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NUMERATION SYSTEMS DEFINED BY ADDITION RULES

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Abstract. We investigate the problem of reconstructing polynomial-based ternary number systems when the expansion of certain elements – the pairwise sums of digits – is given. We call these expansions the "addition rules" for the system. We define a randomized process for expanding *arbitrary* elements based on these addition rules. For some rules, the expansion is not-well defined. We search for addition rules below a certain length that yield well-defined expansion for all elements, which we call the uniqueness property. We also give algorithms for the reconstruction of the base polynomial and the digits of the numeration system from the addition rules. Finally, we formulate some open problems for further research.

1. Introduction

The definition of positional number systems for representing positive integers has many generalizations. In matrix-based numeration systems, vectors are represented as a sum of the powers of a base matrix multiplied by digit vectors. In the present paper, we consider base matrices which are companion matrices of a polynomial and digit sets with three elements. Formally, let n be a positive integer, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $|a_0| = 3$. Let M be the companion matrix of $f(x)$ and $D = \{0, a, b\}$ be a digit set with $a, b \in \mathbb{Z}^n$ such that D is a complete residue system modulo M. Usual

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questions one poses about such a system are the existence of a representation of all integer vectors in base M with digits in D, and how such a representation can be obtained algorithmically.

Here, we focus on the inverse problem: given the representation of some elements of the integer lattice, can we obtain the representation of other elements, and, ultimately, reconstruct f, a and b? Assume that the expansion in base $f(x)$ with digits in D of $2a$, $2b$ and $a+b$ are known. We will show that this can be used to define a nondeterministic “dynamical system”. We investigate which of these dynamical systems satisfy the uniqueness property, which means that each linear combination of the digits a and b with integer coefficients has a unique (but not necessarily finite) representation.

The paper is built up as follows: in Section 2, we give definitions and prior results on numeration systems and define our research problems. In Section 3, we illustrate addition rules and the associated dynamical system through examples. In Section 4, we introduce algorithms for searching, decision and validation of addition rules with certain properties. In Section 5, we give the main idea for the reconstruction of the base of systems from addition rules. In Section 6, we summarize the results and formulate further research questions.

2. Definitions and prior results

The definition of positional number systems for representing positive integers has many generalizations. In [3], a positional representations for complex integers were analyzed. A comprehensive survey in further generalizations can be found in [1]. For the purposes of the present paper, we will need polynomial based systems and matrix numeration systems, see e.g. [5, 4]. For a self-contained presentation, we give the definitions and theorems that will be used throughout the paper.

Definition 2.1. Let $M \in \mathbb{Z}^{n\times n}$ be an invertible matrix and $D \subseteq \mathbb{Z}^n$ is a finite set. The pair (M, D) is called *a matrix numeration system*, if $\forall x \in \mathbb{Z}^n$ has a unique and finite representation in form

$$x = \sum_{i=0}^{n} M^i d_i,$$

where $\forall i \in \{0, 1, \ldots, n\} : d_i \in D$ and $d_n \neq 0$.

Definition 2.2. Let $x, y \in \mathbb{Z}^n$ be integer vectors. *x and y are congruent modulo M* if

$$\exists v \in \mathbb{Z}^n : \ x - y = Mv$$

The congruence is an equivalence relation. Its classes are called *congruence classes modulo M*.

Definition 2.3. A set D is *a complete residue system modulo* M, if it contains exactly one member of each residue class of $\mathbb{Z}^n$.

A few necessary conditions for (M, D) to be a matrix numeration system are as follows. The digit set D needs to be a complete residue system modulo M, thus have cardinality $|\det(M)|$. The digit $0 \in D$. Furthermore M needs to be an expanding matrix (i.e. have eigenvalues with modulus strictly larger than 1).

Definition 2.4. If M is an invertible matrix, then

$$D = \{(i, 0, 0, \ldots, 0)^T : 0 \le i < |\det M|\}$$

is called *the canonical digit set.*

Definition 2.5. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a monic polynomial with integer coefficients. The matrix

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{pmatrix}$$

is called *the companion matrix of* f.

If the constant term $a_0 \neq 0$, then $|\det(M)| = (-1)^{n-1} a_0$, so the matrix M is invertible.

Definition 2.6. A polynomial $f(x)$ is called *a canonical number system polynomial or CNS-polynomial* if its companion matrix with the canonical digit set is a generalized number system.

We note that by the above observations, necessary conditions for f to be a CNS-polynomial, the digit set should contain exactly $|a_0|$ elements and f has to be an expanding polynomial (all roots have modulus greater than 1).

Non-canonical digit sets can also be of interest. Examples of such non-canonical systems can be given by base polynomial $x - 3$ and the digit set $D = \{-1, 0, 1\}$. Another example is $f = x + 3$ with $D = \{0, 2, 7\}$. In the latter example, we have that every integer has a unique representation in base -3 with digits $0, 2, 7$. In [6], the authors give infinite families of non-canonical matrix numeration systems in dimension one.

An important algorithmic problem of numeration systems is the following: given a matrix M and digit set D as inputs, decide if they constitute a matrix numerations system. In [5, 4], methods have been presented to reduce the problem to an exhaustive search over a finite region of vectors. In [2] and independently, in [7], a different approach is introduced. In these papers, the existence of representation using base and digits (M, D) is reduced to the problem of addition in base M representation. This, in turn can be analyzed by understanding the "set of carries" or the "carry automaton" when performing addition on representations. In traditional positional number systems where the carry of 1 is the only possibility when adding two numbers using "paper-and-pencil" addition. In generalized systems, several different carries can occur, influencing digits that are several positions ahead and interacting with each other. How addition is performed is uniquely determined by M and D, and this contains the necessary information to analyze the existence of representations and the numeration system property.

In the present paper we set out to investigate the inverse problem: given the rules how addition works on representations in base M with digits D, can we recover M and D? As it is easy to see, this can only be possible up to a basis change, so instead we try to recover polynomial-based systems and the digits. We investigate ternary systems: the digit set is $D = \{0, a, b\}$, M is unknown (even the dimension is unknown), and we are given the expansion of $a + a$, $b + b$ and $a + b$ as inputs.

3. Reconstruction from addition rules – an example

From now on, a and b will stand for the two nonzero digits of a ternary system. More perecisely, Let M be a base matrix with determinant ± 3, $D = \{0, a, b\}$, and suppose we are given the expansion of $a+a$, $b+b$ and $a+b$ as inputs. We will use the following notation. Instead of writing $M^k e_k + M^{k-1} e_{k-1} + + \cdots + M e_1 + e_0$, we will write the $(k + 1)$ -tuple $(e_k, e_{k-1}, \ldots, e_1, e_0)$. If M is the companion matrix of a polynomial, the same tuple is used to represent $x^k e_k + x^{k-1} e_{k-1} + \cdots + x e_1 + e_0$

Below, we give two example instances of the problem. The inputs are the addition rules, the goal is to recover M and D.

Example 3.1. Let the expansion of pairwise sums of digits be given as follows:

$$\begin{aligned} 2a &\sim (b, 0, 0, b), \\ 2b &\sim (0, b, a), \\ a + b &\sim (a, b, 0). \end{aligned}$$

How can we compute $3a + 2b$? We do not have this in the list, but we can break it down into sums of sums, e.g. as $(a + b) + (a + b) + a$. Now, in the latter sum, the expansion of the summands is known, so we have that $3a + 2b = (2a, 2b, a) = M^2(2a) + M(2b) + a$. This is not a finished expansion because M^2 and M are not multiplied by digits. Nonetheless, the final digit is obtained. After removing the final digit, we may the proceed by finding how (2a, 2b) can be expanded. This "removal of the final digit" step is visualized in the figure below by a horizontal line.

Let us break down the computation of $3a + 2b$. In each step of the calculation we choose a linear combination of $2a$, $2b$, $a + b$, a and b as follows:

$$3a + 2b = c_{2a} \cdot 2a + c_{2b} \cdot 2b + c_{a+b} \cdot (a + b) + c_a \cdot a + c_b \cdot b,$$
$$\text{where } c_a, c_b \in \{0, 1\} \text{ and } c_{2a}, c_{2b}, c_{a+b} \in \mathbb{Z}$$

and with the addition rules we get:

	0	0	0	0	3a+2b	| 3a+2b=(a+b)+(a+b)+a
	0	0	a	b	0	
	0	0	a	b	0	
+	0	0	0	0	a	
	0	0	2a	2b	**a**	| 2b=2b
+	0	0	b	a		
	0	0	2a+b	**a**		| 2a+b=(a+b)+a
	a	b	0			
+	0	0	a			
	a	**b**	**a**	**a**	**a**	

The process stops at this point and we obtain that the expansion of $3a+2b$ is equal to (a, b, a, a, a). Now let see what happens if the linear combinations in steps of the addition are chosen differently.

	0	0	0	0	0	0	3a+2b	| 3a+2b=2a+2b+a
	0	0	0	b	0	0	b	
	0	0	0	0	0	b	a	
+	0	0	0	0	0	0	a	
	0	0	0	b	0	b	2a+b	| 2a+b=(a+b)+a
	0	0	0	0	a	b	0	
+	0	0	0	0	0	0	a	
	0	0	0	b	a	2b	**a**	| 2b=2b
+	0	0	0	0	b	a		
	0	0	0	b	a+b	**a**		| a+b=a+b
+	0	0	a	b	0			
	0	0	a	2b	**0**			| 2b=2b
+	0	0	b	a				
	0	0	a+b	**a**				
+	a	b	0					
	a	**b**	**0**	**a**	**0**	**a**	**a**	

With these steps the expansion of $3a + 2b$ is equal to $(a, b, 0, a, 0, a, a)$, so these rules do not satisfy the uniqueness property of expansions.

Example 3.2. Let $f(x) = x^2 - x - 3$ and the digit set $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. The companion matrix of f is equal to

$$\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}.$$

Let $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $a = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be the digits. In this case

$$2a = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^1 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sim (a, b),$$

$$2b = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^1 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^0 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} \sim (b, 0, a),$$

$$a + b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^1 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^0 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \sim (b, a, 0).$$

Now we assume, that just the expansion of $2a$, $2b$ and $a + b$ are known and try to reconstruct the base of this system. Calculate the expansion of $3a + 2b$ in two ways, as in the previous example.

	0	0	0	0	3a+2b	| 3a+2b=2a+2b+a
	0	0	0	a	b	
	0	0	b	0	a	
+	0	0	0	0	a	
	0	0	b	a	2a+b	| 2a+b=2a+b
	0	0	0	a	b	
+	0	0	0	0	b	
	0	0	b	2a	2b	| 2b=2b
+	0	0	b	0	**a**	
	0	0	2b	2a		| 2a=2a
+	0	0	a	**b**		
	0	0	a+2b			| a+2b=(a+b)+b
	b	a	0			
+	0	0	b			
	b	**a**	**b**	**b**	**a**	

	0	0	0	0	3a+2b	| 3a+2b=(a+b)+(a+b)+a
	0	0	b	a	0	
	0	0	b	a	0	
+	0	0	0	0	a	
	0	0	2b	2a	**a**	| 2a=2a
+	0	0	a	b		
	0	0	a+2b	**b**		| a+2b=2b+a
	b	0	a			
+	0	0	a			
	b	0	2a			| 2a=2a
+	0	a	b			
	b	**a**	**b**	**b**	**a**	

Both calculations give the same expansion, not contradicting the uniqueness property. How can we proceed to recover M? Our strategy (made more systematic in the following sections) is to obtain a nontrivial expression of 0 as a linear combination of powers of x with sums of digits as coefficients. A calculation like the one above shows

$$2a - b \sim (a, 0)$$

which means that

$$b = -x \cdot a + 2a.$$

Now using the expansion of $3a + 2b$, we get

$$3a + 2b = x^4 \cdot b + x^3 \cdot a + x^2 \cdot b + x^1 \cdot b + x^0 \cdot a$$

$$3a+2(-x\cdot a+2a) = x^4(-x\cdot a+2a)+x^3\cdot a+x^2(-x\cdot a+2a)+x^1(-x\cdot a+2a)+x^0\cdot a$$

$$0 = x^5 \cdot a - 2x^4 \cdot a - x^2 \cdot a - 4x \cdot a + 6a = (x^5 - 2x^4 - x^2 - 4x + 6) \cdot a =$$

$$= (x-1)(x^2+2)(x^2-x-3) \cdot a,$$

giving us a hint that the base polynomial might be one of the factors of the right hand side, $x^2 - x - 3$.

The above process may seem ad hoc at first sight. In the following section, we give heuristics that enable us the algorithmic reconstruction of the base polynomial and the digits in most cases we investigated.

4. Decision process and validation

Let n be a positive integer, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, such that $|a_0| = 3$. Let M be the companion matrix of $f(x)$ and $D = \{0, a, b\}$ be the digit set, such that D is a complete residue system modulo M. Furthermore suppose that the expansions of $2a$, $2b$ and $a+b$ are known in the following form:

$$\begin{aligned}2a &= M^p \cdot x_p + M^{p-1} \cdot x_{p-1} + \cdots + M \cdot x_1 + b \sim (x_p, x_{p-1}, \ldots, x_1, b),\\ 2b &= M^n \cdot y_n + M^{n-1} \cdot y_{n-1} + \cdots + M \cdot y_1 + a \sim (y_n, y_{n-1}, \ldots, y_1, a),\\ a+b &= M^m \cdot z_m + M^{m-1} \cdot z_{m-1} + \cdots + M \cdot z_1 + 0 \sim (z_m, z_{m-1}, \ldots, z_1, 0),\end{aligned}$$

where $\forall i, j, k : x_i, y_j, z_k \in D$. We call the above three equations the addition rules for the system. We categorize the systems according to the length of the maximal rule. The set $S(k)$ contains the systems with rules of the form and maximal expansion length k, i.e. $k = \max\{p+1, n+1, m+1\}$. It is easy to calculate that $|S(1)| = 1$ and $|S(k)| = 3^{3k-3} - 3^{3k-6}$, if $k \geq 2$, so the number of systems to be analyzed increases exponentially.

The basic idea was to randomize the algorithm presented in Example 3.2. Let $na + mb$ be a linear combination of digits a and b. We wish to explore its expansion by writing it as a sum or difference of several copies of $2a$, $2b$ and $(a+b)$. In each step of the addition we choose such decompositions randomly and we probabilistically assess uniqueness of the expansion of $na + mb$ by running the algorithm with different decompositions multiple times. We repeat this process for several different choices of n and m in $na + mb$. In order to make the approach formal, we have to address the following questions.

- How many elements are chosen for uniqueness testing and according to what distribution from which set?
- Since the algorithm randomly chooses a linear combination of the given element in each step, it is possible that two runs use the same steps, or just happens to give the same expansion output. How many times do we have to run the algorithm to get reliable results?
- Since there may be cases where no finite expansion of an element exists, it is also necessary to determine how many digits are calculated from the expansion before stopping the expansion process.

To answer these questions, we introduce the following parameters:

$$NoR := \text{number of runs of the algorithm}$$

$$NoD := \text{number of calculated digits of the expansion}$$

$$H(k) := \{(n,m) \in \mathbb{Z}^2 \mid -k \leq n, m \leq k\} \setminus \{(0,0), (1,0), (0,1), (1,1), (2,0), (0,2)\}$$

In order to tune these parameters, we considered systems in the sets $S(2)$, $S(3)$, $S(4)$, $S(5)$ which do not satisfy the uniqueness property. We randomly selected elements $na + mb$ and examined the empirical probability that an addition rule passes the uniqueness test despite not having the uniqueness property. We present our results in the following table.

NoR	2	2	2	2	3	3	3	3
NoD	2	3	4	5	2	3	4	5
$H(10)$	0,54	0,31	0,22	0,19	0,38	0,14	0,09	0,08
$H(20)$	0,52	0,23	0,14	0,1	0,35	0,08	0,03	0,03
$H(30)$	0,5	0,22	0,11	0,07	0,35	0,06	0,02	0,02
$H(50)$	0,51	0,19	0,08	0,05	0,34	0,05	0,01	0,01
NoR	4	4	4	4	5	5	5	5
NoD	2	3	4	5	2	3	4	5
$H(10)$	0,33	0,08	0,06	0,05	0,29	0,05	0,04	0,04
$H(20)$	0,31	0,04	0,02	0,02	0,27	0,02	0,01	0,01
$H(30)$	0,29	0,02	0,01	0,01	0,27	0,01	0,01	0,01
$H(50)$	0,29	0,02	0	0	0,26	0,01	0	0

Table 1. Probability for passing as a false positive for uniqueness in set $S(2)$

NoR	2	2	2	2	3	3	3	3
NoD	2	3	4	5	2	3	4	5
$H(10)$	0,7	0,45	0,23	0,13	0,58	0,34	0,08	0,06
$H(20)$	0,68	0,42	0,16	0,09	0,58	0,31	0,05	0,02
$H(30)$	0,67	0,4	0,15	0,07	0,57	0,3	0,04	0,01
$H(50)$	0,67	0,41	0,14	0,05	0,56	0,3	0,04	0,01
NoR	4	4	4	4	5	5	5	5
NoD	2	3	4	5	2	3	4	5
$H(10)$	0,55	0,3	0,05	0,04	0,53	0,44	0,04	0,03
$H(20)$	0,53	0,28	0,02	0,01	0,51	0,27	0,02	0,01
$H(30)$	0,52	0,27	0,02	0,01	0,51	0,26	0,01	0
$H(50)$	0,52	0,26	0,01	0	0,51	0,26	0,01	0

Table 2. Probability for passing as a false positive for uniqueness in set $S(3)$

NoR	2	2	2	2	3	3	3	3
NoD	2	3	4	5	2	3	4	5
$H(10)$	0,55	0,29	0,17	0,11	0,38	0,12	0,07	0,05
$H(20)$	0,52	0,22	0,11	0,05	0,35	0,07	0,02	0,02
$H(30)$	0,51	0,2	0,09	0,04	0,35	0,06	0,02	0,01
$H(50)$	0,51	0,18	0,07	0,03	0,34	0,05	0,01	0
NoR	4	4	4	4	5	5	5	5
NoD	2	3	4	5	2	3	4	5
$H(10)$	0,33	0,07	0,05	0,04	0,3	0,06	0,04	0,03
$H(20)$	0,3	0,03	0,01	0,01	0,28	0,03	0,01	0,01
$H(30)$	0,28	0,03	0,01	0	0,27	0,01	0	0
$H(50)$	0,29	0,02	0	0	0,26	0,01	0	0

Table 3. Probability for passing as a false positive for uniqueness in set $S(4)$

NoR	2	2	2	2	3	3	3	3
NoD	2	3	4	5	2	3	4	5
$H(10)$	0,69	0,46	0,23	0,14	0,59	0,34	0,1	0,05
$H(20)$	0,67	0,43	0,16	0,08	0,57	0,31	0,07	0,02
$H(30)$	0,67	0,41	0,16	0,06	0,56	0,3	0,04	0,01
$H(50)$	0,67	0,41	0,14	0,06	0,56	0,29	0,04	0,01
NoR	4	4	4	4	5	5	5	5
NoD	2	3	4	5	2	3	4	5
$H(10)$	0,55	0,31	0,06	0,04	0,53	0,28	0,04	0,03
$H(20)$	0,53	0,28	0,03	0,01	0,52	0,26	0,02	0,01
$H(30)$	0,52	0,27	0,02	0,01	0,51	0,26	0,01	0,01
$H(50)$	0,52	0,27	0,02	0	0,51	0,26	0,01	0

Table 4. Probability for passing as a false positive for uniqueness in set $S(5)$

From the results it we concluded that the probabilities depend mostly on the number of runs and the number of digits calculated. Therefore the parameter choices of $NoR = 5$, $NoD = 5$ and $H(k) = 10$ were used in the randomized algorithm, so for a given $k = \{2, 3, 4, 5\}$ and for each system in the set $S(k)$, we choose a random element (n, m) from the set $H(10)$, and we computed the expansion of $na + mb$ exactly five times to the fifth digit. If in all five cases all five digits were the same, a validation process was applied to the system. During validation, we checked the uniqueness of the expansion of all elements of the set $H(10)$ up to the fifth digit. If the validation was successful, the system was declared a candidate to be a matrix numeration system, and we proceeded to reconstruct the base polynomial $f(x)$.

The following pseudo-codes shows the searching and validation process:

```
Procedure GRC (generate random coefficients)
Input: v = an array containing two integers
decomp = NULL
    if v = (1,1) then
        decomp = (0,0,1,0,0)
    else
        decomp[3] = random integer between 0 and min(v[1],v[2]))
        decomp[2] = ceiling of (v[2]-decomp[3])/2
        decomp[1] = ceiling of (v[1]-decomp[3])/2
        decomp[4] = v[2]-decomp[3] mod 2
        decomp[5] = v[1]-decomp[3] mod 2
    end if
    return decomp
end Procedure
```

```
Procedure SearchingProcess
Input: array exp(aa) = expansions of 2a
       array exp(bb) = expansions of 2b
       array exp(ab) = expansions of a+b
result = NULL
    for NoR = 1 to 5 do
        (n,m) = random element of set H(10)
        list = ((n,m))
        digits = NULL
        for i=1 to length of the longest element of expansion + 3 do
            list = append(list,0)
        end for
        for NoD = 1 to 5 do
            while list[1] not equal to "a" or "b" or "0" do
                D = Procedure GCD(list[1]),
                list[1] = D[1]*exp(aa)[1] + D[2]*exp(bb)[1]+
                + D[3]*exp(ab)[1] + D[4]*"a" + D[5]*"b"
                for k=2 to length of the longest expansion do
                    list[k] = list[k] + D[1]*exp(aa)[k]
                    + D[2]*exp(bb)[k] + D[3]*exp(ab)[k]
                end for
            end while
            digits[NoD] = list[1]
            delete list[1]
        end for
        result[NoR] = digits
    end for
    if all elements of result are equal then
       return TRUE
    else
      return FALSE
    end if
end procedure
```

```
Procedure ValidationProcess
    Input: array exp(aa) = expansions of 2a
           array exp(bb) = expansions of 2b
           array exp(ab) = expansions of a+b
    for n = 1 to 10 do
        for m = 1 to 10 do
            if !(SearchingProcess(n,m)) then
                return FALSE
            end if
        end for
    end for
    return TRUE
end Procedure
```

During the research, the sets S(2), S(3), S(4) and S(5) could be fully investigated. The following table shows the number of systems determined by the searching and validation process.

set	systems
$S(2)$	4
$S(3)$	18
$S(4)$	82
$S(5)$	415

Table 5. Number of systems passed the searching and validation process

5. Reconstruction

During the reconstruction, 3 different cases have been identified.

Case 1: All systems passed the validation process where the expansion of $a+b$ is equal to 0, and the expansion of $2a$ and $2b$ are symmetric. Formally this means that

$$\begin{aligned} 2a &\sim (x_n, x_{n-1}, \dots, x_1, b), \\ 2b &\sim (y_n, y_{n-1}, \dots, y_1, a), \\ a+b &\sim (0, 0, \dots, 0) \end{aligned}$$

where for all $i \in \{1, 2, \dots, n\}$:

$$\begin{aligned} x_i = a &\iff y_i = b, \\ x_i = b &\iff y_i = a, \\ x_i = 0 &\iff y_i = 0. \end{aligned}$$

In this case we know that $b = -a$, and if $(n, m) \in H(10)$, then the expansion of $na + mb$ is unique, and therefore we can determine the representation of

0 in polynomial form for all elements of $H(10)$. If all the $f_i(x)$ polynomials represented 0 are calculated then the $\gcd(f_1(x), f_2(x) \ldots, f_{441}(x))$ can be the base of this system.

Case 2: Some systems passed the validation process where the expansion of $2a$ is equal to b. Formally this means that

$$\begin{aligned} 2a &\sim (0, 0, \ldots, b), \\ 2b &\sim (y_n, y_{n-1}, \ldots, y_1, a), \\ a + b &\sim (z_m, z_{m-1}, \ldots, z_1, 0) \end{aligned}$$

where for all $i \in \{1, 2, \ldots, n\}$ and for all $j \in \{1, 2, \ldots, m\} : y_i, z_j \in \{0, a, b\}$. In this case we know that $b = 2a$ and similarly as in the previous case, we can describe the polynomials representing 0 for all $na + mb$, where $(n, m) \in H(10)$, and their greatest common divisor can be the base of these systems.

We used the following algorithm for the reconstruction of the base of these systems:

```
Procedure SystemBase1
    Input: array exp(aa) = expansions of 2a
           array exp(bb) = expansions of 2b
           array exp(ab) = expansions of a+b
    v = list()
    In Case 1: b = -a or In Case 2: b = 2a
    for n = -10 to 10 do
        for m = -10 to 10 do
            v[i]:= representation of 0 from the expansion of na+mb
        end for
    end for
    return GCD(v[1],v[2],...,v[441])
end Procedure
```

Example 5.1. Let

$$\begin{aligned} 2a &\sim (b, 0, a, b), \\ 2b &\sim (a, 0, b, a), \\ a + b &\sim (0, 0, 0, 0) \end{aligned}$$

be the expansions. Since $b = -a$, therefore

$$2a = bx^3 + ax + b = -ax^3 + ax - a \iff a(x^3 - x + 3) = 0,$$

$$2b = ax^3 + bx + a = -bx^3 + bx - b \iff b(x^3 - x + 3) = 0$$

and for example

$$10a + b \sim (a, a, b, a, 0, a, 0, 0),$$

$$9a = 10a + b = ax^7 + ax^6 + bx^5 + ax^4 + ax^2 = ax^7 + ax^6 - ax^5 + ax^4 + ax^2 \iff$$
$$\iff 0 = a(x^7 + x^6 - x^5 + x^4 + x^2 - 9) \iff 0 = a(x^3 - x + 3)(x^4 + x^3 - x - 3)$$

so the polynomial $x^3 - x + 3$ can be the base of this system.

Example 5.2. Let

$$\begin{aligned} 2a &\sim (0,0,0,b), \\ 2b &\sim (a,b,0,a), \\ a+b &\sim (a,b,0,0) \end{aligned}$$

be the expansions. Since $2a = b$, therefore

$$2b = ax^3 + bx^2 + a \iff 4a = ax^3 + 2ax^2 + a \iff$$
$$\iff 0 = a(x^3 + 2x^2 - 3) = a(x-1)(x^3 + 3x + 3)$$

or

$$7a + 4b \sim (a,b,b,0,b,a,0,0)$$
$$15a = 7a + 4b = ax^7 + bx^6 + bx^5 + bx^3 + ax^2 = ax^7 + 2ax^6 + 2ax^5 + 2ax^3 + ax^2 \iff$$
$$\iff 0 = a(x^7 + 2x^6 + 2x^5 + 2x^3 + x^2 - 15) \iff$$
$$\iff 0 = a(x^3 + 3x + 3)(x^5 - x^4 + 2x^3 - 3x^2 + 5x - 5)$$

so the polynomial $x^3 + 3x + 3$ can be the base of this system.

Case 3: In the remaining cases where the rules passed the validation process, we do not have a trivial linear relationship between a and b from the expansions of $2a$, $2b$ and $a + b$. In this case, we used the following idea. Try to find an $(n, m) \in H(10)$ pair such that the expansion of $na + mb$ only conatins one of the the digits a or b. If there exists a pair (n_0, m_0) with this property, and without the loss of generality we assume that the missing digit in the expansion of $n_0a + m_0b$ is a, then we get:

$$n_0a + m_0b = \sum_{i=0}^{k} \epsilon_i x^i, \text{ where } \forall i \in \{1, 2, \dots, k\} : \ \epsilon_i \in \{0, b\}$$

so

$$n_0a = -m_0b + \sum_{i=0}^{k} \epsilon_i x^i = b \underbrace{\left(-m_0 + \sum_{i=0}^{k} \mu_i x^i\right)}_{:=f(x)} = bf(x),$$

$$\text{where } \mu_i = \begin{cases} 0, & \text{if } \epsilon_i = 0 \\ 1, & \text{if } \epsilon_i = b \end{cases}$$

Now we can describe the polynomials representing 0 for all $(n, m) \in H(10)$ as follows:

$$na + mb = \sum_{i=0}^{n} d_i x^i, \text{ where } \forall i \in \{1, 2, \ldots, n\} : d_i \in \{0, a, b\}$$

but assuming the uniqueness property we have that

$$n_0 na + n_0 mb = \sum_{i=0}^{k} n_0 d_i x^i = b \underbrace{\sum_{i=0}^{k} t_i f(x) x^i + (1 - t_i) n_0 x^i}_{:=F(x)} = bF(x),$$

$$\text{where } t_i = \begin{cases} 0, & \text{if } d_i = a \\ 1, & \text{if } d_i = b \end{cases}$$

On the other hand,

$$n_0 na + n_0 mb = nbf(x) + n_0 mb \implies$$

$$nbf(x) + n_0 mb = bF(x) \iff 0 = bF(x) - nbf(x) - n_0 mb =$$
$$= b \underbrace{\left(F(x) - nf(x) - n_0 m\right)}_{P_{n,m}(x)} = bP_{n,m}(x)$$

so in this system the polynomial $P_{n,m}(x)$ represents 0. Similarly, if the missing digit is b, we get the same result and we can describe a polynomial, which represents 0. Using these calculations, we formulated the following algorithm:

```
Procedure SystemBase2
    Input: array exp(aa) = expansions of 2a
           array exp(bb) = expansions of 2b
           array exp(ab) = expansions of a+b
    v = list()
    for n = -10 to 10 do
        for m = -10 to 10 do
            if the expansion of na+mb does not contains a or b
                n_0=n and m_0=m
            end if
            stop
        end for
        stop
    end for
    for n = -10 to 10 do
        for m = -10 to 10 do
           v[i]=P_{n,m}(x)
            end if
        end for
    end for
    return GCD(v[1],v[2],...,v[441])
end Procedure
```

Example 5.3. Let

$$\begin{aligned} 2a &\sim (a,a,b,b),\\ 2b &\sim (b,a,a,a),\\ a+b &\sim (a,b,0,0) \end{aligned}$$

be the expansions. Using the above-mentioned idea, we can get that

$$-3a+3b \sim (b,b,0)$$

which means, that

$$\begin{aligned} -3a+3b = bx^2+bx \iff 3a = -bx^2-bx+3b \iff \\ \iff 6a = -2bx^2-2bx+6b. \end{aligned} \tag{5.1}$$

We know the expansion of $2a$ and $2b$, so

$$2a = ax^3+ax^2+bx+b \iff 6a = 3ax^3+3ax^2+3bx+3b \tag{5.2}$$

and

$$2b = bx^3+ax^2+ax+a \iff 6b = 3bx^3+3ax^2+3ax+3a \tag{5.3}$$

Using the equations 5.1,5.2 and 5.3, we get

$$\begin{aligned} 6a = -2bx^2-2bx+6b = (-bx^2-bx+3b)x^3+(-bx^2-bx+3b)x^2+3bx+3b \iff \\ \iff 0 = -b(x^5+2x^4-2x^3-5x^2-5x+3) \iff \\ \iff 0 = -b(x^3-x-3)(x^2+2x-1) \end{aligned}$$

and

$$\begin{aligned} 6b = 3bx^3+(-bx^2-bx+3b)x^2+(-bx^2-bx+3b)x+(-bx^2-bx+3b) \iff \\ \iff 0 = -b(x^4-x^3-x^2-2x+3) \iff 0 = -b(x^3-x-3)(x-1) \end{aligned}$$

so the polynomial x^3-x-3 can be the base of this system.

A complete list of reconstructed basis polynomials are presented in the Appendix.

6. Summary

We presented the problem of reconstructing numeration systems from the expansions of pairwise sums of digits. We investigated the problem empirically. In order to better understand the connection between the basis and the addition rules, we propose further research directions.

- As a direct extension of the present work, find systems with the uniqueness property in set $S(k)$, if $k > 5$.
- Find algorithms or heuristic approaches for the reconstruct of the digit set of the systems.
- Based on the empirical observations, try to characterize additions rules which fulfill the uniqueness property
- Extend the present work to other numeration systems, including e.g. shift radix systems (SRS).

Conjecture 6.1. *Let* $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x \pm 3$ *be a monic polynomial.*
If $\forall i \in \{1, 2, 3, \ldots, n-1\} : \;\; a_i \in \{0, 1, -1\}$, *then* f *satisfies the uniqueness property.*

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Appendix

$2a$	$2b$	$a+b$	$base$
(a,b)	(0,a)	(a,0)	$2x-3$
(b,b)	(0,a)	(b,0)	3
(a,b)	(b,a)	(0,0)	3
(b,b)	(a,a)	(0,0)	-3

Table 6. Reconstructed basis in $S(2)$

$2a$	$2b$	$a+b$	$base$
(0,a,b)	(0,a,a)	(a,b,0)	
(a,0,b)	(b,0,a)	(0,0,0)	x^2-3
(b,0,b)	(a,0,a)	(0,0,0)	x^2+x-3
(b,a,b)	(a,b,a)	(0,0,0)	x^2-x+3
(a,b,b)	(b,a,a)	(0,0,0)	x^2-x-3
(b,b,b)	(a,a,a)	(0,0,0)	x^2+x+3
(a,a,b)	(b,a,a)	(a,b,0)	x^2-3
(0,0,b)	(a,0,a)	(a,0,0)	x^2-3
(0,0,b)	(b,0,a)	(b,0,0)	$2x^2-3$
(0,0,b)	(a,a,a)	(a,a,0)	x^2+x-3
(0,0,b)	(b,a,a)	(b,a,0)	$2x+3$
(0,0,b)	(a,b,a)	(a,b,0)	-3
(0,0,b)	(b,b,a)	(b,b,0)	$2x^2+2x-3$
(0,a,b)	(a,0,a)	(a,a,0)	-3
(0,a,b)	(b,0,a)	(b,a,0)	x^2-x-3
(0,b,b)	(a,0,a)	(a,b,0)	x^2+2x+3
(0,b,b)	(b,0,a)	(b,b,0)	$2x+3$

Table 7. Reconstructed basis in $S(3)$

$2a$	$2b$	$a+b$	$base$
(a,0,a,b)	(a,b,a,a)	(a,0,b,0)	x^3-x^2-3
(a,a,a,b)	(b,a,a,a)	(a,0,b,0)	x^3-3
(a,b,a,b)	(b,b,a,a)	(a,b,b,0)	x^3-3
(a,a,b,b)	(b,a,a,a)	(a,b,0,0)	x^3-x-3
(0,b,b,b)	(a,a,b,a)	(0,b,a,0)	x^2+3x+3
(a,0,0,b)	(b,0,0,a)	(0,0,0,0)	x^3-3
(b,0,0,b)	(a,0,0,a)	(0,0,0,0)	x^3+3
(a,a,0,b)	(b,b,0,a)	(0,0,0,0)	x^3+x^2-3
(b,a,0,b)	(a,b,0,a)	(0,0,0,0)	x^3-x^2+3
(a,b,0,b)	(b,a,0,a)	(0,0,0,0)	x^3-x^2-3
(b,b,0,b)	(a,a,0,a)	(0,0,0,0)	x^3+x^2+3
(a,0,a,b)	(b,0,b,a)	(0,0,0,0)	x^3+x-3
(b,0,a,b)	(a,0,b,a)	(0,0,0,0)	x^3-x+3
(a,a,a,b)	(b,b,b,a)	(0,0,0,0)	x^3+x^2+x-3
(b,a,a,b)	(a,b,b,a)	(0,0,0,0)	x^3-x^2-x+3
(a,b,a,b)	(b,a,b,a)	(0,0,0,0)	x^3-x^2+x-3
(b,b,a,b)	(a,a,b,a)	(0,0,0,0)	x^3+x^2-x+3
(a,0,b,b)	(b,0,a,a)	(0,0,0,0)	x^3-x-3
(b,0,b,b)	(a,0,a,a)	(0,0,0,0)	x^3+x+3
(a,a,b,b)	(b,b,a,a)	(0,0,0,0)	x^3+x^2-x-3
(b,a,b,b)	(a,b,a,a)	(0,0,0,0)	x^3-x^2+x+3
(a,b,b,b)	(b,a,a,a)	(0,0,0,0)	x^3-x^2-x-3
(b,b,b,b)	(a,a,a,a)	(0,0,0,0)	x^3+x^2+x+3
(0,0,a,b)	(0,a,a,a)	(a,b,b,0)	x^2+3x-3
(0,a,a,b)	(0,a,a,a)	(a,0,b,0)	
(0,a,b,b)	(0,a,a,a)	(a,b,0,0)	-3
(a,0,a,b)	(b,0,a,a)	(0,a,b,0)	x^3-3
(a,a,a,b)	(b,b,a,a)	(0,a,b,0)	x^3+x^2-3
(a,b,a,b)	(a,a,a,a)	(0,a,b,0)	x^2+3
(0,0,0,b)	(a,0,0,a)	(a,0,0,0)	x^3-3
(0,0,0,b)	(b,0,0,a)	(b,0,0,0)	$2x^3-3$
(0,0,0,b)	(a,a,0,a)	(a,a,0,0)	x^3+x^2-3
(0,0,0,b)	(b,a,0,a)	(b,a,0,0)	$2x^2+3x+3$
(0,0,0,b)	(a,b,0,a)	(a,b,0,0)	x^2+3x+3
(0,0,0,b)	(b,b,0,a)	(b,b,0,0)	$2x^3+2x^2-3$
(0,0,0,b)	(a,0,a,a)	(a,0,a,0)	x^3+x-3
(0,0,0,b)	(b,0,a,a)	(b,0,a,0)	$2x^2+2x+3$
(0,0,0,b)	(a,a,a,a)	(a,a,a,0)	x^2+2x+3
(0,0,0,b)	(b,a,a,a)	(b,a,a,0)	$2x^3+x^2+x-3$

Table 8. Reconstructed basis in $S(4)$

$2a$	$2b$	$a+b$	$base$
(0,0,0,b)	(a,b,a,a)	(a,b,a,0)	x^3+2x^2+x-3
(0,0,0,b)	(b,b,a,a)	(b,b,a,0)	$2x^3+2x^2+x-3$
(0,0,0,b)	(a,0,b,a)	(a,0,b,0)	x^2+x+3
(0,0,0,b)	(b,0,b,a)	(b,0,b,0)	$2x^3+2x-3$
(0,0,0,b)	(a,a,b,a)	(a,a,b,0)	x^3+x^2+2x-3
(0,0,0,b)	(b,a,b,a)	(b,a,b,0)	$2x^3+x^2+2x-3$
(0,0,0,b)	(a,b,b,a)	(a,b,b,0)	x^3+2x^2+2x-3
(0,0,0,b)	(b,b,b,a)	(b,b,b,0)	$2x^3+2x^2+2x-3$
(0,0,a,b)	(a,0,0,a)	(a,0,a,0)	x^2+x+3
(0,0,a,b)	(b,0,0,a)	(b,0,a,0)	x^3-x^2-x-3
(0,0,a,b)	(a,a,0,a)	(a,a,a,0)	x^3+x^2+2x-3
(0,0,a,b)	(b,a,0,a)	(b,a,a,0)	$x^4-2x^3-x^2-2x+3$
(0,0,a,b)	(a,b,0,a)	(a,b,a,0)	$2x^2+2x-3$
(0,0,a,b)	(b,b,0,a)	(b,b,a,0)	$x^4-x^3-2x^2-2x+3$
(0,0,a,b)	(a,0,a,a)	(a,a,b,0)	x^3+3x-3
(0,0,a,b)	(b,0,a,a)	(b,a,b,0)	x^4-2x^3-3x+3
(0,0,b,b)	(a,0,0,a)	(a,0,b,0)	x^3+2x^2+2x+3
(0,0,b,b)	(b,0,0,a)	(b,0,b,0)	$2x^2+2x+3$
(0,0,b,b)	(a,a,0,a)	(a,a,b,0)	$x^4+2x^3+x^2+x-3$
(0,0,b,b)	(b,a,0,a)	(b,a,b,0)	$3x^3+x^2+x-3$
(0,0,b,b)	(a,b,0,a)	(a,b,b,0)	$x^4+x^3+2x^2+x-3$
(0,0,b,b)	(b,b,0,a)	(b,b,b,0)	$2x^3+2x^2+x-3$
(0,a,0,b)	(a,0,0,a)	(a,a,0,0)	x^3+3x+3
(0,a,0,b)	(b,0,0,a)	(b,a,0,0)	$x^4+x^3-x^2-3x-3$
(0,a,0,b)	(a,0,a,a)	(a,a,a,0)	x^3+2x^2+x-3
(0,a,0,b)	(b,0,a,a)	(b,a,a,0)	$x^5-2x^3-2x^2-x+3$
(0,a,0,b)	(a,0,b,a)	(a,a,b,0)	$2x^2+2x-3$
(0,a,0,b)	(b,0,b,a)	(b,a,b,0)	$x^5-x^3-2x^2-2x+3$
(0,b,0,b)	(a,0,0,a)	(a,b,0,0)	$x^4+x^3+2x^2+3x+3$
(0,b,0,b)	(b,0,0,a)	(b,b,0,0)	$2x^2+3x+3$
(0,b,0,b)	(a,0,a,a)	(a,b,a,0)	$x^5+2x^3+x^2+x-3$
(0,b,0,b)	(b,0,a,a)	(b,b,a,0)	$3x^3+x^2+x-3$
(0,b,0,b)	(a,0,b,a)	(a,b,b,0)	$x^5+x^3+x^2+2x-3$
(0,b,0,b)	(b,0,b,a)	(b,b,b,0)	$2x^3+x^2+2x-3$
(0,a,a,b)	(a,0,0,a)	(a,a,a,0)	x^3+2x^2+2x-3
(0,a,a,b)	(b,0,0,a)	(b,a,a,0)	$x^5+x^4-2x^3-2x^2-2x+3$
(0,b,a,b)	(a,0,0,a)	(a,b,a,0)	$x^5+x^3+x^2+2x-3$
(0,b,a,b)	(b,0,0,a)	(b,b,a,0)	$x^4-2x^3-x^2-2x+3$
(0,a,b,b)	(a,0,0,a)	(a,a,b,0)	$x^4+x^3+2x^2+x-3$
(0,a,b,b)	(b,0,0,a)	(b,a,b,0)	$x^5-2x^3-2x^2-x+3$
(0,b,b,b)	(a,0,0,a)	(a,b,b,0)	$x^5+x^4+x^3+x^2+x-3$
(0,b,b,b)	(b,0,0,a)	(b,b,b,0)	$2x^3+x^2+x-3$

Table 9. Reconstructed basis in $S(4)$