $(w_n)_{n=0}^{\infty}$ is a random sequence with $s \geq 2$ possible real values. In this work, we investigate the properties of the sequence (G_n).

1. Introduction

In this paper, we investigate inhomogeneous binary recurrences where the inhomogeneous term is a random value from the basic set $\mathcal{A} = \{a_1, \ldots, a_s\} \subset \subset \mathbb{R}$. Our main purpose is to describe the tree induced by the recursive sequence.

We extend the antecedent paper [6] in two directions. Firstly, general binary recurrences are considered instead of the Fibonacci sequence. Secondly, the cardinality s of the set $\mathcal{A}$ is an arbitrary positive integer, not only $s = 2$. There have been articles which deal with random sequences. For example, Embree and Trefethen [2] examined the behaviour of the random sequence $x_n = x_{n-1} \pm \beta x_{n-2}$, where $0 < \beta < 1$ is fixed in advance. They asked how $|x_n|$ depends on β. In fact, [2] is an extension of the paper of Viswanath [8],

Key words and phrases: Inhomogeneous binary recurrence, random sequence, Fibonacci sequence.

2010 Mathematics Subject Classification: 11B37, 05A15.

who considered the random Fibonacci sequence given by $x_0 = x_1 = 1$ and $x_n = \pm x_{n-1} \pm x_{n-2}$, where the signs are chosen independently and with equal probabilities.

In the forthcoming part, we introduce the terminology we will use throughout the paper. Let $s \geq 2$ be a positive integer, and let $a_1 < a_2 < \cdots < a_s$ denote distinct arbitrary real numbers related to the probabilities $p_1, p_2, \ldots, p_s$, respectively, with the property $\sum_i p_i = 1$. Assume that each term $w_n = w_n(a_1, \ldots, a_s)$ of a random sequence $(w_n)_{n=0}^{\infty}$ is provided by a random trial, and takes value a_i with probability p_i. Define an inhomogeneous binary recurrence by

$$G_n = AG_{n-1} + BG_{n-2} + w_{n-2}(a_1, \ldots, a_s), \qquad (n \geq 2) \tag{1.1}$$

with initial values $G_0 = 0$, $G_1 = 1$. Here A and $B \neq 0$ are arbitrary real numbers. Put $D = A^2 + 4B$. We also introduce a sequence $f_n = Af_{n-1} + Bf_{n-2}$ backstage with initial values $f_0 = 0$ and $f_1 = 1$.

Suppose for the moment that $a \in \mathbb{R}$ is fixed and consider the specific sequence

$$G_n = AG_{n-1} + BG_{n-2} + a.$$

Applying Corollary 2 of [1] for the current sequence (G_n) we find that $G_n = f_n + a\sum_{j=0}^{n-1} f_j$. Then Lemma 1 of [5] provides a closed formula for the sum $\sum_{j=0}^{n-1} f_j$. In order to eliminate the term G_n we obtain here from the random case (1.1) we introduce $m_a(n) = G_n$ exclusively for this constant a generated case. The previous arguments lead to

$$m_a(n) = f_n + \begin{cases} \frac{f_n + Bf_{n-1} - 1}{A+B-1} a, & \text{if } A + B \neq 1; \\ \frac{f_n - n}{A-2} a, & \text{if } A + B = 1,\ D \neq 0; \\ \frac{nf_{n-1}}{A} a, & \text{if } A + B = 1,\ D = 0. \end{cases}$$

We anticipate that later we will use the term $m_{a_1}(n)$. Observe that the last branch ($A+B = 1$, $D = 0$) appears only when $A = 2$, $B = -1$, so the sequence (f_n) is the sequence of natural numbers $0, 1, 2, \ldots$, i.e. $f_n = n$.

Returning to (1.1), for $n \geq 2$ the term G_n may take s possible values given by $G_n = G_{n-1} + G_{n-2} + a_j$ $(j = 1, \ldots, s)$. This situation can be precisely figured with a tree. The values of the vertices of the tree are denoted by $\mathcal{T}_{n,k}$, where $n \geq 0$ means the row number (or level) and $k \geq 0$ does the entry position in row n. If $i < j$, then $G_n = G_{n-1} + G_{n-2} + a_i$ is left of $G_n = G_{n-1} + G_{n-2} + a_j$ in the *nth* row of the tree. For instance, Figure 1 illustrates the first few levels of the tree for $s = 3$, using locally the notation $a = a_1$, $b = a_2$, and $c = a_3$.

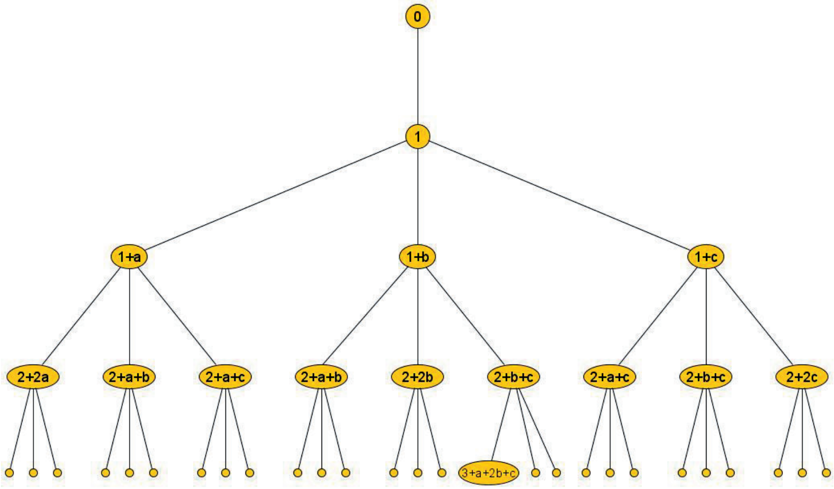

Figure 1. Row $0, 1, 2, 3, 4$ of the tree induced by (1.1) with $s = 3$.

Now we recall a useful lemma (for its proof, see [4, Lemma 2.1]), and one of its outgrowth.

Lemma 1.1. *Assume that N and t are positive integers. Moreover let $c_1, \dots, c_t$ be non-negative integers. The number of solutions to the diophantine equation $x_1 + x_2 + \dots + x_t = N$ with $x_i \geq c_i$ is*

$$\binom{N - \sum_{i=1}^{t} c_i + (t-1)}{t-1}.$$

Lastly, we present a consequence of Lemma 1.1.

Lemma 1.2. *Suppose that N, t and d are positive integers. The number of solutions to the diophantine equation $x_1 + x_2 + \dots + x_t = N$ with $0 \leq x_i \leq d$ is*

$$\binom{td - N + (t-1)}{t-1}.$$

Proof. If $x_1 + x_2 + \dots + x_t = N$, then

$$(d - x_1) + (d - x_2) + \dots + (d - x_t) = td - N$$

holds with the conditions $d - x_i \geq 0$. Now apply Lemma 1.1 to this equation with $c_i = 0$ to obtain the binomial coefficient above. ∎

In this paper, we examine the values what the term G_n can take, and we study some properties of the tree. We disregard the probability questions, and concentrate only on the features of the tree. In the next section, we deal with the general sequence (1.1) and provide Theorem 1, while in Section 3 we restrict ourselves for $f_n = F_n$ (the Fibonacci case). Here the most important result is Theorem 2.

2. Main theorem

First we formulate the principal observation what describes the entries of the tree we have introduced earlier. Obviously, level n contains s^{n-1} vertices.

Theorem 1. *Let $n \geq 2$ and $0 \leq k \leq s^{n-1} - 1$. Assume that the base-$s$ representation of k is $k = \varepsilon_{n-2}\varepsilon_{n-3} \dots \varepsilon_1\varepsilon_{0(s)}$, where $\varepsilon_i \in \{0, \dots, s-1\}$. The entry $\mathcal{T}_{n,k}$ of the kth element of row n is given by*

$$\mathcal{T}_{n,k} = m_{a_1}(n) + \sum_{j=0}^{n-2} \left(a_{\varepsilon_j+1} - a_1\right) f_{j+1}. \tag{2.1}$$

Before the proof we remark that the left winger element of row n of the tree is $m_{a_1}(n)$, the smallest one in the row. Thus (2.1) gives an explicit formula for $\mathcal{T}_{n,k}$ relative to $m_{a_1}(n)$.

Proof. Clearly, by the definition of sequence (f_n) we have $f_2 = A$ and $f_3 = A^2 + B$, moreover $f_{-1} = 1/B$. Put $k_1 = \lfloor k/s \rfloor$ and $k_2 = \lfloor k_1/s \rfloor$. The first observation is that

$$\mathcal{T}_{n,k} = A\mathcal{T}_{n-1,k_1} + B\mathcal{T}_{n-2,k_2} + a_{\varepsilon_0+1} \tag{2.2}$$

holds. It is a consequence of the base-s representation of k.

We split the proof into three parts according to the possible values of $m_{a_1}(n)$.

Case $A + B \neq 1$.

Let first $n = 2$. Hence $k = \varepsilon_{0(s)}$, $0 \leq k = \varepsilon_0 \leq s - 1$. We have

$$m_{a_1}(2) = A + \frac{A+B-1}{A+B-1} a_1 = A + a_1,$$

and

$$\sum_{j=0}^{0} \left(a_{\varepsilon_j+1} - a_1\right) f_{j+1} = a_{\varepsilon_0+1} - a_1,$$

consequently $\mathcal{T}_{2,k} = A + a_{\varepsilon_0+1}$ as we know.

Suppose $n=3$. Thus $k=\varepsilon_1\varepsilon_{0\,\text{(s)}}$, $0\le k=\varepsilon_1 s+\varepsilon_0\le s^2-1$. Since

$$m_{a_1}(3)=(A^2+B)+\frac{(A^2+B)+AB-1}{A+B-1}a_1=(A^2+B)+(A+1)a_1,$$

and

$$\sum_{j=0}^{1}\left(a_{\varepsilon_j+1}-a_1\right)f_{j+1}=(a_{\varepsilon_0+1}-a_1)+(a_{\varepsilon_1+1}-a_1)A,$$

then $\mathcal{T}_{3,k}=A^2+B+Aa_{\varepsilon_1+1}+a_{\varepsilon_0+1}$. This value can be easily checked by considering the sequence (G_n) via the initial values and via (1.1) for $n=2,3$. Hence the first values of (G_n) are

$$G_0=0,\ G_1=1,\ G_2=A+a_{\varepsilon_1+1},\ G_3=A(A+a_{\varepsilon_1+1})+B+a_{\varepsilon_0+1}.$$

Assume now that the statement is true for $2,3,\dots,n-1$. We will justify it for n. Now we apply the construction rule (2.2) of the tree, and then the induction hypothesis, which will be followed by straightforward manipulations. These admit

$$\begin{aligned}
&\mathcal{T}_{n,k}=A\mathcal{T}_{n-1,k_1}+B\mathcal{T}_{n-2,k_2}+a_{\varepsilon_0+1}=\\
=\ & Am_{a_1}(n-1)+A\sum_{j=1}^{n-2}\left(a_{\varepsilon_j+1}-a_1\right)f_j+\\
&+Bm_{a_1}(n-2)+B\sum_{j=2}^{n-2}\left(a_{\varepsilon_j+1}-a_1\right)f_{j-1}+a_{\varepsilon_0+1}=\\
=\ & A\left(f_{n-1}+\frac{f_{n-1}+Bf_{n-2}-1}{A+B-1}a_1\right)+\\
&+A\left(\sum_{j=0}^{n-2}\left(a_{\varepsilon_j+1}-a_1\right)f_j-(a_{\varepsilon_0+1}-a_1)f_0\right)+\\
&+B\left(f_{n-2}+\frac{f_{n-2}+Bf_{n-3}-1}{A+B-1}a_1\right)+\\
&+B\left(\sum_{j=0}^{n-2}\left(a_{\varepsilon_j+1}-a_1\right)f_{j-1}-((a_{\varepsilon_0+1}-a_1)f_{-1}+(a_{\varepsilon_1+1}-a_1)f_0)\right)+\\
&+a_{\varepsilon_0+1}=\\
=\ & f_n+\frac{f_n+Bf_{n-1}-1}{A+B-1}a_1-a_1+\sum_{j=0}^{n-2}\left(a_{\varepsilon_j+1}-a_1\right)f_{j+1}-\\
&-B(a_{\varepsilon_0+1}-a_1)\frac{1}{B}+a_{\varepsilon_0+1}=\\
=\ & m_{a_1}(n)+\sum_{j=0}^{n-2}\left(a_{\varepsilon_j+1}-a_1\right)f_{j+1}.
\end{aligned}$$

Case $A+B=1$, $D\neq 0$.

Clearly, $A\neq 2$ otherwise $D=0$ would fulfil. First we check the statement again for $n=2$ and for $n=3$.

If $n=2$, then

$$m_{a_1}(2)=f_2+\frac{f_2-2}{A-2}a_1=A+\frac{A-2}{A-2}a_1=A+a_1,$$

while $\sum_{j=0}^{0}\left(a_{\varepsilon_j+1}-a_1\right)f_{j+1}=a_{\varepsilon_0+1}-a_1$ holds again. Thus $\mathcal{T}_{2,k}=A+a_{\varepsilon_0+1}$.

Assume that $n=3$. Now

$$m_{a_1}(3)=f_3+\frac{f_3-3}{A-2}a_1=(A^2+B)+\frac{A^2+B-3}{A-2}a_1=(A^2+B)+(A+1)a_1,$$

where we used the equality $B=1-A$. This result and the value $m_{a_1}(3)$ of the previous case coincide. Since there is no change in the sum

$$\sum_{j=0}^{1}\left(a_{\varepsilon_j+1}-a_1\right)f_{j+1}=(a_{\varepsilon_0+1}-a_1)+(a_{\varepsilon_1+1}-a_1)A,$$

then $\mathcal{T}_{3,k}=A^2+B+Aa_{\varepsilon_1+1}+a_{\varepsilon_0+1}$.

In the induction step we must follow only the distinction in $Am_{a_1}(n-1)+$ $+Bm_{a_1}(n-2)$. Now this leads to

$$\begin{aligned}
&A\left(f_{n-1}+\frac{f_{n-1}-(n-1)}{A-2}a_1\right)+B\left(f_{n-2}+\frac{f_{n-2}-(n-2)}{A-2}a_1\right)=\\
=\ &f_n+\frac{f_n}{A-2}a_1-\frac{A(n-1)+B(n-2)}{A-2}a_1=\\
=\ &f_n+\frac{f_n}{A-2}a_1-\left(1+\frac{n}{A-2}\right)a_1=\\
=\ &f_n+\frac{f_n-n}{A-2}a_1-a_1=m_{a_1}(n)-a_1,
\end{aligned}$$

where we used again $B=1-A$. Similarly to the previous case we combine it with the result for the sum, and it immediately leads to

$$\mathcal{T}_{n,k}=m_{a_1}(n)+\sum_{j=0}^{n-2}\left(a_{\varepsilon_j+1}-a_1\right)f_{j+1}.$$

Case $A+B=1$, $D=0$. Recall that now $A=2$, and $B=-1$. Thus $m_{a_1}(2)=2+a_1$, and $m_{a_1}(3)=3+3a_1$, which together with the corresponding sums justify the statement for $n=2$ and $n=3$.

Finally, $Am_{a_1}(n-1)+Bm_{a_1}(n-2)=m_{a_1}(n)-a_1$ follows from

$$\begin{aligned} &= A\left(f_{n-1}+\frac{(n-1)f_{n-2}}{A}a_1\right)+B\left(f_{n-2}+\frac{(n-2)f_{n-3}}{A}a_1\right)= \\ &= f_n+\frac{nf_{n-1}}{A}a_1-\frac{Af_{n-2}+2Bf_{n-3}}{A}a_1=f_n+\frac{nf_{n-1}}{A}a_1-a_1, \end{aligned}$$

where the last term is obvious from $A=2$, $B=-1$, and $f_n=n$. Then the statement is implied by evaluating again the sum as previously. ■

3. Consequences for the Fibonacci case

Suppose that $A=B=1$. Thus sequence (f_n) is the Fibonacci sequence (F_n). Note that

$$m_{a_1}(n)=F_n+\frac{F_n+F_{n-1}-1}{1}a_1=a_1F_{n+1}+F_n-a_1,$$

as we found it in [6].

Structural observations related to the tree

First we simplify the result on $\mathcal{T}_{n,k}$.

Corollary 1. *Let $n\geq 2$ and $0\leq k\leq s^{n-1}-1$. Assume that the base-s representation of k is $k=\varepsilon_{n-2}\varepsilon_{n-3}\ldots\varepsilon_1\varepsilon_{0\,(s)}$, where $\varepsilon_i\in\{0,\ldots,s-1\}$. The entry $\mathcal{T}_{n,k}$ of the kth element of row n is given by*

$$\mathcal{T}_{n,k}=F_n+\sum_{j=0}^{n-2}a_{\varepsilon_j+1}F_{j+1}.$$

Proof. The proof relies on Theorem 1 with $m_{a_1}(n)=F_n+a_1(F_{n+1}-1)$. The straightforward calculations

$$\begin{aligned} \mathcal{T}_{n,k} &= F_n+a_1(F_{n+1}-1)+\sum_{j=0}^{n-2}a_{\varepsilon_j+1}F_{j+1}-\sum_{j=0}^{n-2}a_1F_{j+1} \\ &= F_n+\sum_{j=0}^{n-2}a_{\varepsilon_j+1}F_{j+1} \end{aligned}$$

proves the statement. In the last step the identity $\sum_{j=1}^{t}F_j=F_{t+2}-1$ was used. ■

The next result gives a connection between row $(n-1)$ and the parts of row n (of the tree). Figure 2 makes it really spectacular.

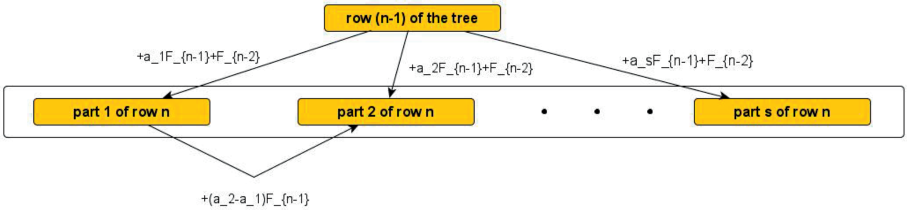

Figure 2. Structural connection between rows $n-1$ and n.

Corollary 2. *If $n \geq 2$ and $0 \leq k \leq s^{n-2}-1$ with $k = \varepsilon_{n-3}\varepsilon_{n-4}\dots\varepsilon_1\varepsilon_{0(s)}$, furthermore $K = \varepsilon_{n-2}s^{n-2} + k$ (where $\varepsilon_{n-2} \in \{0,\dots,s-1\}$, and $K = \varepsilon_{n-2}\varepsilon_{n-3}\varepsilon_{n-4}\dots\varepsilon_1\varepsilon_{0(s)}$), then*

$$\mathcal{T}_{n,K} = \mathcal{T}_{n-1,k} + (a_{\varepsilon_{n-2}+1}F_{n-1} + F_{n-2}).$$

Proof. Apply Corollary 1 as follows.

$$\begin{aligned}
\mathcal{T}_{n,K} &= F_n + \sum_{j=0}^{n-2} a_{\varepsilon_j+1}F_{j+1} = \\
&= F_{n-1} + F_{n-2} + \sum_{j=0}^{n-3} a_{\varepsilon_j+1}F_{j+1} + a_{\varepsilon_{n-2}+1}F_{n-1} = \\
&= \mathcal{T}_{n-1,k} + F_{n-2} + a_{\varepsilon_{n-2}+1}F_{n-1}.
\end{aligned}$$

∎

Bounds on the number of distinct entries of the rows

Assume that n is fixed. In this subsection, we will bound the cardinality $T_n = |\{\mathcal{T}_{n,k}\}|$ which gives the number of distinct values of $\mathcal{T}_{n,k}$ as k goes through the integers $0, 1 \dots, s^{n-2}-1$.

Let the sequence $(C_n)_{n=0}^{\infty}$ be defined as follows. Put $C_0 = C_1 = 0$, and for $n \geq 2$ let $C_n = C_{n-1} + C_{n-2} + 1$. It is easy to show that $C_n = F_{n+1} - 1$ holds. Clearly, in accordance with Corollary 1 each element of row n has the form

$$F_n + u_1a_1 + u_2a_2 + \dots + u_sa_s, \tag{3.1}$$

where $u_1, u_2, \dots, u_s$ are non-negative integers. Using the technique of induction one can immediately prove that $\sum_{i=1}^{s} u_i = C_n$. The main step of the induction depends on (2.2).

Now we record the statement concerning the upper bound on T_n. We assume $s \geq 3$ since the case $s = 2$ has been clarified in [6]. There we found the precise value $T_n = F_{n+1}$.

Theorem 2. *If $s \geq 3$, then the number of distinct entries T_n in row $n \geq 2$ satisfies*

$$T_n \leq \binom{F_{n+1}+s-2}{s-1} - \binom{sF_{n-1}-F_{n+1}}{s-1}.$$

Remark 1. For $s = 2$ the second term on the right-hand side would be zero because it has negative upper index (apart from the case $n = 2$, when we have $\binom{0}{1} = 0$, too), and we simply obtain $T_n \leq \binom{F_{n+1}}{1} = F_{n+1}$. From [6] we know that here equality holds.

Proof. The first term of the upper bound comes directly from the number of solutions to the diophantine equation $\sum_{i=1}^{s} u_i = F_{n+1}-1 = C_n$ in non-negative integers u_i. Applying Lemma 1.1 with $c_i = 0$, we obtain $\binom{F_{n+1}+s-2}{s-1}$.

But not any solution $(u_1, u_2, \ldots, u_s)$ is belonging to the entry set of row n if $s \geq 3$. (For $s = 2$ there is a one to one correspondence between the solutions and the entries, see [6].) For example, consider the case $s = 3$, $n = 6$. Then $F_6 + 4a_1 + 4a_2 + 4a_3$ is not among the entries of row 6 although $4 + 4 + 4 = 12 = F_7 - 1$. In general, if $\sum_{i=1}^{s} u_i = F_{n+1} - 1$ but each u_i is smaller than F_{n-1}, then (3.1) does not appear in row n. Indeed, by Corollary 2 the term $a_{\varepsilon_{n-2}+1}F_{n-1}$ guarantees that at least one coefficient u_i satisfies $u_i \geq F_{n-1}$.

It means that if we calculate those solutions to $\sum_{i=1}^{s} u_i = F_{n+1} - 1$ for which each $u_i < F_{n-1}$, then we can reduce the number of entries in row n. This is the application of Lemma 1.2 with $d = F_{n-1} - 1$ and $N = F_{n+1} - 1$, which provides

$$\binom{s(F_{n-1}-1)-(F_{n+1}-1)+(s-1)}{s-1} = \binom{sF_{n-1}-F_{n+1}}{s-1}.$$

Then the proof is complete. ■

Further "wrong cases" of the solution to $\sum_{i=1}^{s} u_i = F_{n+1}-1$ can be existed, when there is at least one $u_i \geq F_{n-1}$ but looking at row $(n-1)$ there each $w_j < F_{n-2}$ is valid. The coefficients w_j appear in row $n-1$ such that each entry has the form $F_{n-1} + w_1a_1 + w_2a_2 + \cdots + w_sa_s$. Corollary 2 shows that $w_j = u_j$ holds for each j but $w_i = u_i - F_{n-1}$. For instance, let $s = 3$, $n = 7$. Then $13 + 12a_1 + 4a_2 + 4a_3$ is not included in row 7 in spite of the facts that $12 \geq F_6 = 8$ and $12+4+4 = 20 = F_8-1$. It is a consequence of the observation that $(13-5)+(12-8)a_1+4a_2+4a_3$ is not in row 6, hence $13+12a_1+4a_2+4a_3$ has no predecessor.

Finally, we made a computer search to find the factual value of T_n if $s = 3$ and n is small. It makes possible to compare the upper bound of Theorem 2 and T_n. Table 1 contains the cases $n = 2, 3, , \ldots, 9$. We use the abbreviations

$a_n = \binom{F_{n+1}+1}{2}$, $b_n = \binom{2F_{n-1}-F_{n+1}}{2}$ in the table. The upper bound on T_n is given in the row $a_n - b_n$, while $a_n - T_n$ notifies how many of the "candidates" a_n does not appear in row n of the tree. This number is larger than b_n if $n \geq 7$, this phenomenon was forecasted in the previous paragraph.

n	2	3	4	5	6	7	8	9
a_n	3	6	15	36	91	231	595	1540
b_n	0	0	0	0	1	3	10	28
$a_n - b_n$	3	6	15	36	90	228	585	1512
T_n	3	6	15	36	90	225	567	1431
$a_n - T_n$	0	0	0	0	1	6	28	109

Table 1: Upper bounds and factual values for T_n with $s = 3$.

So the upper bound $a_n - b_n$ on $T_n = |\{\mathcal{T}_{n,k}\}|$ is sharp only for a few small integers n, and the exact number of T_n is still an open question.

Sequence $a_n = \binom{F_{n+1}+1}{2}$ appears in the solution of a representation problem (see [3]), and registered in [7] as A033192.

Acknowledgements. For L. Szalay this research was supported by National Research, Development and Innovation Office Grant 2019-2.1.11-TÉT-2020-00165, the Hungarian National Foundation for Scientific Research Grant No. 128088, and No. 130909, and the Slovak Scientific Grant Agency VEGA 1/0776/21.

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