THE REGULARITY INDEX OF s FAT POINTS IN PROJECTIVE SPACE $\mathbb{P}^2$, $s \leq 6$

Phan Quang Nhu Anh and Tran Nam Sinh

(Da Nang, Vietnam)

Communicated by Bui Minh Phong

(Received July 1, 2023; accepted September 5, 2023)

Abstract. In this paper, we compute the regularity index of a set of five almost equimultiple (Theorem 3.2) and a set of six double points in general position in projective space $\mathbb{P}^2$ (Theorem 4.2).

1. Introduction

Let $P_1, \ldots, P_s$ be s distinct points in the projective space $P^n := P^n_K$, with K an arbitrary algebraically closed field. Let $R := K[x_0, x_1, \ldots, x_n]$ be a polynomial ring in $n+1$ variables $x_0, x_1, \ldots, x_n$ with coefficients in K. Let $\wp_1, \wp_2, \ldots, \wp_s$ be the homogeneous prime ideals of the polynomial ring R corresponding to the points $P_1, P_2, \ldots, P_s$.

Let $m_1, m_2, \ldots, m_s$ be positive integers. Then the set of all homogeneous polynomials that vanish at P_i to order m_i for $i = 1, \ldots, s$ is the homogeneous ideal $I = \wp_1^{m_1} \cap \wp_2^{m_2} \cap \ldots \cap \wp_s^{m_s}$. We call the zero-scheme defined by I to be a set of fat points in $\mathbb{P}^n$, and we denote it by

$$Z = m_1P_1 + m_2P_2 + \cdots + m_sP_s.$$

The homogeneous coordinate ring of Z is $A := R/I$. The ring $A = \oplus_{t\geq 0} A_t$ is a graded ring whose multiplicity is

$$e(A) := \sum_{i=1}^{s} \binom{m_i + n - 1}{n}.$$

Key words and phrases: Regularity index, fat points, zero scheme.
2010 Mathematics Subject Classification: Primary 14C20, Secondary 13D40.

The Hilbert function of Z, which defined by $H_Z(t) = \dim_K A_t$, strictly increases until it reaches the multiplicity $e(A)$, at which it stabilizes. The regularity index of Z is defined to be the least integer t such that $H_Z(t) = e(A)$, and we will denote it by $\operatorname{reg}(Z)$. It is well known that $\operatorname{reg}(Z) = \operatorname{reg}(A)$, the Castelnuovo–Mumford regularity of A.

The problem to exactly determine $\operatorname{reg}(Z)$ is more fairly difficult, so one finds an upper bound for $\operatorname{reg}(Z)$. Since 1961, many results on upper bound for $\operatorname{reg}(Z)$ have been published.

For generic fat points $Z = m_1P_1+\cdots+m_sP_s$ with $m_1 \geq \cdots \geq m_s$, Segre [7] showed that

$$\operatorname{reg}(Z) \leq \max\left\{m_1+m_2-1, \left[\frac{m_1+\cdots+m_s}{2}\right]\right\}.$$

Trung [8] conjectured an upper bound for the regularity index of arbitrary fat points in $\mathbb{P}^n$:

$$\operatorname{reg}(Z) \leq \max\{T_j \mid j = 1, ..., n\},$$

where

$$T_j = \max\left\{\left[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j}\right] \;\Big|\; P_{i_1}, \ldots, P_{i_q} \text{ lie on a linear } j\text{-space}\right\}.$$

This upper bound generalizes Segre's upper bound. So, we will call it the Segre bound.

The same conjecture was also given independently by Fatabbi and Lorenzini [3].

In 2020, Nagel and Trok [6] proved the hypothesis of Trung about the upper bound of reg(Z) in the general case.

Currently, the calculation results $\operatorname{reg}(Z)$ are still very modest, only some results have been published in prestigious journals as follows:

For abitrary fat points $Z = m_1P_1+\cdots+m_sP_s$ in $\mathbb{P}^n$, Davis and Geramita [5] proved that

$$\operatorname{reg}(Z) = m_1+\cdots+m_s-1$$

if and only if points $P_1, \ldots, P_s$ lie on a line in $\mathbb{P}^n$.

For fat points $Z = m_1P_1+\cdots+m_sP_s$ in $\mathbb{P}^n$ with $m_1 \geq \cdots \geq m_s$, Catalisano et al. [4] showed formulas to compute $\operatorname{reg}(Z)$ in the following two cases:

- If $s \geq 2,\ 2 \leq m_1 \geq m_2 \geq \cdots \geq m_s$ and $P_1, P_2, \ldots, P_s$ are on a rational normal curve in $\mathbb{P}^n$, then

$$\operatorname{reg}(Z) = \max\left\{m_1+m_2-1, \frac{\sum_{i=1}^{s} m_i + n - 2}{n}\right\}.$$

• If $n \geq 3,\ 2 \leq s \leq n+2,\ 2 \leq m_1 \geq m_2 \geq \cdots \geq m_s$ and $P_1, P_2, \ldots, P_s$ are in general position in $\mathbb{P}^n$, then

$$\operatorname{reg}(Z) = m_1 + m_2 - 1.$$

In 2012, Thien [9] showed a formula to compute $\operatorname{reg}(Z) = \max\{T_j \mid j = 1, \ldots, n\}$ for $Z = m_1P_1 + \cdots + m_{s+2}P_{s+2}$ with $P_1, P_2, \ldots, P_{s+2}$ not in a linear $(s-1)$-space in $\mathbb{P}^n$, $s \leq n$.

In 2017, Thien and Sinh [10] showed a formula to compute $\operatorname{reg}(Z) = \max\{T_j \mid j = 1, \ldots, n\}$ for $Z = mP_1 + \cdots + mP_{s+3}$ with $m \neq 2$ and $P_1, \ldots, P_{s+3}$ not in a linear $(r-1)$-space in $\mathbb{P}^n$, $s \leq r+3$.

It's worth noting that the assumption $\operatorname{reg}(Z) = T = \max\{T_j \mid j = 1, \ldots, n\}$ for the set of arbitrary fat points in $\mathbb{P}^n$ is no longer true because U. Nagel and B. Trok (see [6], Example 5.7) have shown that: If $Z = mP_1 + \cdots + mP_s$ is a set of fat points in $\mathbb{P}^n$ consisting of five points in arbitrary positions and $\binom{d+n}{d}$ points in the general position, $d \geq 5$ then

$$\operatorname{reg}(Z) < T,$$

with d (or n) large enough.

A set of s fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$ is said to be almost equimultiple if $m_i \in \{m-1, m\}$ for all $i = 1, \ldots, s$ and $m \geq 2$.

A set of s fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$ is said to be double point if $m_1 = \cdots = m_s = 2$ for all $i = 1, \ldots, s$.

In this paper, we compute the regularity index of a set of five almost equimultiple and a set of six double points in general position in projective space $\mathbb{P}^2$. These results are stated in Theorem 3.2 and Theorem 4.2.

2. Preliminaries

In proving the main results, we use the lemmas already proven in [2], [5], [9], [11]. The following lemma is used to calculate the value of $\operatorname{reg}(Z)$ when the set of points lie on a line.

Lemma 2.1 ([5], Corollary 2.3)**.** *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of arbitrary fat points in $\mathbb{P}^n$. Then,*

$$\operatorname{reg}(Z) = m_1 + \cdots + m_s - 1$$

if and only if the points $P_1, \ldots, P_s$ lie on a line.

Let $\{i_1, \ldots, i_r\}$ be the subset of the set of indices $\{1, \ldots, s\}$. We call $Y = m_{i_1}P_{i_1} + \cdots + m_{i_r}P_{i_r}$ the subset of fat points of $Z = m_1P_1 + \cdots + m_sP_s$. The following lemma helps us to compare the regularity index of a subset of a given set of fat points.

Lemma 2.2 ([9], Lemma 3.3)**.** *Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in $\mathbb{P}^n$ and $m_1, \dots, m_s$ be positive integers. Put $I = \wp_1^{m_1} \cap \wp_2^{m_2} \cap ... \cap \wp_s^{m_s}$. If $Y = \{P_{i_1}, \dots, P_{i_r}\}$ is a subset of X and $J = \wp_{i_1}^{m_{i_1}} \cap \wp_{i_2}^{m_{i_2}} \cap \dots \cap \wp_{i_r}^{m_{i_r}}$, then*

$$\operatorname{reg}(R/J) \leq \operatorname{reg}(R/I).$$

This implies that if $Z = m_1P_1 + \cdots + m_sP_s$ is the set of fat points defining by I, and $Y = m_{i_1}P_{i_1} + \cdots + m_{i_r}P_{i_r}$ is the set of fat points defining by J, then

$$\operatorname{reg}(Y) \leq \operatorname{reg}(Z).$$

The next two lemmas allow us to compute the regularity index for a given set of points.

Lemma 2.3 ([9], Theorem 3.4)**.** *Let $P_1, \dots, P_{s+2}$ be distinct points not in a linear $(s-1)$-space in $\mathbb{P}^n$, $s \leq n$ and $m_1, \dots, m_s$ are positive integers. Put $I = \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_{s+2}^{m_{s+2}}$, $A = R/I$. Then,*

$$\operatorname{reg}(A) = \max\{T_j \mid j = 1, \dots, n\},$$

where

$$T_j = \max\left\{ \left[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right] \;\middle|\; P_{i_1}, \dots, P_{i_q} \text{ are on a linear} j\text{-space} \right\}.$$

Lemma 2.4 ([11], Theorem 3.1)**.** *Let $X = \{P_1, \dots, P_{s+3}\}$ be a set of distinct points in a general position on a linear s-space, not on a linear $(s-1)$-space in $\mathbb{P}^n$, $s \leq n$, and m_i are positive integers. Let $Z = m_1P_1 + \cdots + m_{s+3}P_{s+3}$ be the set of fat points. Then,*

$$\operatorname{reg}(A) = \max\{T_j | j = 1, \dots, n\},$$

where

$$T_j = \max\left\{ \left[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right] \;\middle|\; P_{i_1}, \dots, P_{i_q} \text{ are on a linear } j\text{-space} \right\}.$$

The following lemma gives the upper bound for the set $n+3$ non-degenerate fat points in $\mathbb{P}^n$.

Lemma 2.5 ([2], Theorem 2.1)**.** *Let $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ be a set of $n+3$ non-degenerate fat points in $\mathbb{P}^n$. Then,*

$$\operatorname{reg}(Z) \leq \max\{T_j | j = 1, \dots, n\},$$

where

$$T_j = \max\left\{ \left[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right] \;\middle|\; P_{i_1}, \dots, P_{i_q} \text{ are on a linear } j\text{-space} \right\}.$$

3. The regularity index of a set of five almost equimultiple in $\mathbb{P}^2$

We will begin this section with the following proposition:

Proposition 3.1. *Let $X = \{P_1, ..., P_5\}$ be a set of five non-degenerate distinct points in $\mathbb{P}^2$. Let $m_1, \ldots, m_5$ be positive integers and a set of five fat points*

$$Z = m_1P_1 + \cdots + m_5P_5.$$

Put

$$T_j = \max\left\{ \left[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right] \;\middle|\; P_{i_1}, \ldots, P_{i_q} \text{ is on a linear } j\text{-space} \right\},$$

and

$$T = \max\{T_j \mid j = 1, 2\}.$$

If $T = T_1$ then

$$\operatorname{reg}(Z) = T.$$

Proof. Since $T = T_1$, there is a line, denoted d, that passes through the points $P_{i_1}, P_{i_2}, \ldots, P_{i_r} \in X$ such that $T_1 = m_{i_1} + m_{i_2} + \cdots + m_{i_r} - 1$. Considering the set of fat points $Y = m_{i_1}P_{i_1} + m_{i_2}P_{i_2} + \cdots + m_{i_r}P_{i_r}$, according to Lemma 2.2, we have

$$\operatorname{reg}(Z) \geq \operatorname{reg}(Y) = T_1 = T.$$

Furthermore, by Lemma 2.5, we have $\operatorname{reg}(Z) \leq T$. Hence $\operatorname{reg}(Z) = T$. ■

Using Proposition 3.1 we calculate the regularity index of the set of five almost equimultiple fat points in $\mathbb{P}^2$, the result is presented in the following theorem:

Theorem 3.2. *Let $X = \{P_1, \ldots, P_5\}$ be a set of five non-degenerate distinct points in $\mathbb{P}^2$, let $m_1, \ldots, m_5$ be positive integers satisfying $m_i \in \{m-1, m\}$, $i = 1, \ldots, 5$, $m \geq 2$ and a set of almost equimultiple fat points*

$$Z = m_1P_1 + \cdots + m_5P_5.$$

Put

$$T_j = \max\left\{ \left[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right] \;\middle|\; P_{i_1}, \ldots, P_{i_q} \text{ are on a linear } j\text{-space} \right\},$$

and

$$T = \max\{T_j \mid j = 1, 2\}.$$

Then

$$\operatorname{reg}(Z) = T.$$

Except for the following cases:

1) $m_1 = m_2 = 1, m_3 = m_4 = m_5 = 2$.

2) $m_1 = m_2 = m_3 = m - 1, m_4 = m_5 = m, m \in \{2, 3\}$.

Proof. According to the assumption of multiples of the fat points Z, we have the following 3 cases:

i) **Case 1:** $Z = (m-1)P_1 + mP_2 + mP_3 + mP_4 + mP_5$.

Now we consider the different positions of the set of points X:

i$_1$) If X is in a general position in $\mathbb{P}^2$ then according to Lemma 2.4,

$$\operatorname{reg}(Z) = T = T_2 = \left[\frac{5m-1}{2}\right].$$

i$_2$) If X has 4 points on a line, then we have $4m - 2 \leq T_1 \leq 4m - 1$, $T_2 = \left[\frac{5m-1}{2}\right]$. So for $m \geq 2$, $T_1 > T_2$, thus $T = \max\{T_1, T_2\} = T_1$. According to Proposition 3.1, then $\operatorname{reg}(Z) = T$.

i$_3$) If X has three points on a line, denote by d.

- If d passes through P_1 then $T_1 = 3m - 2$, $T_2 = \left[\frac{5m-1}{2}\right]$.
  We have $3m - 2 - \frac{5m-1}{2} = \dfrac{m-3}{2}$. So for $m \geq 3$ then $3m - 2 \geq$ $\geq \left[\frac{5m-1}{2}\right]$.
  If $m = 2$ then $T_1 = 4$ and $T_2 = \left[\frac{5.2-1}{2}\right] = 4$. So for $m \geq 2$ we have $T_1 \geq T_2$.
- If d does not go through P_1 then $T_1 = 3m - 1$, $T_2 = \left[\frac{5m-1}{2}\right]$.
  We have $3m-1-\frac{5m-1}{2} = \frac{m-1}{2}$. So for $m \geq 2$ then $3m-1 \geq \left[\frac{5m-1}{2}\right]$.
  That is, for $m \geq 2$, then $T_1 \geq T_2$.

To summarize in all subcases of Case i_3, with $m \geq 2$ we have $T =$ $= \max\{T_1, T_2\} = T_1$. According to Proposition 3.1 we have $\operatorname{reg}(Z) = T$.

ii) **Case 2:** $Z = (m-1)P_1 + (m-1)P_2 + mP_3 + mP_4 + mP_5$.

Now we consider the different positions of the set of points X:

ii$_1$) If X is in the general position in $\mathbb{P}^2$ then according to Lemma 2.4,

$$\operatorname{reg}(Z) = T = T_2 = \left[\frac{5m-2}{2}\right].$$

ii$_2$) If X has 4 points on a line, then

$$4m - 3 \leq T_1 \leq 4m - 2,\ T_2 = \left[\frac{5m-2}{2}\right].$$

So for $m \geq 2$, $T_1 > T_2 \Rightarrow T = \max\{T_1, T_2\} = T_1$. According to Proposition 3.1, we have $\operatorname{reg}(Z) = T$

ii$_3$) X has three points on a line, denote by d. Then:

- If d passes through P_1 and P_2 then $T_1 = 3m-3,\ T_2 = \left[\frac{5m-2}{2}\right]$. We have $3m-3-\frac{5m-2}{2} = \frac{m-4}{2}$. So for $m \geq 4$ then $3m-3 \geq \left[\frac{5m-2}{2}\right]$. For $m=3$ then $T_1 = 6$ and $T_2 = \left[\frac{5.3-2}{2}\right] = 6$. So for $m \geq 3$ then $T_1 \geq T_2$.
- If d only passes through either point P_1 or P_2 then
$$T_1 = 3m-2,\ T_2 = \left[\frac{5m-2}{2}\right].$$
We have $3m-2-\frac{5m-2}{2} = \frac{m-2}{2}$. So for $m \geq 3$ then $3m-1 \geq \left[\frac{5m-1}{2}\right]$. That is, with $m \geq 3$, then $T_1 \geq T_2$.
- If d does not pass through both points P_1 and P_2 then
$$T_1 = 3m-1,\ T_2 = \left[\frac{5m-2}{2}\right].$$
So $T_1 \geq T_2$.

To summarize in all subcases of Case ii$_3$), with $m \geq 3$ we have $T = \max\{T_1, T_2\} = T_1$. According to Proposition 3.1, we have $\operatorname{reg}(Z) = T$. Therefore
$$\operatorname{reg}(Z) = T.$$

iii) **Case 3:** $Z = (m-1)P_1 + (m-1)P_2 + (m-1)P_3 + mP_4 + mP_5$.

Now we also consider the different positions of the set of points X:

iii$_1$) If X is in the general position in $\mathbb{P}^2$ then according to Lemma 2.4,
$$\operatorname{reg}(Z) = T = T_2 = \left[\frac{5m-3}{2}\right].$$

iii$_2$) If X has 4 points on a line, we have
$$4m-4 \leq T_1 \leq 4m-3,\ T_2 = \left[\frac{5m-3}{2}\right].$$
We have $4m-4-\frac{5m-3}{2} = \frac{3m-5}{2} > 0, \forall m \geq 2$. So if $m \geq 2$, then $T_1 > T_2$ $T = \max\{T_1, T_2\} = T_1$. According to Proposition 3.1 we obtain that $\operatorname{reg}(Z) = T$.

iii$_3$) If X has three points on a line, denote by d, we have:

- If d passes through P_1, P_2 and P_3 then $T_1 = 3m-4$, $T_2 = \left[\frac{5m-3}{2}\right]$. We have $3m-4-\frac{5m-3}{2} = \frac{m-5}{2}$.
If $m \geq 5$ then $3m-4 \geq \left[\frac{5m-3}{2}\right]$.
If $m = 4$ then $T_1 = 8$ and $T_2 = \left[\frac{5.4-3}{2}\right] = 8$. So for $m \geq 4$ then $T_1 \geq T_2$.

- If d only passes through either point P_4 or P_5 then
$$T_1 = 3m - 3,\ T_2 = \left[\frac{5m-3}{2}\right].$$
We have $3m - 3 - \frac{5m-3}{2} = \frac{m-3}{2}$. So for $m \geq 3$ we obtain that $3m - 3 \geq \left[\frac{5m-3}{2}\right]$. That is, if $m \geq 3$, then $T_1 \geq T_2$.
- If d passes through both points P_4 and P_4 then
$$T_1 = 3m - 2,\ T_2 = \left[\frac{5m-3}{2}\right].$$
We have $3m - 2 - \frac{5m-3}{2} = \frac{m-1}{2}$. So for $m \geq 2$ we obtain that $T_1 \geq T_2$.

To summarize in all subcases of Case iii$_3$, with $m \geq 4$ we obtain that $T = \max\{T_1, T_2\} = T_1$.

According to Proposition 3.1 we have $\operatorname{reg}(Z) = T$.

Theorem 3.2 is proved. ■

4. The regularity index of a set of six double points in the general position in $\mathbb{P}^2$

Let $X \subset \mathbb{P}^n$ be a zero-dimensional subscheme and I_X be the idean defined by X. We say that X has maximal Hilbert function in degree t if

$$H_{R/I_X}(t) = \min\left\{e(R/I_X),\ \binom{n+t}{t}\right\}.$$

To prove the main result of this section, we use the following lemma:

Lemma 4.1 (Alexander–Hirschowitz, [1])**.** *Let n, t be positive integers. Let X be a general set of r double points in $\mathbb{P}^n$. Then X has maximal Hilbert function in degree t with the following exceptions:*

(1) $t = 2$ *and* $2 \leq r \leq n$*;*

(2) $t = 3, n = 4$ *and* $r = 7$*;*

(3) $t = 4,\ 2 \leq n \leq 4$ *and* $r = \binom{n+2}{2} - 1$.

Next, we consider the set of 6 double points in general positions in P^2 and use Lemma 4.1 to get the following result:

Theorem 4.2. *Let $Z = 2P_1 + \cdots + 2P_6$ be a set of 6 double points in general position in projective space $\mathbb{P}^2$. Put*

$$T_j = \max\left\{ \left[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right] \;\middle|\; P_{i_1}, \ldots, P_{i_q} \text{ are on a linear } j\text{-space} \right\}$$

and

$$T = \max\{T_j | j = 1, 2\}.$$

Then

$$\operatorname{reg}(Z) = T - 1.$$

Proof. Apply Lemma 4.1 to a set of 6 double points in the general position

$$Z = 2P_1 + \cdots + 2P_6$$

in $\mathbb{P}^2$, then Z has maximal Hilbert function in degree t, denote by $H_Z(t)$.

Furthermore, if X is a set of r double points in a general position in the projective space $\mathbb{P}^n$ then $e(A) = e(R/I) = \sum_{i=1}^{r} \binom{2+n-1}{n} = r(n+1)$. Thus, a set of r double points in the general position in $\mathbb{P}^n$ has maximal Hilbert function in degree t if and only if

$$H_Z(t) = \min\left\{ \binom{n+t}{n},\ r(n+1) \right\}.$$

For $n = 2, r = 6$, i.e. with a set of 6 double points in general positions in $\mathbb{P}^2$, we have

$$H_Z(t) = \begin{cases} 1 \ \ if\ t = 0, \\ 3 \ \ if\ t = 1, \\ 6 \ \ if\ t = 2, \\ 10 \ \ if\ t = 3, \\ 15 \ \ if\ t = 4, \\ 18 \ \ if\ t \geq 5. \end{cases}.$$

Thus, when $t = 5$ then $H_Z(t) = e(A) = 18$, at which it stabilizes. Hence

$$\operatorname{reg}(Z) = 5.$$

In this case, $T_1 = 2 * 2 - 1 = 3, T_2 = [\frac{6 * 2}{2}] = 6$.

So $T = \max\{T_1, T_2\} = 6$.

Hence

$$\operatorname{reg}(Z) = T - 1.$$

Theorem 4.2 is proved. ∎

Acknowledgement. The author research is supposed by The University of Danang - University of Science and Education. Under Grant: T2023-TN-01.

References

[1] **Alexander, J. et A. Hirschowitz,** La méthode d'Horace éclatée: application à I'interpolation en degré quatre, *Invent. Math.*, **107** (1992), 585–602.

[2] **Ballico E., O. Dumitrescu and E. Postinghel,** On Segre's bound for fat points in $\mathbb{P}^n$, *Journal of Pure and Applied Algebra*, **220(6)** (2016), 2307–2323.

[3] **Benedetti B., G. Fatabbi and A. Lorenzini,** Segre's bound and the case of $n+2$ fat points of $\mathbb{P}^n$, *Comm. Algebra*, **40** (2012), 395–403.

[4] **Catalisano, M.V., N.V. Trung and G. Valla,** A sharp bound for the regularity index of fat points in general position, *Proc. Amer. Math. Soc.*, **118** (1993), 717–724.

[5] **Davis, E.D. and A.V. Geramita,** The Hilbert function of a special class of 1-dimensional Cohen–Macaulay graded algebras, The Curves Seminar at Queen's, *Queen's Paper in Pure and Appl. Math.*, **67** (1984), 1–29.

[6] **Nagel, U. and B. Trok,** Segre's regularity bound for fat point schemes, *Annali della Scuola normale superiore di Pisa. Classe di scienze*, **20(1)** (2020), 217–237.

[7] **Serge, B.,** Alcune questioni su insiemi finiti di punti in geometria algebrica, *Atti. Convergno. Intern. di Torino*, (1961), 15–33.

[8] **Thien, P.V.,** Segre bound for the regularity index of fat points in P3, *J. Pure and Appl. Algebra*, **151** (2000), 197–214.

[9] **Thien, P.V.,** Regularity index of $s+2$ fat points not on a linear $(s-1)$-space, *Comm. Algebra*, **40** (2012), 3704–3715.

[10] **Thien, P.V. and T.N. Sinh,** On the regularity index of s fat points not on a linear $(r-1)$-space, $s \leq r+3$, *Communications in Algebra*, **45(10)** (2017), 4223–4138.

[11] **Thien, P.V. and T.T.V. Trinh,** An estimate of the regularity index of fat points in some cases, *Annales Univ. Sci. Budapest., Sect. Comp.*, **49** (2019), 399–410.

[12] **Trung, N.V.,** An algebraic approach to the regularity index of fat points in $\mathbb{P}^n$, *Kodai Math. J.*, **17** (1994), 382–389.

P. Q. N. Anh and T. N. Sinh
Department of Mathematics
The University of Danang – University of Science and Education
Vietnam
`pqnanh@ued.udn.vn` and `tnsinh@ued.udn.vn`