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A REMARK ON JOINT APPROXIMATION BY ONE DIRICHLET SERIES

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Abstract. We consider one absolutely convergent series $\zeta_{u_T}(s)$ connected to the Riemann zeta-function, and prove a theorem on joint approximation of analytic functions by shifts $(\zeta_{u_T}(s+it_\tau^{\alpha_1}),\dots,\zeta_{u_T}(s+it_\tau^{\alpha_r}))$ as $T\to\infty$, where $\alpha_1,\dots,\alpha_r$ are fixed different positive numbers, t_τ is the Gram function, and $u_T\to\infty$ and $u_T\ll T^2$ as $T\to\infty$.

1. Introduction

The Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, is defined, for $\sigma>1$, by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}=\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)^{-1},$$

where $\mathbb{P}$ is the set of all prime numbers, and has the analytic continuation to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1. The function $\zeta(s)$ is an important analytic object having a sequence of interesting properties and hypotheses. One property is related to the denseness of the set of values of $\zeta(s)$, and is called universality. More precisely, universality of $\zeta(s)$ means that a wide class of analytic functions is approximated by shifts

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$\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. This was discovered by S.M. Voronin in [15]. For the last version of the Voronin universality theorem, the following notation is convenient. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact sets of the strip D with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Moreover, let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement is valid, see [3, 1, 8, 13, 9, 10].

Theorem 1.1. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Denote by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta. Then Theorem 1.1 shows that the set $\{\zeta(s+i\tau) : \tau \in \mathbb{R}\}$ is dense in the subspace of $H(D)$ of non-vanishing functions.

There are results on approximation of analytic functions by generalized shifts $\zeta(s + i\varphi(\tau))$ for some classes of the function $\varphi(\tau)$. Using generalized shifts allows to consider a joint approximation of a tuple of analytic functions $(f_1(s), \dots, f_r(s))$ by shifts $(\zeta(s+i\varphi_1(\tau)), \dots, \zeta(s+i\varphi_r(\tau)))$. The first result of such a type was obtained in [12] by using the functions $\varphi_j(\tau) = \tau^{\alpha_j}(\log\tau)^{\beta_j}$, $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, \dots, r$. As usual, denote by $\Gamma(s)$ the Euler gamma-function. In [7], the functions $\varphi_j(\tau)$ connected to the functional equation for $\zeta(s)$,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \quad s \in \mathbb{C},$$

was applied. We see that the function $h(s) \overset{\text{def}}{=} \pi^{-s/2}\Gamma(s/2)$ is the main ingredient of the above functional equation. Denote by $\theta(t)$, $t > 0$, the increment of the argument of the function $h(s)$ along the segment connecting the points $1/2$ and $1/2+it$. The function $\theta(t)$ is monotonically increasing and unbounded from above for $t > t_1 = 6.289\dots$. Hence, the equation

$$\theta(t) = (\tau - 1)\pi, \quad \tau \geqslant 0, \tag{1.1}$$

for $t > t_1$ has the unique solution t_τ. The equation (1.1), for $\tau \in \mathbb{N}$, was considered by J.-P. Gram [4] in connection with nontrivial zeros of the Riemann zeta-function. Let γ_n be the imaginary part of the nth positive nontrivial zero of $\zeta(s)$. Then the Riemann-von Mangoldt formula for the number of nontrivial zeros implies that $t_n \sim \gamma_n$ as $n \to \infty$. Gram observed [4] that each interval $[t_{n-1}, t_n]$ with $1 \leqslant n \leqslant 15$, contains one zero of the function $\zeta(1/2+it)$. However, for $n > 15$, this turned out not true. The numbers t_n are called the Gram points, they were studied by various authors, see, for example, [5] and [6]. We call t_τ the Gram function, the theory of t_τ can be found in [5].

For joint universality of $\zeta(s)$, the function t_τ was used in [7].

Theorem 1.2. [7]. *Suppose that $\alpha_1, \dots, \alpha_r$ are fixed different positive numbers. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+it_\tau^{\alpha_j}) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.

Our aim is to show that $\zeta(s+it_\tau^{\alpha_j})$ can be replaced by a certain absolutely convergent Dirichlet series depending on T. Let $\theta > 1/2$ be a fixed number, and, for $u > 0$ and $m \in \mathbb{N}$,

$$v_u(m) = \exp\left\{ -\left(\frac{m}{u}\right)^\theta \right\}.$$

Then the Dirichlet series

$$\zeta_u(s) = \sum_{m=1}^{\infty} \frac{v_u(m)}{m^s}$$

is absolutely convergent for $\sigma > \sigma_0$ with arbitrary σ_0. We will replace the shifts $\zeta(s+it_\tau^{\alpha_j})$ in Theorem 1.2 by $\zeta_{u_T}(s+it_\tau^{\alpha_j})$ for some $u_T \to \infty$ as $T \to \infty$.

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and define the set

$$\Omega = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\}.$$

Moreover, let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then Ω and Ω^r are compact topological Abelian groups, therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined, and we have the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. Denote by $\omega = (\omega_1, \dots, \omega_r)$, $\omega_j \in \Omega_j$, $\omega_j = (\omega_j(p) : p \in \mathbb{P})$, $j = 1, \dots, r$, the elements of Ω^r. On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \omega) = (\zeta(s, \omega_1), \ldots, \zeta(s, \omega_r)),$$

where

$$\zeta(s, \omega_j) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_j(p)}{p^s}\right)^{-1}, \quad j = 1, \ldots, r.$$

Note that the latter product, for almost all $\omega_j \in \Omega_j$, converges uniformly on compact subsets of the strip D, see, for example [8]. The main result of the paper is the following theorem.

Theorem 1.3. *Suppose that $\alpha_1, \ldots, \alpha_r$ are fixed different positive numbers, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then the limit*

$$\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta_{u_T}(s + it_\tau^{\alpha_j}) - f_j(s)| < \varepsilon \right\} =$$
$$= m_H \left\{ \omega \in \Omega^r : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s, \omega_j) - f_j(s)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Theorem 1.3 shows that, for sufficiently large T, there are infinitely many shifts $(\zeta_{u_T}(s + it_\tau^{\alpha_1}), \ldots, \zeta_{u_T}(s + it_\tau^{\alpha_r}))$ of absolutely convergent Dirichlet series $\zeta_{u_T}(s)$ that approximate simultaneously a given tuple $(f_1(s), \ldots, f_r(s))$ of analytic functions. Using for approximation the shifts of absolutely convergent series is a new type of universality, and is main advantage of Theorem 1.3 against Theorem 1.2.

We will derive Theorem 1.3 from a probabilistic limit theorem in the space $H^r(D)$.

2. Distance between $\zeta(s)$ and $\zeta_{u_T}(s)$

Denote by ρ_r the metric in the space $H^r(D)$ which induces its product topology, i. e., for $\underline{g}_k(g_{k1}, \ldots g_{kr}) \in H^r(D)$, $k = 1, 2$,

$$\rho_r(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j}),$$

where ρ is the metric in $H(D)$ inducing the topology of uniform convergence on compacta. For brevity, let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$,

$$\underline{\zeta}(s + it_\tau^{\underline{\alpha}}) = (\zeta(s + it_\tau^{\alpha_1}), \ldots, \zeta(s + it_\tau^{\alpha_r}))$$

and

$$\underline{\zeta}_{u_T}(s + it_\tau^{\underline{\alpha}}) = \left(\zeta_{u_T}(s + it_\tau^{\alpha_1}), \ldots, \zeta_{u_T}(s + it_\tau^{\alpha_r})\right).$$

Lemma 2.1. *Suppose that $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Then, for all $\underline{\alpha}$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \rho_r \left(\underline{\zeta}(s + it_\tau^{\underline{\alpha}}), \underline{\zeta}_{u_T}(s + it_\tau^{\underline{\alpha}}) \right) \mathrm{d}\tau = 0.$$

Proof. From the definition of the metrics ρ_r and ρ, it follows that it suffices to show that, for every compact set $K \subset D$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta(s + it_\tau^{\alpha}) - \zeta_{u_T}(s + it_\tau^{\alpha})| \,\mathrm{d}\tau = 0$$

with every positive α. Let

$$l_{u_T}(s) = \frac{1}{\theta} \Gamma\left(\frac{s}{\theta}\right) u_T^s,$$

where the number θ is from the definition of $v_u(m)$. Then, see, for example, [8], the integral representation

$$\zeta_{u_T}(s) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s+z) l_{u_T}(z) \,\mathrm{d}z \tag{2.1}$$

is valid. Let K be an arbitrary compact set of D. Fix $\varepsilon > 0$ such that, for all $s = \sigma + it \in K$, $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$. Take

$$\theta_1 = \sigma - \frac{1}{2} - \varepsilon > 0 \qquad \text{and} \qquad \theta = \frac{1}{2} + \varepsilon.$$

Then the integrand in (2.1) has simple poles at $z = 0$ and $z = 1 - s$ lying in the strip $-\theta_1 \leqslant \mathrm{Re} z \leqslant \theta$. Therefore, by the residue theorem,

$$\zeta_{u_T}(s) - \zeta(s) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \zeta(s+z) l_{u_T}(z) \,\mathrm{d}z + l_{u_T}(1-s).$$

Hence, for $s \in K$,

$$\begin{aligned} &\zeta_{u_T}(s + it_\tau^{\alpha}) - \zeta(s + it_\tau^{\alpha}) \ll \\ &\ll \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + \varepsilon + it_\tau^{\alpha} + iu\right) \right| \sup_{s\in K} \left| l_{u_T}\left(\frac{1}{2} + \varepsilon - s + iu\right) \right| \mathrm{d}u + \\ &\quad + \sup_{s\in K} |l_{u_T}(1 - s - it_\tau^{\alpha})|. \end{aligned}$$

This gives

$$\frac{1}{T}\int_0^T \sup_{s\in K} |\zeta(s+it_\tau^\alpha)-\zeta_{u_T}(s+it_\tau^\alpha)|\,\mathrm{d}\tau \ll I_1+I_2, \tag{2.2}$$

where

$$I_1=\int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\varepsilon+it_\tau^\alpha+iu\right)\right|\,\mathrm{d}\tau\right)\sup_{s\in K}\left|l_{u_T}\left(\frac{1}{2}+\varepsilon-s+iu\right)\right|\,\mathrm{d}u$$

and

$$I_2=\frac{1}{T}\int_0^T\sup_{s\in K}|l_{u_T}(1-s-it_\tau^\alpha)|\,\mathrm{d}\tau.$$

By the definition of $l_{u_T}(s)$, using the estimate

$$\Gamma(\sigma+it)\ll\exp\{-c|t|\},\quad c>0,$$

we find that, for $s\in K$,

$$\begin{aligned} l_{u_T}\left(\frac{1}{2}+\varepsilon-s+iu\right) &\ll_\varepsilon u_T^{1/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\theta}|u-t|\right\}\ll_{\varepsilon,K} \\ &\ll_{\varepsilon,K} u_T^{-\varepsilon}\exp\{-c_1|u|\},\quad c_1>0. \end{aligned} \tag{2.3}$$

Moreover, Lemma 2.2 of [7] yields the estimate

$$\begin{aligned}\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\varepsilon+it_\tau^\alpha+iu\right)\right|\,\mathrm{d}\tau &\leqslant \left(\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\varepsilon+it_\tau^\alpha+iu\right)\right|^2\,\mathrm{d}\tau\right)^{1/2}\ll_{\varepsilon,\alpha} \\ &\ll_{\varepsilon,\alpha}(1+|u|)^{1/2}.\end{aligned}$$

This and (2.3) show that

$$I_1\ll_{\varepsilon,K,\alpha}u_T^{-\varepsilon}\int_{-\infty}^{\infty}(1+|u|)^{1/2}\exp\{-c_1|u|\}\,\mathrm{d}u\ll_{\varepsilon,K,\alpha}u_T^{-\varepsilon}.$$

Similarly as above, we obtain that, for all $s\in K$,

$$l_{u_T}(1-s-it_\tau^\alpha)\ll_\varepsilon u_T^{1-\sigma}\exp\{-c_2|t+t_\tau^\alpha|\}\ll_{\varepsilon,K}u_T^{1/2-2\varepsilon}\exp\{-c_3t_\tau^\alpha\}$$

with positive c_2 and c_3. Thus, since, by [5],

$$t_\tau=\frac{2\pi\tau}{\log\tau}(1+o(1)),\quad \tau\to\infty,$$

$$(2.4) \qquad I_2 \ll_{\varepsilon,K,\alpha} \frac{u_T^{1/2-2\varepsilon} u_T^{\varepsilon}}{T} + u_T^{1/2-2\varepsilon} \exp\{-c_4 t^{\alpha}_{u_T^{\varepsilon}}\} \ll_{\varepsilon,K,\alpha} u_T^{-\varepsilon}, \quad c_4 > 0.$$

The estimates (2.2) – (2.4) prove the lemma. ∎

3. Limit theorem

In this section, we prove a limit theorem for $\underline{\zeta}_{u_T}(s)$. For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}_{u_T}(s + it^{\underline{\alpha}}_{\tau}) \in A\right\}.$$

Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s,\omega)$, i. e.,

$$P_{\underline{\zeta}}(A) = m_H\left\{\omega \in \Omega^r : \underline{\zeta}(s,\omega) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Theorem 3.1. *Suppose that $\alpha_1, \dots, \alpha_r$ are fixed different positive numbers, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Then, $P_{T,\underline{\alpha}}$ converges weakly to the measure $P_{\underline{\zeta}}$ as $T \to \infty$.*

The proof of Theorem 3.1 is based on a limit lemma for

$$\widehat{P}_{T,\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}(s + it^{\underline{\alpha}}_{\tau}) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)),$$

obtained in [7], Theorem 5.4.

Lemma 3.1. *Suppose that $\alpha_1, \dots, \alpha_r$ are fixed different positive numbers. Then, $\widehat{P}_{T,\underline{\alpha}}$ converges weakly to the measure $P_{\underline{\zeta}}$ as $T \to \infty$.*

Proof of Theorem 3.1. On a certain probability space $(\mathbb{T}, \mathcal{A}, \mu)$ define the random variable ξ_T uniformly distributed on $[0,T]$, and introduce two $H^r(D)$-valued random elements

$$\widehat{X}_{T,\underline{\alpha}} = \widehat{X}_{T,\underline{\alpha}}(s) = \underline{\zeta}(s + it^{\underline{\alpha}}_{\xi_T})$$

and

$$X_{T,\underline{\alpha}} = X_{T,\underline{\alpha}}(s) = \underline{\zeta}_{u_T}(s + it^{\underline{\alpha}}_{\xi_T}).$$

Since the function t_τ is differentiable [5], it is continuous, hence, it is measurable. Therefore t_{ξ_T} is a random variable. In view of Lemma 3.1, the random element converges in distribution to $P_{\underline{\zeta}}$ as $T \to \infty$. We apply the equivalent of weak convergence of probability measures in terms of closed sets, see, for example, Theorem 2.1 of [2]. Let $F \subset H^r(D)$ be an arbitrary closed set,

$\rho_r(g, F)$ the distance between $g \in H^r(D)$ and F, and $\varepsilon > 0$. Then the set $F_\varepsilon = \{g \in H^r(D) : \rho_r(g, F) \leqslant r\}$ is closed as well. Thus, by Lemma 3.1,

$$\limsup_{T\to\infty} \widehat{P}_{T,\underline{\alpha}}(F_\varepsilon) \leqslant P_\zeta(F_\varepsilon). \tag{3.1}$$

Observe that

$$\{X_{T,\underline{\alpha}} \in F\} \subset \{\widehat{X}_{T,\underline{\alpha}} \in F_\varepsilon\} \cup \{\rho_r(X_{T,\underline{\alpha}}, \widehat{X}_{T,\underline{\alpha}}) \geqslant \varepsilon\}.$$

Therefore,

$$\mu\{X_{T,\underline{\alpha}} \in F\} \leqslant \mu\{\widehat{X}_{T,\underline{\alpha}} \in F_\varepsilon\} + \mu\{\rho_r(X_{T,\underline{\alpha}}, \widehat{X}_{T,\underline{\alpha}}) \geqslant \varepsilon\}. \tag{3.2}$$

By the definition of $X_{T,\underline{\alpha}}$ and $\widehat{X}_{T,\underline{\alpha}}$,

$$\mu\{\rho_r(X_{T,\underline{\alpha}}, \widehat{X}_{T,\underline{\alpha}}) \geqslant \varepsilon\} \leqslant \frac{1}{T\varepsilon} \int_0^T \rho_r(X_{T,\underline{\alpha}}, \widehat{X}_{T,\underline{\alpha}})\,\mathrm{d}\tau = o(1)$$

as $T \to \infty$ in view of Lemma 2.1. Since

$$\mu\{\widehat{X}_{T,\underline{\alpha}} \in F_\varepsilon\} = \widehat{P}_{T,\underline{\alpha}}(F_\varepsilon)$$

and

$$\mu\{X_{T,\underline{\alpha}} \in F\} = P_{T,\underline{\alpha}}(F),$$

this and (3.2) show that

$$P_{T,\underline{\alpha}}(F) \leqslant \widehat{P}_{T,\underline{\alpha}}(F_\varepsilon) + o(1), \quad T \to \infty.$$

Thus, by (3.1),

$$\limsup_{T\to\infty} P_{T,\underline{\alpha}}(F) \leqslant \limsup_{T\to\infty} \widehat{P}_{T,\underline{\alpha}}(F_\varepsilon) = P_\zeta(F_\varepsilon)$$

Now let $\varepsilon \to +0$. Then $F_\varepsilon \to F$, and we have, by the last inequality,

$$\limsup_{T\to\infty} P_{T,\underline{\alpha}}(F) \leqslant P_\zeta(F),$$

i. e.,$P_{T,\underline{\alpha}}$ converges weakly to P_ζ as $T \to \infty$. ∎

For $A \in \mathcal{B}(\mathbb{R})$), define

$$Q_{T,\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta_{u_T}(s + it_\tau^{\alpha_j}) - f_j(s)| \in A\right\}.$$

Lemma 3.2. *Suppose that $\alpha_1, \dots, \alpha_r$ are fixed different positive numbers, and $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then $Q_{T,\underline{\alpha}}$ converges weakly to*

$$m_H \left\{ \omega \in \Omega^r : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s, \omega_j) - f_j(s)| \in A \right\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

as $T \to \infty$.

Proof. The mapping $v : H^r(D) \to \mathbb{R}$ given by

$$v(g_1, \dots, g_r) = \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - f_j(s)|, \quad (g_1, \dots, g_r) \in H^r(D),$$

is continuous, and, for every $A \in \mathcal{B}(\mathbb{R})$,

$$Q_{T,\underline{\alpha}}(A) = P_{T,\underline{\alpha}}(v^{-1}A).$$

This means that $Q_{T,\underline{\alpha}} = P_{T,\underline{\alpha}} v^{-1}$. Therefore, Theorem 5.1 of [2] and Theorem 3.1 imply that $Q_{T,\underline{\alpha}}$ converges weakly to $P_\zeta v^{-1}$ as $T \to \infty$, and the definition of the mapping v gives the assertion of the lemma. ■

For the proof of Theorem 1.3, we will use the language of distribution functions. Therefore, we rewrite Lemma 3.2 in terms of distribution functions. We recall that the weak convergence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is equivalent to that of corresponding distribution functions, and the distribution function $F_n(x)$ converges weakly to a distribution function $F(x)$ as $n \to \infty$ if

$$\lim_{n \to \infty} F_n(x) = F(x)$$

for every continuity point x of the function $F(x)$.

The corresponding distribution functions of the measures $P_{T,\underline{\alpha}}$ and P_ζ are

$$F_{T,\underline{\alpha}}(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta_{u_T}(s + it_\tau^{\alpha_j}) - f_j(s)| < \varepsilon \right\}$$

and

$$F_\zeta(\varepsilon) \stackrel{\text{def}}{=} m_H \left\{ \omega \in \Omega^r : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s, \omega_j) - f_j(s)| < \varepsilon \right\},$$

respectively. Therefore from Lemma 3.2, we have

Lemma 3.3. *Suppose that the hypotheses of Lemma 3.2 are satisfied. Then $F_{T,\underline{\alpha}}(\varepsilon)$ converges weakly to $F_\zeta(\varepsilon)$ as $T \to \infty$.*

4. Proof of Theorem 1.3

We will use the explicit form of the support of the measure P_ζ. We recall that the support of P_ζ is a minimal closed set $S_\zeta \subset \mathcal{B}(H^r(D))$ such that $P_\zeta(S_\zeta)=1$. Set

$$S=\{g \in H(D):\ \text{either } g(s)\neq 0, \text{ or } g(s)\equiv 0\}.$$

Lemma 4.1. *The support S_ζ coincides with S^r.*

Proof. The lemma is Lemma 6.8 of [7]. ■

Proof of Theorem 1.3. By the Mergelyan theorem on approximation of analytic functions by polynomials [11], there exist polynomials $p_1(s),\dots,p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|f_j(s)-\mathrm{e}^{p_j(s)}\right|<\frac{\varepsilon}{2}. \tag{4.1}$$

The requirements on the sets K_j and $f_j(s)$ come from using of the Mergelyan theorem. Clearly, $(\mathrm{e}^{p_1(s)},\dots,\mathrm{e}^{p_r(s)})\in S_\zeta$ in view of Lemma 4.1. Hence, by the support property,

$$m_H\left\{\omega\in\Omega^r:\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|\zeta(s,\omega_j)-\mathrm{e}^{p_j(s)}\right|<\frac{\varepsilon}{2}\right\}>0. \tag{4.2}$$

Inequality (4.1) shows that

$$\left\{\omega\in\Omega^r:\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s,\omega_j)-f_j(s)|<\varepsilon\right\}\supset$$
$$\supset\left\{\omega\in\Omega^r:\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\left|\zeta(s,\omega_j)-\mathrm{e}^{p_j(s)}\right|<\frac{\varepsilon}{2}\right\}.$$

Hence, by (4.2),

$$m_H\left\{\omega\in\Omega^r:\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s,\omega_j)-f_j(s)|<\varepsilon\right\}>0. \tag{4.3}$$

It is well known that the set of discontinuity points of distribution functions is at most countable. This, Lemma 3.3 and (4.3) prove that the limit

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta_{u_T}(s+it_\tau^{\alpha_j})-f_j(s)|<\varepsilon\right\}=$$
$$=m_H\left\{\omega\in\Omega^r:\ \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s,\omega_j)-f_j(s)|<\varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon>0$. ■

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