IN MEMORIAM FOR PROFESSOR EDUARD WIRSING

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Abstract. We mention some results of Professor Eduard Wirsing concerning the first author's conjectures. We also list some our results for arithmetical functions and prove three theorems.

1. On the characterization of $\log n$

Professor E. Wirsing gave very important results for some of our conjectures. In this short survey paper we would like to elaborate on that.

In the following let $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of primes, positive integers, integers, real numbers and complex numbers, respectively. An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ is said to be additive if $(n, m) = 1$ implies that $f(nm) = f(n) + f(m)$ and it is completely additive if this equality holds for all positive integers n and m. Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of all real-valued additive and completely additive functions, respectively. Similarly, an arithmetic function $g : \mathbb{N} \to \mathbb{C}$ is said to be multiplicative if $(n, m) = 1$ implies that $g(nm) = g(n)g(m)$ and it is completely multiplicative if this equality holds for all positive integers n and m. We denote by $\mathcal{M}$ and $\mathcal{M}^*$ the set of all complex-valued multiplicative and completely multiplicative functions, respectively.

The problem concerning the characterization of functions $f(n) = U \log n$ ($f = U \log$) as additive arithmetic functions was studied by several authors. It

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is clear that $f = U \log$ belongs to $\mathcal{A}^*$. Normally *log* is considered as a mapping $\mathbb{R}^+ \to \mathbb{R}$ and in this context it is well known that continuity along with the functional equation $f(xy) = f(x) + f(y)$ characterizes the logarithm up to a constant factor. Restricting the domain from $\mathbb{R}^+$ to $\mathbb{N}$ creates an interesting question: What further properties along with (complete) additivity will ensure that an arithmetic function f is in fact $U \log$? Most of the sufficient conditions that are known can be formulated in terms of their differences.

The first result is from P. Erdős [8]. Here we shall list the most important results on this topic:

1. If $f \in \mathcal{A}$, $\Delta f(n) := f(n+1) - f(n) \geq 0$ $(n \in \mathbb{N})$, then $f = U \log$ (P. Erdős [8])

2. If $f \in \mathcal{A}$, $\Delta f(n) = o(1)$ $(n \to \infty)$, then $f = U \log$ (P. Erdős [8])

3. If $f \in \mathcal{A}^*$, $\Delta f(n) = o(\log n)$ $(n \to \infty)$, then $f = U \log$ (E. Wirsing [36])

4. If $f \in \mathcal{A}$, $\Delta f(n) = o(1)$ $(n \to \infty)$ through a set of density 1, then $f = U \log$ (A. Hildebrand [11]).

Since the appearance of Erdős' paper several new characterizations of the logarithm have been found that generalize or sharpen Erdős' original results in a variety of ways. I. Kátai [20], [21] proposed the problem to obtain similar characterizations when n and $n+1$ are replaced by two linear forms $an+b$ and $cn+d$, where $a > 0$, b, $c > 0$, d are integers with $ad - bc \neq 0$. Specifically, I. Kátai asked for a characterization of those real-valued additive functions f which satisfy

$$f(an+b) - f(cn+d) - D \to 0 \quad \text{as} \quad n \to \infty$$

for a real number D. I. Kátai considered this problem with $d = 0$ and small values of a and b in [20], [21]. The general case has been treated and completely solved by P. D. T. A. Elliott [4], [5], [7]. Namely, P. D. T. A. Elliott [7], showed that if a real-valued additive function f satisfies the above relation, then $f(n) = U \log n$ holds for all positive integers n which are prime to $ac(ad - bc)$.

Conjecture 1. (P. Erdős, [8]) *Is it true that* $f \in \mathcal{A}$ *and*

$$\frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| \to 0$$

implies that $f(n) = U \log n$ *for every* $n \in \mathbb{N}$*?*

This has been proved independently by I. Kátai [21] and E. Wirsing [36].

Later E. Wirsing gave many interesting generalizations of this assertion. We mention the next assertion. Let $x_i \nearrow \infty$, $\gamma > 0$. If $f \in \mathcal{A}$ and

$$\lim_{i\to\infty} \frac{1}{x_i} \sum_{x_i < n \le (1+\gamma)x_i} |f(n+1) - f(n)| = 0,$$

then $f(n) = U \log n$ for every $n \in \mathbb{N}$.

On the other hand, the second author obtained in [28], [29] a complete characterization of those functions $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ for which the relation

$$\sum_{n\le x} |f_1(an+b) - f_2(n) - D| = o(x) \quad \text{as} \quad x \to \infty$$

holds for some fixed positive integers a, b and for a real constant D. It was proved that the above relation implies that there are a real constant U and functions $F_1 \in \mathcal{A}$, $F_2 \in \mathcal{A}$ such that

$$f_1(n) = U \log n + F_1(n), \qquad f_2(n) = U \log n + F_2(n)$$

and

$$F_1(an+b) - F_2(n) - D + U \log a = 0$$

hold for all positive integers n. In B. M. Phong [29] the same result has been proved, if the relation

$$\sum_{n\le x} \frac{1}{n}\Big|f_1(an+b) - f_2(n) - D\Big| = o(\log x) \quad \text{as} \quad x \to \infty$$

is satisfied.

With a little modification of [28], [29] the second author improved in [32] some results mentioned above. It was proved that if $a, b, c \in \mathbb{N}$, $D \in \mathbb{R}$ and $f_1, f_2 \in \mathcal{A}$ satisfy the condition

$$\liminf_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{n}\Big|f_1(an+b) - f_2(cn) - D\Big| = 0,$$

then there are a real constant U and functions $F_1 \in \mathcal{A}$, $F_2 \in \mathcal{A}$ such that

$$f_1(n) = U \log n + F_1(n), \qquad f_2(n) = U \log n + F_2(n)$$

and

$$F_1(an+b) - F_2(cn) - D + U \log\left(\frac{a}{c}\right) = 0$$

hold for all positive integers n.

2. Sets of uniqueness

Conjecture 2. (I. Kátai, [19]) *If $f \in \mathcal{A}^*$ and $f(p+1) = 0$ for every $p \in \mathcal{P}$, then $f(n) = 0$ for every $n \in \mathbb{N}$.*

Definition 1. (Set of uniqueness for completely additive functions) We say that $A \subseteq \mathbb{N}$ is *a set of uniqueness for completely additive functions* if $f \in \mathcal{A}^*$, $f(a) = 0$ for every $a \in A$ implies that $f(n) = 0$ for all $n \in \mathbb{N}$.

Definition 2. (Set of uniqueness for completely additive functions $\pmod 1$) We say that $B \subseteq \mathbb{N}$ is *a set of uniqueness for completely additive functions* $\pmod 1$ if $f \in \mathcal{A}^*$, $f(b) \equiv 0 \pmod 1$ for all $b \in B$ implies that $f(n) \equiv 0 \pmod 1$ for every $n \in \mathbb{N}$.

D. Wolke [40] proved that A is a set of uniqueness if and only if every $n \in \mathbb{N}$ can be written as $n = \prod_{i=1}^{k} a_i^{r_i}$, where $a_i \in A$ and $r_i \in \mathbb{Q}$.

K.-H. Indlekofer [15, 16], P. Hoffman [14], F. Dress and B. Volkmann [2] proved independently that B is a set of uniqueness for completely additive functions $\pmod 1$ if and only if every $n \in \mathbb{N}$ can be written as

$$n = \prod_{j=1}^{k} b_j^{\ell_j}, \text{where } \ \ell_j \in \mathbb{Z}, \ b_j \in B.$$

I. Kátai [18, 19] formulated the conjecture in 1969 that

$$\mathcal{P}_{+1} = \{p + 1 | p \in \mathcal{P}\}$$

is a set of uniqueness for additive functions, and proved that there exists such a finite set Q of primes for which $\mathcal{P}_{+1} \cup Q$ is a set of uniqueness $\pmod 1$. P.D.T.A. Elliott [3] proved that $Q = \{p | p \leq 10^{387}, p \in \mathcal{P}\}$ is an appropriate choice, that is every $n \in \mathbb{N}$ can be written as

$$n = t \cdot \prod_{i=1}^{k} (p_i + 1)^{\epsilon_i}, \quad \epsilon_i \in \{-1, 1\},$$

and t is such a rational number the largest prime factor of which does not exceed 10^{387}.

Furthermore, in [6] P.D.T.A. Elliott proved that for every rational number r, one can find such primes $p_1, \ldots, p_k$ and $\epsilon_j \in \{-1, 1\}$ $(j = 1, 2, \ldots, k)$, for which

$$r^g = \prod_{i=1}^{k} (p_i + 1)^{\epsilon_i}.$$

Here g is a constant, $g \in \{1, 2, 3\}$.

A direct consequence of this assertion is the following result: *Let* $f \in \mathcal{M}^*$, $f(p+1) = p+1 \quad (\forall p \in \mathcal{P})$. *Then*

$$f(n) = nH(n), \quad H \in \mathcal{M}^*, \quad H(n)^g = 1 \quad \text{for every} \quad n \in \mathbb{N}.$$

Especially, if $f(n)$ *is a positive real number for every* $n \in \mathbb{N}$, *then*

$$f(n) = n \quad \text{for every} \quad n \in \mathbb{N}.$$

T. Csajbók, A. Járai and J. Kasza [1] proved that every integer $n \in [2, 10^{14}]$ can be written as $\frac{p+1}{q+1}$ $(p, q \in \mathcal{P})$.

In [34] E. Wirsing proved:

(1) *There are constants* A, B *such that for any additive function* f *and all* $n \in \mathbb{N}$ *the inequality*

$$|f(n)| \leq A \max_{n \leq p+1 \leq n^B} |f(p+1)|$$

holds.

(2) *There are constants* c_1, c_2 *such that for every* $n \in \mathbb{N}$ *there is a representation*

$$n^a = \prod_{i=1}^{b} (p_i + 1)^{\epsilon_i}, \quad p_i \in \mathcal{P},$$

where $a \leq c_1$, $b \leq c_1$, $\epsilon_i \in \{-1, 1\}$, $n \leq p_i + 1 \leq n^{c_2}$.

As a corollary he mentioned that:

•*If* $f \in \mathcal{A}^*$ *and* $f(p+1) = 0$ *for every prime* p, *then* $f(n) = 0$ *for every* $n \in \mathbb{N}$.

•*If* $f \in \mathcal{A}^*$ *and* $f(p+1) = o(\log(p+1))$ *for every* $p \in \mathcal{P}$, *then* $f(n) = = o(\log n)$, *consequently* $f(n) = 0$ *for every* $n \in \mathbb{N}$.

J. Mehta, G.K. Viswanadham [27] proved that the set $\mathcal{P}[i] + 1$ is a set of uniqueness for additive functions over the set of Gaussian integers, where $\mathcal{P}[i]$ denotes the set of all Gaussian primes.

3. Some conjectures concerning multiplicative functions

A function g is said to be unimodular if g satisfies the condition $|g(n)| = = 1$ for all positive integers n. In the following we shall denote by $\mathcal{M}_1$ and ${\mathcal{M}^*}_1$ the class of all unimodular functions $g \in \mathcal{M}$ and $g \in \mathcal{M}^*$, respectively.

The classes $\mathcal{M}_1$ and ${\mathcal{M}^*}_1$ are very important subclasses in $\mathcal{M}$ and $\mathcal{M}^*$. For each real-valued additive function f the function $g(n) = e^{2\pi i f(n)}$ $(n \in \mathbb{N})$

belongs to $\mathcal{M}_1$, and so the results for unimodular multiplicative functions can be used to obtain information for the distribution of additive functions.

The functions of the form $g(n) = n^s$ $(n \in \mathbb{N})$ belong to $\mathcal{M}^*$ for all fixed complex numbers s. These functions play a similar exceptional role among multiplicative functions as the functions $U \log$ among additive functions. This raises the question: Can one characterize the functions of the type $g(n) = n^s$ as multiplicative functions by imposing suitable regularity conditions on g? It turns out that this leads to problems that are much more difficult than those arising in the case of additive functions.

More than 41 years ago I. Kátai [22] stated conjectures concerning multiplicative functions. For $\mathcal{M}_1$, the conjectures of I. Kátai are:

Conjecture 3. (I. Kátai, [22]) *If $g \in \mathcal{M}_1$ and*

$$g(n+1) - g(n) = o(1) \quad \textit{as} \quad n \to \infty,$$

then $g(n) = n^{i\tau}$ $(\forall n \in \mathbb{N})$ for some real number τ.

I. Kátai mentioned this conjecture in his talk in Ooty (India) in a conference celebrating the 75th anniversary of Professor P. Erdős. Some weeks later E. Wirsing wrote him a letter, giving the proof of this conjecture. Several years later Tang Yuansheng and Shao Pintsung proved independently the conjecture. The three authors published a joint paper [38] containing two proofs. The proof of Conjecture 3 also was given in [33, 39].

It is not hard to deduce from this result that if $f,\ g \in \mathcal{M}_1$, $g(n+1) - f(n) = o(1)$ as $n \to \infty$, then $f(n) = g(n) = n^{i\tau}$ $(n \in \mathbb{N})$.

Conjecture 4. (I. Kátai [22]) *If $g \in \mathcal{M}_1$ satisfies*

$$(*) \qquad \sum_{n \le x} |g(n+1) - g(n)| = o(1) \quad \textit{as} \quad x \to \infty,$$

or if

$$(**) \qquad \frac{1}{\log x} \sum_{n \le x} \frac{|g(n+1) - g(n)|}{n} = o(1) \quad \textit{as} \quad x \to \infty,$$

then $g(n) = n^{i\tau}$ $(\forall n \in \mathbb{N})$ for some real number τ.

I. Kátai considered functions $g \in \mathcal{M}$ under the conditions that $g(n+1) - g(n)$ tends to zero in some sense. For example, it follows from Theorem 3 of I. Kátai [23] that a function $g \in \mathcal{M}_1$ satisfying the condition

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n+1) - g(n)| < \infty$$

must be of the form $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

Finally, O. Klurman [26] proved that Conjecture 4 is true.

Hence we can deduce the following

Theorem 1. *Assume that* $f \in \mathcal{A}$, $\Delta f(n) = f(n+1) - f(n)$, *and that for every* $\varepsilon > 0$

$$\frac{1}{x}\sharp\{n \le x| \quad \|\Delta f(n)\| > \varepsilon\} \to 0 \quad as \quad x \to \infty \tag{3.1}$$

or if

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ \|\Delta f(n)\| > \varepsilon}} \frac{1}{n} \to 0 \quad as \quad x \to \infty, \tag{3.2}$$

where $\|x\|$ *is the distance of the real number* x *from a nearest integer.*

Then $f(n) = \lambda \log n + E(n)$, *where* $\lambda \in \mathbb{R}, E \in \mathcal{A}, E(n) \in \mathbb{Z}$ *for every* $n \in \mathbb{N}$.

Proof. Let $e(n) := e^{2\pi i n}$ and let $g(n) := e(f(n))$. Then $g(n+1)\overline{g}(n) = e(\Delta f(n))$.

Consequently, if $\{\Delta f(n)\} < c \cdot \varepsilon$, then $|g(n+1)\overline{g}(n) - 1| \asymp \varepsilon$, while if $1 - c \cdot \varepsilon < \{\Delta f(n)\}$, then the same is true.

From the condition (3.1) or (3.2) we obtain that $(*)$ or $(**)$ holds, consequently

$$g(n) = e^{i\tau \log n} = e\Big(\frac{\tau}{2\pi} \log n\Big) = e(f(n)).$$

Let $E(n) := f(n) - \frac{\tau}{2\pi} \log n$. Then $E \in \mathcal{A}$ and $E(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$.

The proof of Theorem 1 is thus complete. ■

Theorem 2. *Assume that* $f \in \mathcal{A}$ *and that for every* $\varepsilon > 0$ *either*

$$\frac{1}{x}\sharp\{n \le x| \quad |\Delta f(n)| > \varepsilon\} \to 0 \quad as \quad x \to \infty \tag{3.3}$$

or

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ |\Delta f(n)| > \varepsilon}} \frac{1}{n} \to 0 \quad as \quad x \to \infty. \tag{3.4}$$

Then $f(n) = c \log n$ *for some real number* c.

Proof. From Theorem 1 we obtain that if (3.3) or (3.4) is satisfied, then

$$f(n) = c \log n + E(n)$$

and that

$$\frac{1}{x}\sharp\{n \le x| \quad E(n) \ne E(n+1)\} \to 0 \quad \text{as} \quad x \to \infty \tag{3.5}$$

or

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ E(n) \ne E(n+1)}} \frac{1}{n} \to 0 \quad \text{as} \quad x \to \infty. \tag{3.6}$$

Assume that $E(n) \ne 0$ for at least one n. Let

$$g_\lambda(n) := e^{2\pi i \lambda E(n)},$$

where λ is an irrational number. Since

$$g_\lambda(n+1)\overline{g_\lambda}(n) = 1 \quad \text{if} \quad E(n+1) = E(n)$$

and so for some exceptional set of n we have

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ E(n) \ne E(n+1)}} \frac{1}{n} \to 0 \quad \text{as} \quad x \to \infty,$$

we obtain that

$$\lambda E(n) = \tau(\lambda)\log n + V(n), V(n) \in \mathbb{Z} \quad \text{for every} \quad n \in \mathbb{N}.$$

If $\tau(\lambda) = 0$ and there is a number $M \in \mathbb{N}$ for which $E(M) \ne 0$, then $\lambda = \frac{V(M)}{E(M)}$ would be a rational number, which contradicts our assumption.

Assume that $\tau(\lambda) \ne 0$. Since

$$\lambda\Delta E(n) - \Delta V(n) = \tau(\lambda)\log(1 + \tfrac{1}{n}),$$

and $\Delta E(n) = 0$, $\Delta V(n) = 0$ for almost all $n \in \mathbb{N}$, the left-hand side of the above equality is 0, the right-hand side being $\tau(\lambda)\log(1 + \frac{1}{n}) \ne 0$. This is a contradiction.

The proof of Theorem 2 is thus complete. ■

4. A conjecture of Kátai and Subbarao

For any function $f \in \mathcal{M}^*{}_1$ let the quotient operator $\mathcal{S}_f$ be defined by

$$\mathcal{S}_f(n) := f(n+1)\overline{f}(n)$$

and let $\mathcal{A}_f$ be the set of limit points of $\mathcal{S}_f(n)$. Let $\mathcal{U}_k$ be the set of all k-th roots of unity, i.e. $\mathcal{U}_k = \{\omega \in \mathbb{C}| \;\; \omega^k = 1\}$.

I. Kátai and M. V. Subbarao in their papers [24] and [25] formulated three conjectures, two of which are:

Conjecture 5. *Let* $f \in \mathcal{M}^*{}_1$ *and* $\mathcal{A}_f = \{\alpha_1, \cdots, \alpha_k\}$. *Then* $\mathcal{A}_f = \mathcal{U}_k$. *and* $f(n) = n^{i\tau}F(n)$ *with some* $\tau \in \mathbb{R}$, *where* $F(n)^k = 1$ *for all* $n \in \mathbb{N}$.

Conjecture 6. *If* $f \in \mathcal{M}^*{}_1$ *and* $F(\mathbb{N}) = \mathcal{U}_k$, *then* $\mathcal{A}_F = \mathcal{U}_k$. *In other words:* $\mathcal{S}_F(n)$ *attains every* $\omega \in \mathcal{U}_k$ *infinitely often.*

We note that Conjecture 5 for $\mathcal{A}_f = \{1\}$ has been proposed by the first author, and solved by E. Wirsing in 1984.

In [24], I. Kátai and M. V. Subbarao proved Conjecture 5 for $k \in \{1, 2, 3\}$. For the case $k = 4$, they proved in [25] the following

Theorem A. (I. Kátai and M. V. Subbarao [25]) *If* $f \in \mathcal{M}^*{}_1$ *and* $|\mathcal{A}_f| = 4$, *then there is some constant* $\tau \in \mathbb{R}$ *such that* $f(n) = n^{i\tau}F(n)$ *and either* $\mathcal{A}_f = \mathcal{U}_4, F(n) \in \mathcal{U}_4$ *for every* $n \in \mathbb{N}$ *or* $\mathcal{A}_f = \mathcal{U}_5, F(n) \in \mathcal{U}_5$ *for every* $n \in \mathbb{N}$ *and* $F(n+1)\overline{F}(n) \in \mathcal{A}_f$ *for every large* $n \in \mathbb{N}$.

Recently E. Wirsing [37], using the results and methods of [35], [38] and [39], proved the following two results:

Theorem B. (E. Wirsing [37]) *If* $f \in \mathcal{M}^*{}_1$ *and* $\mathcal{A}_f$ *is finite, then there are constants* $\tau \in \mathbb{R}$ *and* $\ell \in \mathbb{N}$ *such that* $f^\ell(n) = n^{i\tau}$.

Theorem C. (E. Wirsing [37]) *Conjecture 6 implies Conjecture 5.*

Theorem B is a weak version of Conjecture 5.

For each function $F \in \mathcal{M}^*{}_1$ let

$$\mathcal{B}_F = \{F(n+1)\overline{F}(n) |\ n \in \mathbb{N}\}.$$

A weaker form of Conjecture 6 can be formulated as follows.

Conjecture 7. *If* $F \in \mathcal{M}^*{}_1$ *and* $F(\mathbb{N}) = \mathcal{U}_\ell$, *then* $\mathcal{B}_F = \mathcal{U}_\ell$.

The second author proved in [30] the following result:

Theorem D. (B. M. Phong [30]) *Conjecture 7 is true for* $\ell \leq 5$, *i.e. if* $F \in \mathcal{M}^*{}_1$ *and* $F(\mathbb{N}) = \mathcal{U}_\ell$, *then* $\mathcal{B}_F = \mathcal{U}_\ell$ *for every* $\ell \leq 5$.

Now we prove that if $f \in \mathcal{M}^*{}_1$ and $|\mathcal{A}_f| < \infty$, then $1 \in \mathcal{A}_f$.

Theorem 3. *If* $f \in \mathcal{M}^*{}_1$ *and* $|\mathcal{A}_f| < \infty$, *then* $1 \in \mathcal{A}_f$.

Proof. Assume $f \in \mathcal{M}^*{}_1$ and $\mathcal{A}_f = \{\alpha_1, \cdots, \alpha_k\}$, where $k \in \mathbb{N}$. Let $\delta = \min_{i \neq j} |\alpha_i - \alpha_j|$. For every large n $(n > N_0$, say), there is exactly one

$\alpha \in \mathcal{A}_f$ for which $|f(n+1)\overline{f}(n) - \alpha| < \delta$. Let $C(n) := \alpha$, i.e. $C(n)$ is that element of $\mathcal{A}_f$ which is the closest to $f(n+1)\overline{f}(n)$.

First we infer from the fact $f \in \mathcal{M}^*{}_1$ that

$$\text{(4.1)} \qquad \text{If} \quad k|n(n+1) \quad \text{and} \quad n, k \in \mathbb{N}, \ n > N_0, \quad \text{then} \quad C(n)\overline{C}(n+k) \in \mathcal{A}_f.$$

Indeed, if positive integers $k, n \in \mathbb{N}$, $n > N_0$ satisfy the condition $k|n(n+1)$, then $k|n(n+k+1)$ and

$$\frac{f(n+1)}{f(n)} \frac{f(n+k)}{f(n+k+1)} = \frac{f\Big[(n+1)(n+k)\Big]}{f\Big[n(n+k+1)\Big]} = \frac{f\Big[\frac{n(n+k+1)}{k} + 1\Big]}{f\Big[\frac{n(n+k+1)}{k}\Big]},$$

which proves that

$$C(n)\overline{C}(n+k) = C\Big[\frac{n(n+k+1)}{k}\Big] \in \mathcal{A}_f.$$

Consequently, the relation (4.1) is true.

We note from (4.1) that

$$\text{(4.2)} \qquad \text{If} \quad (m-n)|n(n+1) \quad \text{and} \quad m > n > N_0, \quad \text{then} \quad C(n)\overline{C}(m) \in \mathcal{A}_f.$$

A key element in the proof of Theorem 3 is played by (4.2) and by the sets $\{n_1 < n_2 < \cdots < n_r\}$ of positive integers having the property

$$\text{(4.3)} \qquad n_j - n_i = (n_i, n_j) \quad \text{for every} \quad 1 \le i < j \le r$$

The existence of such integers for every $r \ge 2$ has been first proved by Heath-Brown [9], a simple construction is given in [10]. For generalizations and applications of such sets we refer to works of Hildebrand [12] and [13].

Now we prove that $1 \in \mathcal{A}_f$. Assume by contradiction that $1 \notin \mathcal{A}_f$. Then we choose a positive integer $r > |\mathcal{A}_f|$ and a sequence $n_1 < n_2 < \cdots < n_r$ of positive integers satisfying (4.3) and $n_1 > N_0$. Hence

$$n_j - n_i = (n_i, n_j) \quad \text{and} \quad (n_j - n_i)|n_i(n_j+1) \quad \text{for every} \quad 1 \le i < j \le r,$$

consequently we get from (4.2) that $C(n_i)\overline{C}(n_j) \in \mathcal{A}_f$, which implies

$$C(n_i)\overline{C}(n_j) \ne 1, \text{i. e.} \quad C(n_i) \ne C(n_j) \quad \text{for every} \quad 1 \le i < j \le r.$$

This is impossible, because $C(n_i) \in \mathcal{A}_f \ \ (i = 1, 2, \cdots, r)$ and $|\mathcal{A}_f| < r$.

Hence $1 \in \mathcal{A}_f$ and the proof of Theorem 3 is complete. ∎

For an arbitrary, multiplicatively written commutative group $\mathbb{G}$ let $\mathcal{M}(\mathbb{G})$, resp. $\mathcal{M}^*(\mathbb{G})$ denote the classes of multiplicative, resp. completely multiplicative functions. A function $f : \mathbb{N} \to \mathbb{G}$ belongs to $\mathcal{M}(\mathbb{G})$ if $f(mn) = f(m)f(n)$ holds for each pair of coprime m, n, and it belongs to $\mathcal{M}^*(\mathbb{G})$ if the above equation holds for all $m, n \in \mathbb{N}$.

In [31] the second author proved the following results.

Theorem E. (B. M. Phong [31]) *Assume that* $\mathbb{G}$ *is any commutative multiplicative group and* $F_1, F_2 \in \mathcal{M}^*(\mathbb{G})$. *If*

$$\mathcal{B}(F_1, F_2, A, B) := \{F_1(An + B)(F_2(An))^{-1} \mid n \in \mathbb{N}\}$$

is a finite set, then there are finite subgroups $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *of* $\mathbb{G}$ *such that* $\mathbb{G}_2 \subseteq \mathbb{G}_1$ *and* $F_i(\mathbb{N})$ *is a subgroup of* $\mathbb{G}_i$ $(i = 1, 2)$.

The proof of Theorem E is based on the similar result concerning the case when $F_1 = F_2$.

Theorem F. (B. M. Phong [31]) *Assume that* $\mathbb{G}$ *is any commutative multiplicative group and* $F \in \mathcal{M}^*(\mathbb{G})$. *If*

$$\mathcal{B}(F, F, A, B) := \{F(An + B)(F(An))^{-1} \mid n \in \mathbb{N}\}$$

is a finite set, then there is a finite subgroup $\mathbb{G}_0$ *of* $\mathbb{G}$ *such that* $F(\mathbb{N})$ *is a subgroup of* $\mathbb{G}_0$.

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