Annales Univ. Sci. Budapest., Sect. Comp. 54 (2023) 33–50

CONTINUATION OF THE LAUDATION TO Professor Jean-Marie De Koninck on his 75th birthday

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In *Annales Univ. Sci. Budapest., Sect. Comp.*, **47** (2018) 47–61 I wrote the Laudation to Professor Jean-Marie De Koninck. Here, I provide the highlights of his mathematical results from 2018 to 2023.

1. On the consecutive prime divisors of an integer

Let $\omega(n)$ stand for the number of distinct prime factors of an integer $n \geq 2$, so that we may list the distinct prime factors of n as $p_1(n) < p_2(n) < \cdots < p_{\omega(n)}(n)$. The growth rates of the functions $\omega(n)$ and $p_j(n)$ have been the subject of much research in the last century, as they possess surprisingly regular behavior. One such result is an interesting 1946 theorem of Paul Erdős which asserts that if $\xi(n) \to \infty$ as $n \to \infty$, then for any small $\varepsilon > 0$ and $\xi(n) \leq k \leq \omega(n)$ the inequalities

$$e^{e^{k(1-\varepsilon)}} < p_k(n) < e^{e^{k(1+\varepsilon)}}$$

are satisfied for almost all $n \leq x$. This was strengthened in 1976 by János Galambos as he showed that if $\varepsilon > 0$ is sufficiently small, and $j = j(x)$ tends to infinity with x such that $j(x) \leq (1-\varepsilon)\log\log x$, then for any fixed real number $z > 1$,

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \leq x : \frac{\log p_{j+1}(n)}{\log p_j(n)} < z\right\} = 1 - \frac{1}{z}.$$

In [168], De Koninck and Kátai expanded on these results by studying the distribution of the consecutive spacings between the prime factors of an integer. More precisely, letting

$$\nu_p = \nu_p(n) := \min\{q \mid n : q > p\}$$

and $\lambda \in (0,1]$, the authors studied the function

$$U_\lambda(n) := \sum_{\substack{p\mid n \\ \frac{\log p}{\log \nu_p(n)} < \lambda}} 1.$$

They first proved that given an arbitrary real number $\lambda \in (0,1]$,

$$\sum_{n\le x} U_\lambda(n) = (1+o(1))\lambda\, x \,\log\log x \qquad (x\to\infty) \tag{1}$$

and moreover that for every $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n\le x : \left|\frac{U_\lambda(n)}{\omega(n)} - \lambda\right| > \varepsilon\right\} = 0. \tag{2}$$

In that same paper, they considered the "shifted prime version" of the above by establishing that given an arbitrary real number $\lambda \in (0,1]$,

$$\sum_{p\le x} U_\lambda(p+1) = (1+o(1))\lambda\,\mathrm{li}(x)\log\log x \qquad (x\to\infty),$$

where $\mathrm{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$ is the logarithmic integral. Moreover, they proved that for every $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)}\#\left\{p\le x : \left|\frac{U_\lambda(p+1)}{\omega(p+1)} - \lambda\right| > \varepsilon\right\} = 0,$$

where as usual $\pi(x)$ stands for the number of primes not exceeding x.

In a subsequent paper, De Koninck and Kátai [169] examined other functions which provide further information on the spacings between the prime divisors of an integer. First, given $\lambda \in (0,1)$ and $p \in \wp$ (here $\wp$ stands for the set of all primes), consider the set

$$\mathcal{B}_\lambda(p) := \left\{q\in\wp : \lambda < \frac{\log q}{\log p} < 1/\lambda\right\}.$$

We will say that a positive integer m is coprime to a set of primes A and write $(m,A)=1$ if $(m,p)=1$ for every $p\in A$. De Koninck and Kátai studied the arithmetic function

$$u_\lambda(n) := \#\{p \mid n : (n/p, \mathcal{B}_\lambda(p)) = 1\}.$$

Observe that it follows from (1) and (2) that

$$\sum_{n\le x} u_\lambda(n) = (1+o(1))\lambda^2\, x\,\log\log x \qquad (x\to\infty), \tag{3}$$

and

$$\sum_{n\le x} u_\lambda(n)^2 = \lambda^4\, x\,(\log\log x)^2 + O\left(x\log\log x\right). \tag{4}$$

By improving the above, they managed to show that, as $x \to \infty$,

$$\frac{1}{x}\sum_{n\le x}\left(u_\lambda(n) - \lambda^2 \log\log x\right)^2 = (1+o(1))\psi(\lambda)\log\log x,$$

where

$$\psi(\lambda) = \lambda^2 + 2\lambda^2(1-\lambda^2) - 4\lambda^4 \log\frac{1}{\lambda}.$$

They also obtained an analogue result for shifted primes, namely that, as $x \to \infty$,

$$\frac{1}{\pi(x)}\sum_{p\le x} u_\lambda(p+1) = (1+o(1))\lambda^2 \log\log x,$$

$$\frac{1}{\pi(x)}\sum_{p\le x}\left(u_\lambda(p+1) - \lambda^2 \log\log x\right)^2 = (1+o(1))\psi(\lambda)\log\log x.$$

Finally, in a paper to appear in 2023, De Koninck and Kátai [172] examined the spacings between particular types of prime divisors of n, such as certain congruence classes of primes and various other subsets of $\wp$.

Let $\mathcal{B}$ be a set of primes whose corresponding counting function $B(x) :=$ $:= \#\{p \le x : p \in \mathcal{B}\}$ is such that, for some real number $\beta > 0$, we have

$$B(x) = \beta\,\mathrm{li}(x) + O\left(\frac{x}{\log^3 x}\right). \tag{5}$$

Any such set $\mathcal{B}$ satisfying (5) is called a *B-set.*

Now, consider the arithmetic function

$$U_{\lambda,\mathcal{B}}(n) := \sum_{\substack{p\mid n\\ p\in\mathcal{B}\\ \frac{\log p}{\log \nu_p(n)}<\lambda}} 1.$$

Using the techniques developed to establish (1) and (2), one will easily establish that given $\lambda \in (0,1]$ and a set of primes $\mathcal{B}$ satisfying estimate (5),

$$\sum_{n\le x} U_{\lambda,\mathcal{B}}(n) = (1+o(1))\lambda\beta x \log\log x \qquad (x\to\infty) \tag{6}$$

and, for an arbitrarily small $\varepsilon > 0$,

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n\le x : \left|\frac{U_{\lambda,\mathcal{B}}(n)}{\omega(n)} - \lambda\beta\right| \ge \varepsilon\right\} = 0. \tag{7}$$

Moreover, estimates (6) and (7) can be modified to hold for shifted primes. Indeed, given $\lambda \in (0,1]$, a set of primes $\mathcal{B}$ satisfying estimate (5) and a fixed integer $a \neq 0$, we have

$$\frac{1}{\pi(x)} \sum_{p \le x} U_{\lambda,\mathcal{B}}(p+a) = (1+o(1))\lambda\beta \log\log x \qquad (x \to \infty)$$

and moreover, for any arbitrarily small $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \left| \frac{U_{\lambda,\mathcal{B}}(p+a)}{\omega(p+a)} - \lambda\beta \right| \ge \varepsilon \right\} = 0.$$

De Koninck and Kátai further extended the above results as follows. Given $\lambda \in (0,1]$ and a set of primes $\mathcal{B}$ satisfying estimate (5), consider the arithmetic function $\widetilde{U}_{\lambda,\mathcal{B}}(n)$ which counts the number of prime divisors p of n which belong to $\mathcal{B}$ and for which the next prime divisor q of n also belonging to $\mathcal{B}$ satisfies $\log p / \log q < \lambda$. In short, setting $Q_{\mathcal{B}}(u,v) := \prod_{\substack{u<p<v \\ p\in\mathcal{B}}} p$, one can define this arithmetic function formally as

$$\widetilde{U}_{\lambda,\mathcal{B}}(n) := \sum_{\substack{p \mid n \\ \log p/\log q < \lambda \\ p,q \in \mathcal{B} \\ \left(\frac{n}{pq}, Q_{\mathcal{B}}(p,q)\right)=1}} 1.$$

In the case of the function $\widetilde{U}_{\lambda,\mathcal{B}}(n)$, one can obtain an asymptotic formula for its average value with greater accuracy than the one obtained for the function $U_{\lambda,\mathcal{B}}(n)$ (whose average value was revealed through estimate (6)). Indeed, we have the following result.

Let $\lambda \in (0,1]$ and $\mathcal{B}$ a set of primes satisfying estimate (5)*. Then,*

$$\sum_{n \le x} \widetilde{U}_{\lambda,\mathcal{B}}(n) = \beta\lambda^{\beta} x \log\log x + O(x \log\log\log x).$$

Moreover, for any arbitrarily small $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \left| \frac{\widetilde{U}_{\lambda,\mathcal{B}}(n)}{\omega(n)} - \beta\lambda^{\beta} \right| \ge \varepsilon \right\} = 0.$$

The authors then provided several examples of B-sets to which one can apply the above results. Two of these are as follows.

Given an integer $k \geq 3$, let $\ell_1, \dots, \ell_r$ be the reduced residue system modulo k, with $r = \phi(k)$ (here ϕ stands for the Euler totient function). Then, it is clear that each of the residue classes

$$\mathcal{B}_{\ell_j} := \{p \in \wp : p \equiv \ell_j \pmod{k}\} \qquad (j = 1, \dots, r)$$

satisfies condition (5) and is therefore a B-set.

A second interesting B-set involves the sum-of-digits function $s_q(n)$ which stands for the sum of the base q digits of an integer n (here, $q \geq 2$). First note that it is known (for instance from the work of Mauduit and Rivat) that if $(k, q-1) = 1$, then there exists a real number $\sigma = \sigma_{q,k} > 0$ such that

$$\frac{1}{\pi(x)} \#\{p \leq x : s_q(p) \equiv \ell \pmod{k}\} = \frac{1}{k} + O_{q,k}\left(x^{-\sigma} \log x\right). \tag{8}$$

Clearly, estimate (8) guarantees that if for a given integer $k \geq 3$, we set

$$\mathcal{B}_\ell := \{p \in \wp : s_q(p) \equiv \ell \pmod{k}\} \qquad (\ell = 0, 1, \dots, k-1),$$

then $\mathcal{B}_\ell$ is indeed a B-set for each $\ell = 0, 1, \dots, k-1$.

2. Consecutive integers divisible by a power of their largest prime factors

Given an integer $n \geq 2$, let $P(n)$ stand for its largest prime factor. Given integers $k \geq 2$ and $\ell \geq 2$, consider the set $E_{k,\ell}$ of those integers $n \geq 2$ for which $P(n+i)^\ell \mid n+i$ for $i = 0, 1, \dots, k-1$. These sets are very thin. For instance, the smallest element of $E_{3,2}$ is 1 294 298. The study of the sets $E_{k,\ell}$ originated in 2009 and was later pursued by several authors. In 2018, De Koninck and Moineau [146] used an approach based on polynomials to find several elements appearing in various $E_{k,\ell}$ sets. For instance, by considering the system of consecutive polynomials

$$\begin{aligned} g(x) - 1 &= (2x+1)^2(x-1), \\ g(x) &= x(4x^2 - 3), \\ g(x) + 1 &= (2x-1)^2(x+1), \end{aligned}$$

they could find thousands of elements of $E_{3,2}$, and by considering the system

$$\begin{aligned} g(x) - 1 &= (2x-1)^3(30x^4 + 45x^3 + 24x^2 + 6x + 1), \\ g(x) &= x^3(240x^4 - 168x^2 + 35), \\ g(x) + 1 &= (2x+1)^3(30x^4 - 45x^3 + 24x^2 - 6x + 1), \end{aligned}$$

they could identify integers belonging of $E_{3,3}$ with no less than 77 digits. In order to find elements belonging to $E_{2,\ell}$, they proved that, given any fixed integer $\ell \geq 2$, there exist $g_1(x), g_2(x) \in \mathbb{Z}[x]$ each of degree $\ell - 1$ such that

$$x^\ell \cdot g_1(x) + (-1)^\ell = (x-1)^\ell \cdot g_2(x).$$

Using this approach, they could find a 116-digit integer belonging to $E_{2,6}$.

More generally, given k integers $\ell_0, \ell_1, \ldots, \ell_{k-1}$, each ≥ 2, consider the set

$$F(\ell_0, \ell_1, \ldots, \ell_{k-1}) := \{n \in \mathbb{N} : P(n+i)^{\ell_i} \mid n+i \text{ for } i = 0, 1, \ldots, k-1\},$$

so that in particular $E_{k,\ell} = F(\underbrace{\ell, \ldots, \ell}_{k})$. Most likely, each set $F(\ell_0, \ell_1, \ldots, \ell_{k-1})$ is infinite, but besides the set $F(2,2)$, no such statement has been proved.

In their 2018 paper, De Koninck and Moineau also showed that if we assume that there exist infinitely many primes of the form $9k^2 + 6k + 2$ (respectively $4k^2 + 2k + 1$), then the set $F(3,2)$ (respectively $F(4,2)$) is infinite. They then explored some identities involving consecutive polynomials whose algebraic structure provides the potential for revealing infinitely many members of $F(\ell_0, \ell_1, \ldots, \ell_{k-1})$ for any given k-tuple of integers $\ell_0 \geq 2, \ell_1 \geq 2, \ldots, \ell_{k-1} \geq 2$.

3. On the middle divisors of an integer

Given a positive integer n, let

$$\rho_1(n) := \max\{d \mid n : d \leq \sqrt{n}\} \qquad \text{and} \qquad \rho_2(n) := \min\{d \mid n : d \geq \sqrt{n}\}$$

stand for the *middle divisors* of n.

The mean value of $\rho_2(n)$ has been established more than 40 years ago as Tenenbaum proved that

$$\sum_{n \leq x} \rho_2(n) = \frac{\pi^2}{12} \frac{x^2}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

In 2020, De Koninck and Razafindrasoanaivolala [158] generalised this last estimate by showing that, given any real number $a > 0$ and any integer $k \geq 1$,

$$\sum_{n \leq x} \rho_2(n)^a = c_0 \frac{x^{a+1}}{\log x} + c_1 \frac{x^{a+1}}{\log^2 x} + \cdots + c_{k-1} \frac{x^{a+1}}{\log^k x} + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right),$$

where, for $\ell = 0, 1, \ldots, k-1$,

$$c_\ell = c_\ell(a) = \frac{\ell!}{(a+1)^{\ell+1}} \sum_{j=0}^{\ell} \frac{(a+1)^j (-1)^j \zeta^{(j)}(a+1)}{j!}$$

with ζ standing for the Riemann zeta function. They also proved that given any integer $k \geq 1$ and any real number $r > -1$,

$$\sum_{n\leq x} \frac{\rho_2(n)}{\rho_1(n)^r} = e_0 \frac{x^2}{\log x} + e_1 \frac{x^2}{\log^2 x} + \cdots + e_{k-1} \frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right)$$

where $e_0 = \dfrac{\zeta(r+2)}{2}$ and for each $1 \leq \ell \leq k-1$,

$$e_\ell = \left(\frac{r+2}{2}\right) c_\ell + \sum_{\nu=0}^{\ell-1} \frac{r c_\nu}{2} \prod_{m=\nu}^{\ell-1} \left(\frac{m+1}{2}\right),$$

with, for each $\nu = 0, 1, \ldots, \ell$,

$$c_\nu = \frac{\nu!}{(r+2)^{\nu+1}} \sum_{j=0}^{\nu} \frac{(r+2)^j(-1)^j \zeta^{(j)}(r+2)}{j!}.$$

Interestingly, as a consequence of this last result,

$$T_r(x) := \sum_{n\leq x} \frac{\rho_2(n)}{\rho_1(n)^r} \sim \frac{\zeta(r+2)}{2} \frac{x^2}{\log x} \qquad \text{as } x \to \infty,$$

implying that all sums $T_r(x)$ are of the same order, independently of the chosen number $r > -1$. For instance, although it may at first appear counterintuitive, we do have that $\displaystyle\sum_{n\leq x} \rho_2(n)\sqrt{\rho_1(n)} \asymp \sum_{n\leq x} \frac{\rho_2(n)}{\sqrt{\rho_1(n)}}$.

Later, in 2023, De Koninck and Razafindrasoanaivolala [173] proved that

$$\sum_{\substack{4\leq n\leq x\\ n\neq \text{prime}}} \frac{\log \rho_2(n)}{\log \rho_1(n)} = x \log\log x + O(x)$$

and also that for all x sufficient large,

$$c_1\, x < \sum_{2\leq n\leq x} \frac{\log \rho_1(n)}{\log \rho_2(n)} < c_2\, x,$$

where

$$\begin{aligned} c_1 &= 1 - \log 2 + \int_1^2 \frac{1-\log u}{u(u+1)}\,du + \int_3^\infty \frac{u-1}{u+1} \frac{\rho(u-1)}{u}\,du \approx 0.528087, \\ c_2 &= 2 - 2\log 2 \approx 0.613706. \end{aligned}$$

4. Bounds for the counting function of the Jordan-Pólya numbers

A positive integer n is said to be a *Jordan-Pólya number* if it can be written as a product of factorials. Jordan-Pólya numbers arise naturally in a simple combinatorial problem. Given k groups of $n_1, n_2, \dots, n_k$ distinct objects, then the number of distinct permutations of these $n_1 + n_2 + \dots + n_k$ objects which maintain objects of the same group adjacent is equal to $k! \cdot n_1! \cdot n_2! \cdots n_k!$, a Jordan-Pólya number.

The smallest 20 Jordan-Pólya numbers are

$$1, 2, 4, 6, 8, 12, 16, 24, 32, 36, 48, 64, 72, 96, 120, 128, 144, 192, 216, 240.$$

Much study has been done on a particular subset of the Jordan-Pólya numbers, namely those which are themselves factorials. In particular, consider the equation

$$(*) \qquad n! = a_1!a_2! \cdots a_r! \quad \text{in integers } n > a_1 \geq a_2 \geq \dots \geq a_r \geq 2,\ r \geq 2.$$

It is easy to show that this equation has infinitely many "trivial" solutions. Indeed, choose any integers $a_2 \geq \dots \geq a_r \geq 2$ and set $n = a_2! \cdots a_r!$. Then, choose $a_1 = n - 1$. One can easily see that $n! = n \cdot (n-1)! = a_1!a_2! \cdots a_r!$. Besides these trivial solutions of equation (*), we find the non-trivial solutions

$$(**) \qquad 9! = 2! \cdot 3!^2 \cdot 7!, \qquad 10! = 6! \cdot 7! = 3! \cdot 5! \cdot 7!, \qquad 16! = 2! \cdot 5! \cdot 14!.$$

According to Hickerson's conjecture, there are no other non-trivial solutions for equation (*). In 2007, Luca showed that if the *abc* conjecture holds, then equation (*) has only a finite number of non-trivial solutions. In 2016, Nair and Shorey proved that any other non-trivial solution n of (*), besides those in (**), must satisfy $n > e^{80}$.

On the other hand, more than 40 years ago, Erdős and Graham showed that the number of distinct integers of the form $a_1!a_2! \cdots a_r!$, where $a_1 < a_2 < \dots < a_r \leq y$ is $\exp\{(1+o(1))y(\log\log y)/\log y\}$ as $y \to \infty$.

In 2020, letting $\mathcal{J}$ stand for the set of Jordan-Pólya numbers and $\mathcal{J}(x)$ for its counting function, De Koninck, Doyon, Verreault, and Razafindrasoanaivolala [157] showed that $\mathcal{J}(x) = o(x)$ and in fact that, given any small $\varepsilon > 0$, the much stronger estimate

$$\mathcal{J}(x) < \exp\left\{(4+\varepsilon)\frac{\sqrt{\log x}\log\log\log x}{\log\log x}\right\} \qquad (x \geq x_1)$$

holds for some $x_1 = x_1(\varepsilon) > 0$. They also showed that, for any given $\varepsilon > 0$, there exists $x_2 = x_2(\varepsilon)$ such that

$$\mathcal{J}(x) > \exp\left\{(2-\varepsilon)\frac{\sqrt{\log x}}{\log\log x}\right\} \qquad (x \geq x_2).$$

5. New upper bounds for the number of divisors function

Let $\tau(n)$ stand for the number of positive divisors of n. In 1915, Ramanujan obtained the upper bound

$$\tau(n) \le \left(\frac{\log(n\gamma(n))}{\omega(n)}\right)^{\omega(n)} \cdot \beta(n) \qquad (n \ge 2),$$

where

$$\gamma(n) = \prod_{p|n} p \qquad \text{and} \qquad \beta(n) = \prod_{p|n} \frac{1}{\log p}.$$

In 2020, De Koninck and Letendre [155] obtained several new optimal pointwise upper bounds for the number-of-divisors function $\tau(n)$. In particular, they proved the following:

- For all $n \ge 2$,
$$\tau(n) \le \left(\frac{\eta_2 \log n}{\omega(n)\log\max(2,\omega(n))}\right)^{\omega(n)},$$
where $\eta_2 = \exp\left(\frac{1}{6}\log 96 - \log\left(\frac{\log 60060}{6\log 6}\right)\right) = 2.0907132\ldots$ is optimal. Moreover, the constant η_2 can be replaced by 2 if $n \ge 782139803452561073520$.
- For each integer $n > 782139803452561073520$,
$$\tau(n) \le \left(\frac{2\log n}{\omega(n)\log\max(2,\omega(n))}\right)^{\omega(n)}.$$
Moreover, the above inequality is true for all $n \ge 2$ with $\omega(n) \le 3$.
- For all $n \ge 2$,
$$\tau(n) \le \left(1 + \eta_3 \frac{\log n}{\omega(n)\log\max(2,\omega(n))}\right)^{\omega(n)},$$
where $\eta_3 = \dfrac{(1152^{1/7}-1)7\log 7}{\log 367567200} = 1.1999953\ldots$ is optimal. Moreover, the constant η_3 can be replaced by 1 if $\omega(n) \ge 74$.
- For each positive integer n with $k = \omega(n) \ge 74$,
$$\tau(n) < \left(1 + \frac{\log n}{k\log k}\right)^k.$$

Finally, an interesting consequence of some of their results is that $n \ge \omega(n)^{\omega(n)}$ whenever $\omega(n) \notin [4, 12]$ or $n > 43 \cdot 2 \cdot 3 \cdot 5 \cdots 31 = 8\,624\,101\,075\,590$.

6. Characteristics of multiplicative functions at consecutive arguments

De Koninck, Kátai and Phong recently published five joint papers ([150], [153], [160], [164], [166]) regarding families of multiplicative functions which are characterised by their behaviour at consecutive arguments.

In [160], the bulk of their results can be summarized as follows.

> *Let $\mathcal{M}_1^*$ stand for the set of completely multiplicative functions f such that $|f(n)| = 1$ for all positive integers n and let c_0, c_1, c_2 be three complex numbers such that $(c_0, c_1, c_2) \neq (0,0,0)$. Given $f \in \mathcal{M}_1^*$ and setting $s(n) := c_0 f(n-1) + c_1 f(n) + c_2 f(n+1)$, and assuming that $\lim_{x\to\infty} \frac{1}{x}\sum_{n\leq x} |s(n)| = 0$, then $c_0 + c_1 + c_2 = 0$ and there exists a real number τ such that $f(n) = n^{i\tau}$ for all positive integers n. Moreover, let $f_0, f_1, f_2 \in \mathcal{M}_1^*$ and consider the sum $s(n) := c_0 f_0(n-1) + c_1 f_1(n) + c_2 f_2(n+1)$. Assuming that $\lim_{n\to\infty} s(n) = 0$ and assuming also that either $f_0(n) = f_1(n)$ or $f_0(n) = f_2(n)$ or $f_1(n) = f_2(n)$, then $c_0 + c_1 + c_2 = 0$ and there exists $\tau \in \mathbb{R}$ such that $f_0(n) = f_1(n) = f_2(n) = n^{i\tau}$ for all positive integers n.*

To facilitate the presentation of the results obtained by De Koninck, Kátai and Phong [166] on the same topic, let us first introduce additional notation. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ stand for the set of points on the unit circle and given $f \in \mathcal{M}_1^*$, let $\Delta f(n) := f(n+1) - f(n)$.

In 2017, Klurman proved a 1983 conjecture of Kátai, namely that given $f \in \mathcal{M}_1^*$ such that $\sum_{n\leq x} |\Delta f(n)| = o(x)$ as $x \to \infty$ (or such that $\sum_{n\leq x} \frac{|\Delta f(n)|}{n} = o(\log x)$ as $x \to \infty$), then there exists some real number t such that $f(n) = n^{it}$ for all $n \in \mathbb{N}$.

Given $f \in \mathcal{M}_1^*$, we shall denote by $S(f)$ the set of limit points of the set $\{f(n) : n \in \mathbb{N}\}$ and by $R(f)$ the set $\{p \in \wp : f(p) \neq 1\}$, where $\wp$ stands for the set of all primes. Also, given $k \in \mathbb{N}$, we set $W_k := \{e(a/k) : a = 0, 1, \ldots, k-1\} = \{\omega \in \mathbb{C} : \omega^k = 1\}$, where $e(y) := e^{2\pi i y}$. Finally, given a set of complex numbers $\{a_n : n \in \mathbb{N}\}$, we denote its closure by $\overline{\{a_n : n \in \mathbb{N}\}}$. In 2018, Klurman and Mangerel proved the following.

Theorem A. (Klurman and Mangerel) *Assume that $f, g \in \mathcal{M}_1^*$ are such that $S(f) = S(g) = \mathbb{T}$ and also that $\overline{\{(f(n), g(n+1)) : n \in \mathbb{N}\}} \neq \mathbb{T} \times \mathbb{T}$. Further assume that for infinitely many $j \in \mathbb{N}$, either $|R(f^j)| \cdot |R(g^j)| > 1$ or $R(f^j) \neq R(g^j)$. Then, for some real number t and positive integers k and ℓ, we have $f(n) = n^{it/k} F(n)$ and $g(n) = n^{it/\ell} G(n)$, where $F(\mathbb{N}) \in W_k$ and $G(\mathbb{N}) \in W_\ell$.*

This last theorem motivates the introduction of the set $\mathcal{H}$, namely the set made up of those pairs (f, g) of functions in $\mathcal{M}_1^*$ for which there exist infinitely many $j \in \mathbb{N}$ for which either $|R(f^j)| \cdot |R(g^j)| > 1$ or $R(f^j) \neq R(g^j)$.

In their 2020 paper, De Koninck, Kátai and Phong [166] applied the above results of Klurman and Mangerel to characterise those triplets of multiplicative functions f, g, h with unusually small gaps between their consecutive values, and they also considered the higher iterations $\Delta^m f(n)$ for each of the integers $m = 2, 3, 4, 5, 6, 7$ and obtained bounds for $|\Delta^m f(n)|$.

More precisely, they proved the following.

Theorem 1. *Let $f, g, h \in \mathcal{M}_1^*$ be such that the function $s(n) := g(n+2) - -2h(n+1) + f(n)$ satisfies*

$$\sum_{n \le x} \frac{|s(n)|}{n} = o(\log x) \qquad (x \to \infty).$$

Then, there exists a real number t such that $f(n) = g(n) = h(n) = n^{it}$ for all $n \in \mathbb{N}$.

Theorem 2. *Let $f, g, h \in \mathcal{M}_1^*$ be such that $S(f) = S(g) = S(h) = \mathbb{T}$. Assume also that*

$$\overline{\{(g(n+1), h(n)) : n \in \mathbb{N}\}} \neq \mathbb{T} \times \mathbb{T} \quad \text{and} \quad \overline{\{(h(n+1), f(n)) : n \in \mathbb{N}\}} \neq \mathbb{T} \times \mathbb{T}$$

and that (f, h), $(h, g) \in \mathcal{H}$. Finally, let $\omega, \kappa \in \mathbb{T}$ be such that

$$(*) \qquad s(n) := g(n+2)\omega - 2h(n+1) + f(n)\kappa$$

satisfies $\lim_{n \to \infty} s(n) = 0$. Then, there exists a real number t such that $f(n) = = g(n) = h(n) = n^{it}$ for all $n \in \mathbb{N}$ and moreover $\omega = \kappa = 1$.

Interestingly, the situation is much simpler if at least one of the three sets $S(f)$, $S(g)$, $S(h)$ is not equal to $\mathbb{T}$, as can be seen in the following theorem.

Theorem 3. *Let $f, g, h \in \mathcal{M}_1^*$, where at least one of the three sets $S(f)$, $S(g)$, $S(h)$ is not equal to $\mathbb{T}$. Letting $s(n)$ be as in Theorem 1 and assuming that relation $(*)$ of Theorem 2 holds, then*

$$\omega = \kappa = 1 \quad \text{and} \quad f(n) = g(n) = h(n) = 1 \ \text{ for all } n \in \mathbb{N}.$$

Regarding iterations of the $\Delta f(n)$, we only mention two of the results obtained by De Koninck, Kátai and Phong in [160].

In the next theorems, we always assume that $f \in \mathcal{M}_1^*$, $S(f) = \mathbb{T}$ and also that $|R(f^m)| = \infty$ for infinitely many positive integers m. Moreover, we set

$$\xi(n) := f(n+1)\overline{f(n)}.$$

We then have the following result.

Theorem 4. *Assume that there exist $\delta > 0$, $\omega \in \mathbb{T}$ and some $n_0 \in \mathbb{N}$ such that*

$$|\xi(n)\omega - 1| < 2 - \delta \qquad (n \geq n_0).$$

Then, there exists a real number t such that $f(n) = n^{it}F(n)$ for all $n \in \mathbb{N}$, where $F(\mathbb{N}) \subseteq W_k$ and $\omega \in W_k$ for some positive integer k.

Considering iterations of $\Delta f(n)$, let

$$\Delta^2 f(n) := \Delta\Delta f(n) = \Delta(f(n+1) - f(n)) = f(n+2) - 2f(n+1) + f(n),$$

and for an arbitrary integer $k \geq 3$, let $\Delta^k f(n) := \Delta\Delta^{k-1} f(n)$. Observe that we have the trivial bound $|\Delta^k f(n)| \leq 2^k$, with equality achieved in the case of the multiplicative function $f(n) = (-1)^{n+1}$.

In each of the following theorems, the real numbers $\varepsilon > 0$ and $\delta > 0$ are arbitrary but fixed.

Theorem 5. *Assume that $|\Delta^2 f(n)| \leq K := 2 - \delta$ for all $n \geq n_0$ for some positive integer n_0. Then, there exists a real number t and some positive integer k such that $f(n) = n^{it/k}F(n)$ for all $n \in \mathbb{N}$, where $F^k(n) = 1$ for all $n \geq 1$. Moreover, setting $E(n) := F(n+2) - 2F(n+1) + F(n)$, we have $|E(n)| \leq K + \varepsilon$ for all $n \geq n_0$.*

Theorem 6. *Assume that $|\Delta^3 f(n)| \leq K := 4 - \delta$ for all $n \geq n_0$ for some positive integer n_0. Then, there exists some real number t such that $f(n) = n^{it}F(n)$ for all $n \in \mathbb{N}$, where $F^\ell(n) = 1$ for all $n \in \mathbb{N}$ and $|\Delta^3 F(n)| \leq K + \varepsilon$ provided $n \geq n_1(\varepsilon)$.*

The authors then moved on to establish upper bounds for $|\Delta^m f(n)$ for $m = 4, 5, 6, 7$. They concluded their paper by the following interesting conjecture.

Conjecture. *Theorem* A *remains true if the condition*

$$\overline{\{(f(n), g(n+1)) : n \in \mathbb{N}\}} \neq \mathbb{T} \times \mathbb{T}$$

is weakened and replaced by the following: there exists a pair of points ξ, η located on the unit circle for which

$$\sum_{\substack{n \leq x \\ |f(n) - \xi| < \varepsilon \\ |g(n+1) - \eta| < \varepsilon}} \frac{1}{n} = o(\log x) \quad \textit{as} \quad x \to \infty,$$

provided $\varepsilon > 0$ is sufficiently small.

In 1985, M. V. Subbarao introduced the concept of *weakly multiplicative* arithmetic function (later renamed *quasi-multiplicative*) as those functions f for which

$$f(np) = f(n)f(p)$$

for every $p \in \wp$ and $n \in \mathbb{N}$ coprime to p.

Similarly, g is said to be *quasi-additive* if

$$g(np) = g(n) + g(p)$$

for every $p \in \wp$ and $n \in \mathbb{N}$ coprime to p.

Clearly, multiplicative (resp. additive) functions are quasi-multiplicative (resp. quasi-additive) functions.

Many interesting papers have been published on this topic, in particular those of J. Fabrykowski and M. V. Subbarao, J. Fehér and B. M. Phong , as well as B. M. Phong.

First, some known results. The following is an old result proved independently in 1970 by I. Kátai and E. Wirsing.

Proposition 1. *Let f be an additive function satisfying*

$$\frac{1}{x}\sum_{n\le x}|f(n+1)-f(n)| \to 0 \quad \textit{as} \quad x\to\infty,$$

Then there exists a constant c such that $f(n) = c\log n$ for all positive integers n.

In 2000, I. Kátai and M. V. Subbarao proved the following four results regarding wider classes of arithmetical functions.

Theorem A. *If a quasi-additive function f is monotonic, then it is a constant multiple of $\log n$.*

Theorem B. *If f is a quasi-additive function and*

$$\frac{1}{x}\sum_{n\le x}|f(n+1)-f(n)| \to 0 \quad \textit{as} \quad x\to\infty,$$

then there exists a constant C such that $f(n) = C\log n$.

Theorem C. *If g is a quasi-multiplicative function, $|g(n)| = 1$ and*

$$\Delta g(n) := g(n+1) - g(n) \to 0 \quad \textit{as} \quad n\to\infty,$$

then $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

Theorem D. *If g is a quasi-multiplicative function, $|g(n)| = 1$ and*

$$\frac{1}{x}\sum_{n\le x}|g(n+1)-g(n)| \to 0 \quad as \quad x\to\infty,$$

then g is a completely multiplicative function.

Observe that Theorem C and Theorem D also hold for multiplicative functions.

Let $\mathcal{B}$ be a set of primes for which

$$\sum_{p\in\mathcal{B}}\frac{1}{p} < \infty$$

and let $\mathcal{B}^*$ be the multiplicative semigroup generated by $\mathcal{B}$. Moreover, let $\mathcal{M}$ be the set of squarefree numbers coprime to $\mathcal{B}^*$. It is clear that every integer n can be uniquely written in the form $n = Km$, $(K, m) = 1$, where m is the largest divisor of n that belongs to $\mathcal{M}$ and for which $(n/m, m) = 1$.

Definition. Let $f : \mathbb{N} \to \mathbb{R}$. We say that f is almost-additive if for every $p \in \wp$ and $n \in \mathbb{N}$ with $(p, n) = 1$ and $(p, \mathcal{B}) = 1$, we have

$$f(np) = f(n) + f(p).$$

Definition. Let $g : \mathbb{N} \to \mathbb{C}$. We say that g is almost-multiplicative if for every $p \in \wp$ and $n \in \mathbb{N}$ with $(p, n) = 1$ and $(p, \mathcal{B}) = 1$, we have

$$g(np) = g(n)g(p).$$

In 2021, De Koninck, Kátai and Phong [166] generalized Theorems A–D of I. Kátai and M. V. Subbarao by proving the following results.

Theorem 1. *If some given almost-additive function f is monotonic, then $f(n) = C\log n$ for some $C \in \mathbb{R}$.*

Theorem 2. *If f is an almost-additive function and*

$$\frac{1}{x}\sum_{n\le x}|f(n+1)-f(n)| \to 0 \quad as \quad x\to\infty,$$

then $f(n) = C\log n$ for some $C \in \mathbb{R}$.

Theorem 3. *If g is an almost-multiplicative function, $|g(n)| = 1$ and*

$$\frac{1}{x}\sum_{n\le x}|g(n+1)-g(n)| \to 0 \quad as \quad x\to\infty,$$

then $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

From these, follows the following.

Corollary. *If g is an almost-multiplicative function, $|g(n)| = 1$ and*

$$\Delta g(n) := g(n+1) - g(n) \to 0 \quad as \quad n \to \infty,$$

then $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

7. On the behaviour of certain arithmetic functions at shifted primes

Let φ stand for the Euler totient function. Garcia and Luca have proved that, given any positive integer ℓ, the set of those primes p such that $\varphi(p + +\ell)/\varphi(p - \ell) > 1$ has the same density as the set of those primes p for which $\varphi(p + \ell)/\varphi(p - \ell) < 1$. In 2018, De Koninck and Kátai [147] proved this result using classical results from probabilistic and analytic number theory, and thereafter established similar results for the sum of divisors function and for the k-fold iterate of the Euler function. They also examined the modulus of continuity of some arithmetical functions and finally provided a general result regarding the existence of the distribution function for the function $s(p) := f(p+\ell) - f(p-\ell)$ for any fixed positive integer ℓ provided the additive function f satisfies certain conditions.

Let $\tau(n)$ stand for the number of positive divisors of n. Given an additive function f and a real number $\alpha \in [0, 1)$, let

$$h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d|n \\ \{f(d)\}<\alpha}} 1,$$

where $\{y\}$ stands for the fractional part of y, and consider the discrepancy $\Delta(n) := \sup_{0\leq\alpha<\beta<1} |h_n(\beta) - h_n(\alpha) - (\beta - \alpha)|$. In 2019, De Koninck and Kátai [152] showed that $\Delta(p + 1) \to 0$ for almost all primes p if and only if $\sum_{q\in\wp} \frac{\|mf(q)\|^2}{q} = \infty$ for every positive integer m, where $\|x\|$ stands for the distance between x and its nearest integer and where the sum runs over all primes q.

CONTINUATION OF LIST OF PUBLICATIONS

Jean-Marie De Koninck

The first part of the list of publications is published in *Annales Univ. Sci. Budapest., Sect. Comp.*, **47** (2018) 63–74.

Papers published in refereed journals

[143] On properties of sharp normal numbers and of non-Liouville numbers, *Annales mathématiques du Québec* **42(1)** (2018), 31–47. (with I. Kátai)

[144] Distinguishing between sharp and non-sharp normal numbers, *Math. Pannon.*, **26(1)** (2017/2018), 3–14. (with I. Kátai)

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[149] Distribution of arithmetic functions on particular subsets of integers, *Lithuanian Mathematical Journal*, **59(1)** (2019), 24–38. (with I. Kátai)

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[157] Bounds for the counting function of the Jordan-Pólya numbers, *Archivum Mathematicum*, **56** (2020), 135–146. (with N. Doyon, W. Verreault and A. A. B. Razafindrasoanaivolala)

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[159] Digit patterns in real numbers created from permutations, *Annales Univ. Sci. Budapest., Sect. Comp.*, **50** (2020), 93–100. (with I. Kátai)

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