ON THE EQUATION $G(n) = F(n^2 - 1) + D$

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Abstract. We give all solutions (D, G, F) of the equation

$$G(n) = F(n^2 - 1) + D \quad \text{for every} \quad n \in \mathbb{N},$$

where G, F are completely multiplicative functions and $D \in \mathbb{C}$.

1. Introduction

In the following let $\mathcal{P}$, $\mathbb{N}$, and $\mathbb{C}$ denote the set of primes, positive integers and complex numbers, respectively. Let $\mathbb{N}_1 := \mathbb{N} \setminus \{1\}$. We denote by $\mathcal{M}$ $(\mathcal{M}^*)$ the set of all complex-valued multiplicative (completely multiplicative) functions, respectively. For each $m \in \mathbb{N}, m \geq 2$ let $\chi_m(n)$ $(\chi_m^*(n))$ be the real principal (non-principal) Dirichlet character $\pmod{m}$, respectively.

Let $\mathbb{E}(n) = 1$, $\mathbb{I}(n) = n$ for every $n \in \mathbb{N}$ and $\mathbb{O}(1) = 1, \mathbb{O}(n) = 0$ if $n \geq 2$. For each $\omega \in \mathbb{C}$ with $\omega^3 = -1$ we define the function $\Psi_\omega : \mathbb{N} \to \mathbb{C}$ such that

$$\Psi_\omega(n) = \begin{cases} 0 & \text{if} \quad 3|n \\ \omega^{\frac{2(n-1)}{3}} & \text{if} \quad n \equiv 1 \pmod{3} \\ \omega^{\frac{4(n-2)}{3}+1} & \text{if} \quad n \equiv 2 \pmod{3}. \end{cases}$$

One can check that $\Psi_\omega \in \mathcal{M}^*$ and $\Psi_{-1} = \chi_3^*$.

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The problem concerning the characterization of the identity function as multiplicative arithmetical function with some equation was studied by several authors. C. Spiro proved that if $f \in \mathcal{M}$ satisfies the following relations

$$f(p+q) = f(p) + f(q) \quad (\forall p, q \in \mathcal{P}) \quad \text{and} \quad f(p_0) \neq 0 \quad \text{for some} \quad p_0 \in \mathcal{P},$$

then f is the identity function.

Recently, in [1] we gave all solutions of the equation

$$F(n^2 + m^2 + k) = H(n) + H(m) + K \quad \text{for every} \quad n \in \mathbb{N},$$

where $k \in \mathbb{N}$ is the sum of two fixed squares, $K \in \mathbb{C}$ and F, H are completely multiplicative functions. The equation

$$f(n^2 + Dm^2) = f^2(n) + Df^2(m)$$

was completely solved for arithmetic functions f in [3] and [4].

Here we shall prove the following

Theorem 1. *Assume that $D \in \mathbb{C}$ and the functions $G, F \in \mathcal{M}^*$ satisfy the relation*

$$G(n) = F(n^2 - 1) + D \quad \textit{for every} \quad n \in \mathbb{N}_1. \tag{1.1}$$

Then the following assertions hold.

(a) *If $D = 0$ and $F(3) \neq 0$, then $G = F = \mathbb{E}$.*

(b) *If $D = 0$ and $F(3) = 0$, then $G = \mathbb{O}$ and*

$$F(2) = F(3) = F(5) = F(7) = 0, \quad F(n^2 - 1) = 0 \quad \textit{for every} \quad n \in \mathbb{N}_1.$$

(c) *If $D \neq 0, F(2) = F(3) = 0$, then $D = 1$, $G(n) = 1$ for every $n \in \mathbb{N}$ and $F(n^2 - 1) = 0$ for every $n \in \mathbb{N}_1$.*

(d) *If $D \neq 0, F(2) = 0$ and $F(3) \neq 0$, then $(D, G, F) = (1, \chi_2, \chi_4^*)$.*

(e) *If $D \neq 0, \omega = F(2) \neq 0$ and $F(3) = 0$, then $\omega^3 = -1$ and $(D, G, F) = (1, \chi_3, \Psi_\omega)\}$.*

(f) *If $D \neq 0$ and $F(2)F(3) \neq 0$, then $(D, G, F) = \{(-1, \mathbb{O}, \mathbb{E}), (1, \mathbb{I}^2, \mathbb{I})\}$.*

Remark 1. Let $\mathcal{F}$ be the set of all $F \in \mathcal{M}^*$, for which $F(n^2 - 1) = 0$ holds for every $n \in \mathbb{N}_1$. Then $|\mathcal{F}| = \infty$. For the proof of this assertion, we consider $k \in \mathbb{N}$ primes $p_1 < p_2 < \ldots < p_k$ of the form $4t + 1$. Let

$$\mathcal{B} := \{p_1^{\alpha_1} \cdots p_k^{\alpha_k} \mid \ \alpha_i \in \mathbb{Z}, \alpha_i \geq 0 \ \ (i = 1, \ldots, k)\}.$$

We define $f \in \mathcal{M}^*$ as follows:

$$f(n) = \begin{cases} 1 & \text{if} \quad n \in \mathcal{B} \\ 0 & \text{if} \quad n \notin \mathcal{B} \end{cases} .$$

Then $f \in \mathcal{F}$. Indirectly, assume that $f(n^2-1) = f(n-1)f(n+1) \neq 0$, then $n-1 \in \mathcal{B}$, $n+1 \in \mathcal{B}$, which imply $n-1 = 4m+1$ and $n+1 = 4\ell+1$ for some $m, \ell \in \mathbb{N}$. Then $2 = 4(\ell - m)$, which is impossible.

2. Lemmas

In this section we assume that $D \in \mathbb{C}$ and the functions $G, F \in \mathcal{M}^*$ satisfy the relation (1.1) for every $n \in \mathbb{N}_1$.

Let $F(2) = \omega$ and $F(3) = \mu$. First we apply (1.1) with $n \in \{2, 3, 5, 7, 11, 13\}$. We have

$$\begin{cases} G(2) &= F(3) + D = \mu + D, \\ G(3) &= F(2)^3 + D = \omega^3 + D, \\ G(5) &= F(2)^3 F(3) + D = \omega^3 \mu + D, \\ G(7) &= F(2)^4 F(3) + D = \omega^4 \mu + D, \\ G(11) &= F(2)^3 F(3) F(5) + D = \omega^3 \mu F(5) + D, \\ G(13) &= F(2)^3 F(3) F(7) + D = \omega^3 \mu F(7) + D. \end{cases} \tag{2.1}$$

By using this system and by applying (1.1) with $n \in \{4, 6, 8, 9, 15, 26, 49, 55\}$, we obtain that

$$E_1 = G(2)^2 - F(3)F(5) - D = D^2 + 2\mu D - \mu F(5) + \mu^2 - D = 0,$$

$$\begin{aligned} E_2 &= G(2)G(3) - F(5)F(7) - D = \\ &= D\omega^3 + \omega^3\mu + \mu D - F(5)F(7) + D^2 - D = 0, \end{aligned}$$

$$E_3 = G(2)^3 - F(3)^2 F(7) - D = D^3 + 3D^2\mu + 3D\mu^2 - \mu^2 F(7) + \mu^3 - D = 0,$$

$$E_4 = G(3)^2 - F(2)^4 F(5) - D = \omega^6 - \omega^4 F(5) + 2D\omega^3 + D^2 - D = 0,$$

$$\begin{aligned} E_5 &= G(3)G(5) - F(2)^5 F(7) - D = \\ &= \mu\omega^6 - \omega^5 F(7) + D\mu\omega^3 + D\omega^3 + D^2 - D = 0, \end{aligned}$$

$$\begin{aligned} E_6 &= G(2)G(13) - F(3)^3 F(5)^2 - D = \\ &= DF(7)\mu\omega^3 + F(7)\mu^2\omega^3 - \mu^3 F(5)^2 + D^2 + \mu D - D = 0, \end{aligned}$$

$$\begin{aligned} E_7 &= G(7)^2 - F(2)^5 F(3) F(5)^2 - D = \\ &= \mu^2\omega^8 - \omega^5\mu F(5)^2 + 2D\mu\omega^4 + D^2 - D = 0, \end{aligned}$$

$$\begin{aligned} E_8 &= G(5)G(11) - F(2)^4 F(3)^3 F(7) - D = \\ &= F(5)\mu^2\omega^6 - \omega^4\mu^3 F(7) + DF(5)\mu\omega^3 + D\mu\omega^3 + D^2 - D = 0. \end{aligned}$$

Lemma 1. *Assume that $D = 0$ and the functions $G, F \in \mathcal{M}^*$ satisfy the relation (1.1). Then we have*

(a) *If $F(3) \neq 0$, then $G = F = \mathbb{E}$,*

(b) *If $F(3) = 0$, then $G = \mathbb{O}$ and*

$$F(2) = F(3) = F(5) = F(7) = 0, \quad F(n^2 - 1) = 0 \quad \text{for every} \quad n \in \mathbb{N}_1.$$

Proof. (a) Assume that $\mu = F(3) \neq 0$. We shall prove that $G = F = \mathbb{E}$.

Since $D = 0$ and $\mu \neq 0$, the equations

$$E_1 = -\mu(F(5) - \mu) = 0 \quad \text{and} \quad E_3 = -\mu^2(F(7) - \mu) = 0$$

imply that

$$F(5) = \mu \quad \text{and} \quad F(7) = \mu.$$

Then

$$E_2 = D\omega^3 + \omega^3\mu + \mu D - F(5)F(7) + D^2 - D = \omega^3\mu - F(5)F(7) = \mu(\omega^3 - \mu) = 0,$$

which shows that

$$\mu = \omega^3 \quad \text{and} \quad \omega \neq 0.$$

Consequently

$$F(7) = \mu = \omega^3,$$

$$E_5 = \mu\omega^6 - \omega^5 F(7) + D\mu\omega^3 + D\omega^3 + D^2 - D = \omega^6\mu - \omega^5 F(7) = \omega^8(\omega - 1) = 0$$

from which $\omega = 1$ follows. Thus the above relations with (2.1) imply that $\mu = \omega^3 = 1$, $F(2) = F(3) = F(5) = F(7) = \mu = 1$ and $G(2) = G(3) = G(5) = 1$.

Now we shall prove that $G = F = \mathbb{E}$.

Assume that $G(n) = F(n) = 1$ for every $n < P$, where $P > 5$. It is obvious that $G(P) = F(P) = 1$ if $P \notin \mathcal{P}$. Thus we may assume that $P \in \mathcal{P}, P \geq 7$. Then we infer from (1.1) that

$$G(P) = F(P-1)F(P+1) + D = F(2)F(P-1)F\Big(\frac{P+1}{2}\Big) = 1$$

and

$$1 = G(P-1) = F(P-2)F(P) + D = F(P),$$

which proves that $G(P) = F(P) = 1$. The proof of (a) is complete.

(b) Assume that $D = 0$ and $\mu = F(3) = 0$. In this case we prove that $\omega = F(2) = 0$.

Assume in contradiction that $\omega \neq 0$. Then it follows from E_4 that

$$\begin{aligned} E_4 &= \omega^6 - \omega^4 F(5) + 2D\omega^3 + D^2 - D = \\ &= \omega^6 - \omega^4 F(5) = \omega^4(\omega^2 - F(5)) = 0, \end{aligned}$$

which implies that

$$F(5) = \omega^2 \neq 0.$$

Since

$$E_2 = D\omega^3 + \omega^3\mu + \mu D - F(5)F(7) + D^2 - D = -F(5)F(7) = -\omega^2 F(7) = 0$$

we have $F(7) = 0$. These are impossible, because

$$G(3) = F(2)F(4) = \omega^3$$

and

$$0 = G(27) - F(26)F(28) - D = G(3)^3 = F(2)^9 = \omega^9 \neq 0.$$

Thus, we proved that $\omega = \mu = 0$. Then $G(3) = \omega^3 = 0$ and $F(2) = F(3) = F(5) = F(7) = 0$. By using the fact $n^2 - 1 \equiv 0 \pmod 3$ if $(n, 3) = 1$, we have

$$G(n) = F(n^2 - 1) + D = 0 \quad \text{if} \quad n \in \mathbb{N}_1 \ \text{and} \ (n, 3) = 1.$$

Therefore $G(n) = 0$ for every $n \in \mathbb{N}_1$, that is $G = \mathbb{O}$. Then $F(n^2 - 1) = G(n) - D = 0$ for every $n \in \mathbb{N}_1$. The proof of Lemma 1 is complete. ∎

Lemma 2. *Assume that $D \in \mathbb{C} \setminus \{0\}$ and the functions $G, F \in \mathcal{M}^*$ satisfy the relation (1.1). Then we have*

(c) *If $F(2) = F(3) = 0$, then $D = 1$, $G(n) = 1$ for every $n \in \mathbb{N}$ and $F(n^2 - 1) = 0$ for every $n \in \mathbb{N}_1$.*

(d) *If $F(2) = 0$ and $F(3) \neq 0$, then $(D, G, F) = (1, \chi_2, \chi_4^*)$.*

(e) *If $\omega = F(2) \neq 0$ and $F(3) = 0$, then $\omega^3 = -1$ and $(D, G, F) = (1, \chi_3, \Psi_\omega)$.*

Proof. (c) Assume that $F(2) = F(3) = 0$, i.e. $\omega = \mu = 0$. Then we have

$$E_4 = \omega^6 - \omega^4 F(5) + 2D\omega^3 + D^2 - D = D^2 - D = 0,$$

which with $D \neq 0$ implies that $D = 1$. Then $G(n) = F(n^2 - 1) + 1$ for every $n \in \mathbb{N}_1$.

Since $\omega = F(2) = 0$, we have $G(3) = F(2)^3 + 1 = 1$. It is well-known that $n^2 - 1 \equiv 0 \pmod 3$ if $(n, 3) = 1$, therefore we infer from $\mu = F(3) = 0$ that

$$G(n) = F(n^2 - 1) + 1 = 1 \quad \text{if} \quad (n, 3) = 1.$$

Consequently

$$G(n)=1 \quad \text{for every} \quad n\in\mathbb{N} \quad \text{and} \quad F(n^2-1)=0 \quad \text{for every} \quad n\in\mathbb{N}_1.$$

The proof of (c) is finished.

(d) Now assume that $F(2)=\omega=0$ and $F(3)=\mu\neq 0$. Then

$$E_4=\omega^6-\omega^4F(5)+2D\omega^3+D^2-D=D^2-D=0,$$

which with $D\neq 0$ gives $D=1$. Now we have

$$E_1=D^2+2\mu D-\mu F(5)+\mu^2-D=-\mu(-\mu+F(5)-2)=0$$

and

$$E_3=D^3+3D^2\mu+3D\mu^2-\mu^2F(7)+\mu^3-D=-\mu(-\mu^2+F(7)\mu-3\mu-3)=0,$$

which imply

$$F(5)=\mu+2 \quad \text{and} \quad F(7)=\frac{\mu^2+3\mu+3}{\mu}. \tag{2.2}$$

Therefore

$$\begin{aligned}E_2&=D\omega^3+\omega^3\mu+\mu D-F(5)F(7)+D^2-D=\\&=-\frac{\mu^3+5\mu^2+9\mu-\mu^2+6}{\mu}=-\frac{(\mu+1)(\mu^2+3\mu+6)}{\mu}=0\end{aligned}$$

and

$$\begin{aligned}E_6&=DF(7)\mu\omega^3+F(7)\mu^2\omega^3-\mu^3F(5)^2+D^2+\mu D-D=\\&=-\mu(\mu+1)^2(\mu^2+2\mu-1)=0.\end{aligned}$$

These with $\mu\neq 0$ imply that $\mu=-1$, because in the other case $\mu^2+3\mu+6= =0$ and $\mu^2+2\mu-1=0$ would satisfied, which are impossible. Therefore, we infer from (2.1) and (2.2) that

$$F(2)=0, F(3)=-1, F(5)=1, F(7)=-1$$

and

$$G(2)=0, G(3)=G(5)=G(7)=1.$$

We shall prove that in this case $(G,F)=(\chi_2,\chi_4^*)$.

We have $F(n)=\chi_4^*(n)$ and $G(n)=\chi_2(n)$ for $n\in\mathbb{N}, n\leq 7$. Assume that $F(n)=\chi_4^*(n)$ and $G(n)=\chi_2(n)$ hold for every $n<P$, where $P\in\mathbb{N}, P>7$.

It is obvious that $F(P) = \chi_4^*(P)$ and $G(P) = \chi_2(P)$ if $P \notin \mathcal{P}$. Thus we may assume that $P \in \mathcal{P}$. Then

$$G(P) = F(P-1)F(P+1) + D = F(2)F(P-1)F\Big(\frac{P+1}{2}\Big) + 1 = 1 = \chi_2(P)$$

and

$$\begin{aligned} 0 = \chi_2(P-1) = G(P-1) = F(P-2)F(P) + D = \\ = \chi_4^*(P-2)F(P) + 1 = -\chi_4^*(P)F(P) + 1, \end{aligned}$$

because $\chi_4^*(k-2) = -\chi_4^*(k)$ for every $k \in \mathbb{N}$. The above relation shows that $\chi_4^*(P)F(P) = 1$, which with $(\chi_4^*(P))^2 = 1$ implies that $F(P) = \chi_4^*(P)$.

The proof of (d) is complete.

(e) Assume that $\omega \neq 0$ and $\mu = 0$. First we prove that

$$\omega^3 = -1. \tag{2.3}$$

Since $\mu = 0$, we have

$$E_1 = D^2 + 2\mu D - \mu F(5) + \mu^2 - D = D^2 - D = 0,$$

which with $D \neq 0$ shows that $D = 1$. Then

$$E_5 = \mu\omega^6 - \omega^5 F(7) + D\mu\omega^3 + D\omega^3 + D^2 - D = \omega^3\Big(1 - \omega^2 F(7)\Big) = 0,$$

consequently

$$F(7) = \frac{1}{\omega^2} \tag{2.4}$$

and

$$\begin{aligned} E_2 = D\omega^3 + \omega^3\mu + \mu D - F(5)F(7) + D^2 - D = \omega^3 - F(5)F(7) = \\ = \omega^3 - \frac{F(5)}{\omega^2} = \frac{\omega^5 - F(5)}{\omega^2} = 0. \end{aligned}$$

The last relation implies that

$$F(5) = \omega^5. \tag{2.5}$$

Now we apply (1.1) for $n \in \{21, 351\}$, we shall obtain from (2.1), (2.4) and (2.5) that

$$G(3) = \omega^3 + 1,\ G(7) = G(13) = 1,\ F(5) = \omega^5,\ F(7) = \frac{1}{\omega^2} \tag{2.6}$$

and so

$$0 = G(21) - bF(20)F(22) - D = G(3)G(7) - F(2)^3F(5)F(11) - 1 =$$
$$= (\omega^3 + 1) - \omega^8 F(11) - 1 = \omega^3(1 - \omega^5 F(11)).$$

This with $\omega \neq 0$ implies that $F(11) = \frac{1}{\omega^5}$, consequently

$$\begin{aligned} 0 &= G(351) - F(350)F(352) - D = \\ &= G(3)^3 G(13) - F(2)^6 F(5)^2 F(7) F(11) - 1 = \\ &= (\omega^3 + 1)^3 - \frac{\omega^{16}}{\omega^7} - 1 = 3\omega^3(\omega + 1)(\omega^2 - \omega + 1). \end{aligned}$$

Since $\omega \neq 0$, the last relation shows that

$$\omega^3 + 1 = (\omega + 1)(\omega^2 - \omega + 1) = 0,$$

which proves (2.3).

It follows from (2.3) that $G(3) = \omega^3 + 1 = 0$. Since $\mu = F(3) = 0$ and $n^2 - 1 \equiv 0 \pmod 3$ if $(n, 3) = 1$, we infer from (1.1) that

$$G(n) = F(n^2 - 1) + D = D = 1 \quad \text{if} \quad (n, 3) = 1.$$

This implies that $G = \chi_3$.

Now we prove that $F(n) = \Psi_\omega(n)$ for every $n \in \mathbb{N}$. Since $F(3) = \Psi_\omega(3) = 0$ and $F, \Psi_\omega \in \mathcal{M}^*$, we have $F(n) = \Psi_\omega(n) = 0$ if $3|n$.

It follows from (1.1) that

$$0 = \chi_3(3m) = G(3m) = F(3m - 1)F(3m + 1) + 1 \quad \text{for every} \quad m \in \mathbb{N},$$

which implies that

$$F(3m - 1)F(3m + 1) = -1 = \omega^3 \quad \text{for every} \quad m \in \mathbb{N}. \tag{2.7}$$

It is clear to check from (2.7) that

$$\begin{aligned} F(3m + 1)F(3m + 2) &= \frac{F(6m + 2)F(6m + 4)}{\omega^2} = \\ &= \frac{F\Big(3(2m + 1) - 1\Big)F\Big(3(2m + 1) + 1\Big)}{\omega^2} = \frac{\omega^3}{\omega^2} = \omega \end{aligned} \tag{2.8}$$

holds for every $m \in \mathbb{N}$. Thus we infer from (2.7) and (2.8) that

$$F(3m - 1)F(3m + 1) = \omega^3 = \omega^2 \cdot \omega = \omega^2 F(3m + 1)F(3m + 2),$$

which with $\omega^6 = 1$ implies that

$$F\big(3(m+1)-1\big) = F(3m+2) = \omega^4 F(3m-1) \quad \text{for every} \quad m \in \mathbb{N}.$$

Hence

$$F(3m+2) = \omega^{4m+1} \quad \text{for every} \quad m \in \mathbb{N},$$

and so (2.8) shows that

$$\omega = F(3m+1)F(3m+2) = \omega^{4m+1}F(3m+1) = \omega^{-2m+1}F(3m+1).$$

Since $\omega^{6m} = 1$ for every $m \in \mathbb{N}$, we obtain from the above relation that

$$F(3m+1) = \omega^{2m} \quad \text{for every} \quad m \in \mathbb{N}.$$

Thus, we have proved that $F = \Psi_\omega$.

The proof of (e) is complete. ∎

Lemma 3. *Assume that $D \in \mathbb{C} \setminus \{0\}$ and the functions $G, F \in \mathcal{M}^*$ satisfy the relation (1.1). If $F(2)F(3) \neq 0$, then*

$$\Big(D, F(2), F(3)\Big) \in \Big\{(-1,1,1), (1,2,3)\Big\}.$$

Proof. Let $F(2) = \omega$ and $F(3) = \mu$. Then it follows from our assumptions that $D\omega\mu \neq 0$.

Now we infer from E_1 and E_3 that

$$F(5) = \frac{\mu^2 + 2D\mu + D^2 - D}{\mu} \quad \text{and} \quad F(7) = \frac{\mu^3 + 3D\mu^2 + 3D^2\mu + D^3 - D}{\mu^2}.$$

By using a computer, we obtain the following equations:

$$F_1 = \mu E_4 = \mu\omega^6 - D^2\omega^4 - 2D\mu\omega^4 - \mu^2\omega^4 + 2D\mu\omega^3 + D\omega^4 + D^2\mu - D\mu = 0$$

$$\begin{aligned} F_2 = \mu^2 E_5 = \mu^3\omega^6 &- D^3\omega^5 - 3D^2\mu\omega^5 - 3D\mu^2\omega^5 - \mu^3\omega^5 + D\mu^3\omega^3 + D\mu^2\omega^3 + \\ &+ D\omega^5 + D^2\mu^2 - D\mu^2 = 0 \end{aligned}$$

$$\begin{aligned} F_3 = \mu E_7 = \mu^3\omega^8 &- D^4\omega^5 - 4D^3\mu\omega^5 - 6D^2\mu^2\omega^5 - 4D\mu^3\omega^5 - \mu^4\omega^5 + \\ &+ 2D^3\omega^5 + 4D^2\mu\omega^5 + 2D\mu^2\omega^5 - D^2\omega^5 + 2D\mu^2\omega^4 + \\ &+ D^2\mu - D\mu = 0 \end{aligned}$$

$$\begin{aligned} F_4 = E_8 = D^2\mu\omega^6 &+ 2D\mu^2\omega^6 + \mu^3\omega^6 - D^3\mu\omega^4 - 3D^2\mu^2\omega^4 - 3D\mu^3\omega^4 - \\ &- D\mu\omega^6 - \mu^4\omega^4 + D^3\omega^3 + 2D^2\mu\omega^3 + D\mu^2\omega^3 + D\mu\omega^4 - D^2\omega^3 + \\ &+ D\mu\omega^3 + D^2 - D = 0. \end{aligned}$$

We now follow the method which was used in [5] to prove Lemma 3.

Let $f(x,y,z)\in\mathbb{Q}[x,y,z]$ be a polynomial in three variables x,y,z. Then we can write the polynomial $f(x,y,z)$ in the following form

$$f(x,y,z)=a_0(x,y)z^k+a_1(x,y)z^{k-1}+\ldots+a_k(x,y),$$

where $a_i(x,y)\in\mathbb{Q}[x,y]\ \ (i=1,\ldots,k)$ and we write $f(x,y,z)\in(\mathbb{Q}[x,y])[z]$.

Let $a(x,y,z),b(x,y,z)$ be polynomials in $(\mathbb{Q}[x,y])[z]$ with $b(x,y,z)\neq 0$. Then unique polynomials exist $q(x,y,z)$ and $r(x,y,z)$ in $(\mathbb{Q}[x,y])[z]$ with

$$a(x,y,z)=q(x,y,z)b(x,y,z)+r(x,y,z)$$

and such that the degree of $r(x,y,z)$ is smaller than the degree of $b(x,y,z)$ in z. The polynomials $q(x,y,z)$ and $r(x,y,z)$ are uniquely determined by $a(x,y,z)$ and $b(x,y,z)$. Similarly, we can define $rem(a(x,y,z),b(x,y,z),x)$ and $rem(a(x,y,z),b(x,y,z),y)$. Now let

$$\begin{cases} F_1(x,y,z)=x^6y-x^4z^2-2x^4yz-x^4y^2+x^4z+2x^3yz+yz^2-yz,\\ F_2(x,y,z)=x^6y^3-x^5z^3-3x^5yz^2-3x^5y^2z-x^5y^3+x^3y^3z+x^5z+\\ \qquad +x^3y^2z+y^2z^2-y^2z,\\ F_3(x,y,z)=x^8y^3-x^5z^4-4x^5yz^3-6x^5y^2z^2-4x^5y^3z-x^5y^4+2x^5z^3+\\ \qquad +4x^5yz^2+2x^5y^2z-x^5z^2+2x^4y^2z+yz^2-yz,\\ F_4(x,y,z)=x^6yz^2+2x^6y^2z+x^6y^3-x^4yz^3-3x^4y^2z^2-x^6yz-\\ \qquad -3x^4y^3z-x^4y^4+x^3z^3+2x^3yz^2+x^4yz+x^3y^2z-x^3z^2+\\ \qquad +x^3yz+z^2-z.\end{cases}$$

Let

$$\begin{cases} a_1(x,y,z)=rem\left(\frac{-(x^4-y)^2}{x^3y}F_2(x,y,z),F_1(x,y,z),z\right),\\ a_2(x,y,z)=rem\left(\frac{-(x^4-y)^3}{x^3y^2}F_3(x,y,z),F_1(x,y,z),z\right),\\ a_3(x,y,z)=rem\left(\frac{(x^4-y)^2}{x^3y}F_4(x,y,z),F_1(x,y,z),z\right).\end{cases}$$

By using a computer we can determinate the polynomials $a_1(x,y,z),a_2(x,y,z)$ and $a_3(x,y,z)$ as follows:

$$\begin{aligned} a_1(x,y,z)&=x^{12}y+x^{12}z-x^{11}y^2+x^{12}+2x^{11}y-x^{10}y-x^{10}z-2x^9y^2-\\ &-2x^9yz-x^8y^2z+4x^9z-3x^8y^2+2x^8yz+2x^7y^3-x^8y-x^7y^2+x^6y^3+\\ &+x^6y^2z+x^6y^2-2x^6yz+x^5y^3-4x^5y^2z+2x^4y^3z-4x^5yz-x^3y^4+x^3y^3+\\ &+x^2y^4+3x^2y^3z+3x^2y^2z-xy^4-2xy^3z-y^4z+y^3z,\end{aligned}$$

$$a_2(x,y,z) = x^{18} - x^{17}y + 4x^{15}z + 3x^{14}y + 3x^{13}y^2 - 2x^{13}z - 6x^{12}y^2 - \\ - 12x^{12}yz + 4x^{12}z - 8x^{11}y^2 + 4x^{11}yz + 5x^9y^3 + 12x^9y^2z - x^{11} - 2x^9yz + \\ + 6x^8y^3 - 12x^8y^2z + x^9y + 2x^9z - 4x^8yz - 3x^6y^4 - 4x^6y^3z - 2x^8z + \\ + 4x^6y^2z + x^5y^4 + 12x^5y^3z + 2x^7y + 2x^5y^2z - 2x^5y^2 - 4x^5yz - x^2y^5 - \\ - 4x^2y^4z + 4x^4yz - 4x^2y^3z - x^3y^2 + 2xy^3z + xy^3 + 2xy^2z - 2y^2z,$$

$$a_3(x,y,z) = x^{13}y - x^{11}y^2 - x^{11}yz - x^{11}y - 2x^{10}y^2 + 2x^{10}yz + x^{10}z + x^9yz + \\ + 2x^8y^3 + 2x^8y^2z + 2x^9y - 4x^8yz + 2x^7y^3 - 5x^7y^2z + x^8z - x^7y^2 - 4x^7yz - \\ - 2x^6y^2z - x^5y^4 - x^5y^3z + 2x^7z - 2x^6y^2 + 3x^6yz - x^5y^3 + 2x^5y^2z + \\ + 6x^4y^3z + x^7 + 2x^4y^3 + 7x^4y^2z + x^3y^4 + 2x^3y^3z - x^5y - 2x^5z - 4x^4yz - \\ - 4x^3y^2z - xy^5 - 3xy^4z + 2x^4z - 2x^3yz - 3xy^3z - x^3y + y^3z + xy^2 + \\ + 2xyz + 3y^2z - 2yz.$$

In the next step, we compute the following polynomials:

$$b_1(x,y,z) = rem\big((x^{12} - x^{10} - 2x^9y - x^8y^2 + 4x^9 + 2x^8y + x^6y^2 - 2x^6y - \\ - 4x^5y^2 + 2x^4y^3 - 4x^5y + 3x^2y^3 + 3x^2y^2 - \\ - 2xy^3 - y^4 + y^3)^2F_1(x,y,z), a_1(x,y,z), z\big) = \\ = x(x^4-y)^2(x^{21}y - x^{17}y^4 - 3x^{19}y - 3x^{18}y^2 + 4x^{17}y^3 - 2x^{19} - 2x^{17}y^2 + \\ + 2x^{14}y^5 + 3x^{17}y + 11x^{16}y^2 + 4x^{15}y^3 - 15x^{14}y^4 + 2x^{13}y^5 + x^{17} + 2x^{16}y + \\ + 9x^{15}y^2 + 7x^{14}y^3 - 2x^{13}y^4 - x^{11}y^6 - 4x^{16} - 2x^{15}y - 7x^{14}y^2 - 18x^{13}y^3 - \\ - 7x^{12}y^4 + 20x^{11}y^5 - 4x^{10}y^6 + 4x^{14}y - x^{13}y^2 - 14x^{12}y^3 - x^{11}y^4 + 5x^{10}y^5 - \\ - x^9y^6 + 2x^{13}y + 3x^{12}y^2 - 7x^{11}y^3 + 5x^{10}y^4 + 12x^9y^5 - 14x^8y^6 + 2x^7y^7 + \\ + 4x^{12}y + 2x^{11}y^2 + 9x^{10}y^3 + 18x^9y^4 - 11x^8y^5 + 2x^7y^6 + 2x^6y^7 - 4x^{10}y^2 - \\ - 7x^9y^3 + 4x^8y^4 + 4x^7y^5 - 13x^6y^6 + 4x^5y^7 - 3x^9y^2 - 4x^7y^4 - 6x^6y^5 + \\ + 11x^5y^6 - 2x^4y^7 - x^3y^8 - 2x^7y^3 - 4x^6y^4 + 5x^5y^5 - 7x^4y^6 + 3x^3y^7 + \\ + 5x^5y^4 + 4x^3y^6 - x^2y^7 + 3x^4y^4 + x^3y^5 - 7x^2y^6 - 2x^2y^5 + 3xy^6 + y^7 - y^6),$$

$$b_2(x,y,z) = rem\big(-4(2x^{15} - x^{13} - 6x^{12}y + 2x^{12} + 2x^{11}y + 6x^9y^2 - x^9y - \\ - 6x^8y^2 + x^9 - 2x^8y - 2x^6y^3 - x^8 + 2x^6y^2 + 6x^5y^3 + x^5y^2 - 2x^5y - 2x^2y^4 + \\ + 2x^4y - 2x^2y^3 + xy^3 + xy^2 - y^2)^2F_1(x,y,z), a_2(x,y,z), z)\big) = \\ = x(x^4-y)^3\big(x^{27} - 2x^{26}y + x^25y^2 - 8x^{24}y + 8x^{23}y^2 + 4x^{24} - 4x^{23}y - \\ - 12x^{22}y^2 - 4x^{21}y^3 + 4x^{22}y + 24x^{21}y^2 - 12x^{20}y^3 - 2x^{22} - 6x^{21}y + 24x^{20}y^2 + \\ + 48x^{19}y^3 + 4x^{21} - 4x^{20}y - 24x^{19}y^2 - 58x^{18}y^3 + 14x^{17}y^4 - 2x^{20} - 2x^{19}y - \\ - 16x^{17}y^3 - 72x^{16}y^4 + 2x^{18}y + 10x^{17}y^2 + 20x^{16}y^3 + 78x^{15}y^4 - 2x^{14}y^5 + \\ + 2x^{18} - 2x^{17}y - 4x^{16}y^2 + 16x^{15}y^3 + 12x^{14}y^4 + 44x^{13}y^5 - 6x^{17} - 2x^{15}y^2 -$$

$$
\begin{aligned}
&- 20x^{14}y^3 - 20x^{13}y^4 - 68x^{12}y^5 + 8x^{15}y + 8x^{14}y^2 - 2x^{13}y^3 - 12x^{12}y^4 - \\
&- 12x^{10}y^6 + 2x^{15} + 6x^{14}y + 10x^{13}y^2 + 24x^{12}y^3 + 24x^{11}y^4 - 4x^{10}y^5 + 37x^9y^6 - \\
&- 4x^{14} - 4x^{13}y - 18x^{12}y^2 - 30x^{11}y^3 - 2x^{10}y^4 + 24x^9y^5 + x^{13} + 4x^{12}y - \\
&- 12x^{11}y^2 - 12x^{10}y^3 - 2x^9y^4 - 20x^8y^5 - 10x^6y^7 + 20x^10y^2 + 14x^9y^3 + \\
&+ 6x^8y^4 - 12x^6y^6 - 2x^11 + 4x^{10}y - x^9y^2 - 6x^8y^3 + 14x^7y^4 - 2x^6y^5 + 4x^5y^6 + \\
&+ x^3y^8 + 2x^{10} - 8x^8y^2 - 12x^7y^3 - 2x^6y^4 + 4x^3y^7 - 2x^8y + 2x^7y^2 + 8x^5y^4 - \\
&- 2x^4y^5 + 4x^3y^6 + 4x^7y + 6x^4y^4 - 4x^2y^6 - 4x^6y - 3x^5y^2 + 4x^4y^3 - 6x^2y^5 + \\
&+ 4x^4y^2 - 4x^3y^3 - 4x^2y^4 + 2xy^5 - 2x^3y^2 + 3xy^4 + 2x^2y^2 + 2xy^3 - 2y^3\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
b_3(x,y,z) &= rem\Big(-(x^{11}y - 2x^{10}y - x^{10} - x^9y - 2x^8y^2 + 4x^8y + 5x^7y^2 - x^8 + \\
&+ 4x^7y + 2x^6y^2 + x^5y^3 - 2x^7 - 3x^6y - 2x^5y^2 - 6x^4y^3 - 7x^4y^2 - 2x^3y^3 + \\
&+ 2x^5 + 4x^4y + 4x^3y^2 + 3xy^4 - 2x^4 + 2x^3y + 3xy^3 - y^3 - 2xy - \\
&- 3y^2 + 2y)^2 F_1(x,y,z), a_3(x,y,z), z\Big) = \\
&= x(x^2 - y)(x^4 - y)^2\Big(x^{19}y^2 - 3x^{17}y^2 - 6x^{16}y^3 + 2x^{16}y^2 - x^{15}y^3 + x^{16}y + \\
&+ 5x^{15}y^2 + 12x^{14}y^3 + 15x^{13}y^4 - x^{15}y - 6x^{14}y^2 - 3x^{13}y^3 + 4x^{12}y^4 - 9x^{13}y^2 - \\
&- 16x^{12}y^3 - 18x^{11}y^4 - 20x^{10}y^5 + 2x^{13}y + 11x^{12}y^2 + 6x^{11}y^3 - 6x^{10}y^4 - \\
&- 6x^9y^5 - 3x^{12}y + 10x^{11}y^2 + 26x^{10}y^3 + 17x^9y^4 + 12x^8y^5 + 15x^7y^6 - \\
&- 8x^{11}y - 9x^{10}y^2 - 6x^9y^3 + 2x^8y^4 + 16x^7y^5 + 4x^6y^6 + 4x^{10}y - 4x^9y^2 - \\
&- 14x^8y^3 - 25x^7y^4 - 6x^6y^5 - 3x^5y^6 - 6x^4y^7 + x^{10} + 5x^9y + 8x^8y^2 - 2x^7y^3 - \\
&- 10x^6y^4 - 2x^5y^5 - 12x^4y^6 - x^3y^7 - 4x^8y + x^7y^2 + 13x^6y^3 + 3x^5y^4 + 3x^4y^5 + \\
&+ xy^8 + x^8 - 6x^7y - 6x^6y^2 + x^5y^3 + 11x^4y^4 + 9x^3y^5 + 3xy^7 + 3x^7 + 3x^6y + \\
&+ x^4y^3 - 4x^3y^4 + 3xy^6 - x^5y + 7x^4y^2 + x^3y^3 - 2xy^5 - y^6 - 2x^5 - 4x^4y - \\
&- 2x^3y^2 - 5xy^4 - 3y^5 + 2x^4 - 3x^3y - 3xy^3 + 2y^4 + xy^2 + y^3 + \\
&+ 2xy + 3y^2 - 2y\Big).
\end{aligned}
$$

Since $F_1(\omega,\mu,D) = F_1 = 0$, $F_2(\omega,\mu,D) = F_2 = 0$, $F_3(\omega,\mu,D) = F_3 = 0$ and $F_4(\omega,\mu,D) = F_4 = 0$, it easy to check that

$$a_1(\omega,\mu,D) = 0, a_2(\omega,\mu,D) = 0, a_3(\omega,\mu,D) = 0$$

and

$$b_1(\omega,\mu,D) = 0, b_2(\omega,\mu,D) = 0, b_3(\omega,\mu,D) = 0.$$

If $\mu = \omega^4$, then $F_1(\omega,\mu,D) = -\omega^7(\omega-1)(\omega^4+\omega^3+2D) = 0$, which implies either $\omega = 1$ or $Q = \omega^4+\omega^3+2D = 0$. If $\omega = 1$, then $F_2 = -D(D+1)^2 = 0$, which with $D \neq 0$ implies that $D = -1$. Thus we have $(D, F(2), F(3)) = (-1, 1, 1)$.

Now assume that $\mu = \omega^4$ and $Q = \omega^4 + \omega^3 + 2D = 0$. Let $Q(x, z) = x^4 + x^3 + 2z$. Then we infer from $F_2(x, y, z), F_3(x, y, z), F_4(x, y, z)$ with $y = x^4$ that

$$\begin{aligned} P_1(x) &= rem(-8F_2(x, y, z), Q(x, z), z) = \\ &= x^8(x - 1)(4x^{10} + x^8 - 4x^7 - x^6 - 4x^3 - 8x^2 - 8x - 4) \end{aligned}$$

and

$$\begin{aligned} P_2(x) &= rem(-16F_3(x, y, z), Q(x, z), z) = \\ &= x^7(x - 1)(x^{13} - 19x^{12} - 13x^{11} - 17x^{10} - 12x^9 + \\ &+ 12x^7 + 16x^6 + 20x^5 + 24x^4 + 20x^3 + 16x^2 + 16x + 8). \end{aligned}$$

Since $P_1(\omega) = P_2(\omega) = 0$ and $gcd(P_1(x), P_2(x)) = x^7(x - 1)$, therefore these relations with $\omega \neq 0$ imply that $\omega = 1$ and $Q = 2 + 2D = 0, D = -1$. Thus we also have $(D, F(2), F(3)) = (-1, 1, 1)$.

It can check as above that if $\mu = \omega^2$, then $(D, F(2), F(3)) = (-1, 1, 1)$.

In the following we can assume that $\omega(\omega^4 - \mu)(\omega^2 - \mu) \neq 0$. We consider the following system of equations

$$\text{(2.9)} \qquad \begin{cases} B_1(x, y, z) = \dfrac{b_1(x, y, z)}{x(x^4 - y)^2} = 0, \\ B_2(x, y, z) = \dfrac{b_2(x, y, z)}{x(x^4 - y)^3} = 0, \\ B_3(x, y, z) = \dfrac{b_3(x, y, z)}{x(x^2 - y)(x^4 - y)^2} = 0. \end{cases}$$

With help of a computer and Maple program, the system (2.9) has three solutions: $(x, y) \in \{(0, 0), (1, 1), (2, 3)\}$. Since

$$B_1(\omega, \mu, D) = B_2(\omega, \mu, D) = B_3(\omega, \mu, D) = 0 \quad \text{and} \quad \omega\mu \neq 0,$$

we have $(\omega, \mu) \in \{(1, 1), (2, 3)\}$.

If $(\omega, \mu) = (1, 1)$, then $F_2 = -D(D + 1)^2 = 0$, consequently $D = -1$.

If $(\omega, \mu) = (2, 3)$, then

$$F_1 = -(13D + 48)(D - 1) = 0 \quad \text{and} \quad F_2 = -(D - 1)(32D^2 + 311D + 864) = 0.$$

These imply that $D = 1$, and so $(D, \omega, \mu) \in \{(-1, 1, 1), (1, 2, 3)\}$.

Lemma 3 is proved. ∎

3. Proof of Theorem 1

To complete the proof of Theorem 1, we need to prove the assertion (f).

By using Lemma 3, we have $(D, \omega, \mu) \in \{(-1, 1, 1), (1, 2, 3)\}$.

◦ Assume that $(D, \omega, \mu) = (-1, 1, 1)$. Then $G(2) = F(3) - 1 = 0$ and $G(3) = F(2)F(4) - 1 = F(2)^3 - 1 = 0$. Assume that $G(n) = 0$ and $F(n) = 1$ for every $n < P$, where $P > 3$. It is obvious that $G(P) = 0$ and $F(P) = 1$ if $P \notin \mathcal{P}$. Therefore we may assume that $P \in \mathcal{P}, P \geq 5$. Then

$$G(P) = F(P-1)F(P+1) - 1 = F(2)F(P-1)F\Big(\frac{P-1}{2}\Big) - 1 = 1 - 1 = 0$$

and

$$0 = G(P-1) = F(P-2)F(P) - 1 = F(P) - 1, \ F(P) = 1.$$

Thus $(D, G, F) = (1, \mathbb{O}, \mathbb{E})$.

◦ Assume that $(D, \omega, \mu) = (1, 2, 3)$. Then we infer from (2.1) that $G(2) = \mu + D = 2^2, G(3) = \omega^3 + 1 = 3^2$ and $G(7) = \omega^4\mu + D = 7^2$. On the other hand, we obtain from E_1, E_2 that $F(5) = 5, F(7) = 7$. Assume that $G(n) = n^2$ and $F(n) = n$ for every $n < P$, where $P > 7$. Similarly as above, we can assume that $P \in \mathcal{P}, P \geq 11$. Then

$$G(P) = F(P-1)F(P+1) + 1 = F(2)F(P-1)F\Big(\frac{P-1}{2}\Big) + 1 = \\ = 2(P-1)\Big(\frac{P-1}{2}\Big) + 1 = P^2$$

and

$$(P-1)^2 = G(P-1) = F(P-2)F(P) + 1 = (P-2)F(P) + 1, \ F(P) = P.$$

Thus we proved that $(D, G, F) = (1, \mathbb{I}^2, \mathbb{I})$.

The proof of Theorem 1 is complete. ■

References

[1] **Kátai, I., B.M.M. Khanh and B.M. Phong,** On the equation $F(n^2 + m^2 + k) = H(n) + H(m) + K$, *J. Math. Math. Sci.*, **2022(1)**, 132–148.

[2] **Kátai, I. and B.M. Phong,** Completely multiplicative functions with special properties , *Mathematica Pannonica New Series*, (2022), (submitted).

[3] **Khanh, B.M.M.,** On the equation $f(n + Dm) = f(n) + Df(m)$, *Annales Univ. Sci. Budapest., Sect. Comp.*, **44** (2015), 59–68.

[4] **Khanh, B.M.M.,** On conjecture concerning the functional equation, *Annales Univ. Sci. Budapest., Sect. Comp.*, **46** (2017), 123–135.

[5] **Khanh, B.M.M.,** On the equation $f(n^2+Dm^2+k) = f(n)^2+Df(m)^2+k$, *Annales Univ. Sci. Budapest., Sect. Comp.*, **52** (2021), 195–216.

[6] **Spiro, C.,** Additive uniqueness sets for arithmetic functions, *J. Number Theory*, **42** (1992), 232–246.

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