*Annales Univ. Sci. Budapest., Sect. Comp. 53 (2022) 123–135*

# IN MEMORIAM FOR PROFESSOR EDUARD WIRSING

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Communicated by Jean-Marie De Koninck

(Received April 8, 2022; accepted July 10, 2022)

**Abstract.** We mention some results of Professor Eduard Wirsing concerning the first author's conjectures. We also list some our results for arithmetical functions and prove three theorems.

## 1. On the characterization of $\log n$

Professor E. Wirsing gave very important results for some of our conjectures. In this short survey paper we would like to elaborate on that.

In the following let $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of primes, positive integers, integers, real numbers and complex numbers, respectively. An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ is said to be additive if $(n, m) = 1$ implies that $f(nm) = f(n) + f(m)$ and it is completely additive if this equality holds for all positive integers $n$ and $m$. Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of all real-valued additive and completely additive functions, respectively. Similarly, an arithmetic function $g : \mathbb{N} \to \mathbb{C}$ is said to be multiplicative if $(n, m) = 1$ implies that $g(nm) = g(n)g(m)$ and it is completely multiplicative if this equality holds for all positive integers $n$ and $m$. We denote by $\mathcal{M}$ and $\mathcal{M}^*$ the set of all complex-valued multiplicative and completely multiplicative functions, respectively.

The problem concerning the characterization of functions $f(n) = U \log n$ ($f = U \log$) as additive arithmetic functions was studied by several authors. It

*Key words and phrases*: Arithmetical function, sets of uniqueness, Abelian groups.
*2010 Mathematics Subject Classification*: Primary 11A07, 11A25, Secondary 11N25, 11N64.

is clear that $f = U\log$ belongs to $\mathcal{A}^*$. Normally *log* is considered as a mapping $\mathbb{R}^+ \to \mathbb{R}$ and in this context it is well known that continuity along with the functional equation $f(xy) = f(x) + f(y)$ characterizes the logarithm up to a constant factor. Restricting the domain from $\mathbb{R}^+$ to $\mathbb{N}$ creates an interesting question: What further properties along with (complete) additivity will ensure that an arithmetic function $f$ is in fact $U\log$? Most of the sufficient conditions that are known can be formulated in terms of their differences.

The first result is from P. Erdős [8]. Here we shall list the most important results on this topic:

1. If $f \in \mathcal{A}$, $\Delta f(n) := f(n+1) - f(n) \geq 0$ $(n \in \mathbb{N})$, then $f = U\log$ (P. Erdős [8])

2. If $f \in \mathcal{A}$, $\Delta f(n) = o(1)$ $(n \to \infty)$, then $f = U\log$ (P. Erdős [8])

3. If $f \in \mathcal{A}^*$, $\Delta f(n) = o(\log n)$ $(n \to \infty)$, then $f = U\log$ (E. Wirsing [36])

4. If $f \in \mathcal{A}$, $\Delta f(n) = o(1)$ $(n \to \infty)$ through a set of density 1, then $f = U\log$ (A. Hildebrand [11]).

Since the appearance of Erdős' paper several new characterizations of the logarithm have been found that generalize or sharpen Erdős' original results in a variety of ways. I. Kátai [20], [21] proposed the problem to obtain similar characterizations when $n$ and $n+1$ are replaced by two linear forms $an+b$ and $cn+d$, where $a > 0$, $b$, $c > 0$, $d$ are integers with $ad - bc \neq 0$. Specifically, I. Kátai asked for a characterization of those real-valued additive functions $f$ which satisfy

$$f(an+b) - f(cn+d) - D \to 0 \quad \text{as} \quad n \to \infty$$

for a real number $D$. I. Kátai considered this problem with $d = 0$ and small values of $a$ and $b$ in [20], [21]. The general case has been treated and completely solved by P. D. T. A. Elliott [4], [5], [7]. Namely, P. D. T. A. Elliott [7], showed that if a real-valued additive function $f$ satisfies the above relation, then $f(n) = U\log n$ holds for all positive integers $n$ which are prime to $ac(ad-bc)$.

**Conjecture 1.** (P. Erdős, [8]) *Is it true that $f \in \mathcal{A}$ and*

$$\frac{1}{x}\sum_{n\leq x} |f(n+1) - f(n)| \to 0$$

*implies that $f(n) = U\log n$ for every $n \in \mathbb{N}$?*

This has been proved independently by I. Kátai [21] and E. Wirsing [36].

Later E. Wirsing gave many interesting generalizations of this assertion. We mention the next assertion. Let $x_i \nearrow \infty$, $\gamma > 0$. If $f \in \mathcal{A}$ and

$$\lim_{i\to\infty} \frac{1}{x_i} \sum_{x_i < n \le (1+\gamma)x_i} |f(n+1) - f(n)| = 0,$$

then $f(n) = U \log n$ for every $n \in \mathbb{N}$.

On the other hand, the second author obtained in [28], [29] a complete characterization of those functions $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ for which the relation

$$\sum_{n\le x} |f_1(an+b) - f_2(n) - D| = o(x) \quad \text{as} \quad x \to \infty$$

holds for some fixed positive integers $a$, $b$ and for a real constant $D$. It was proved that the above relation implies that there are a real constant $U$ and functions $F_1 \in \mathcal{A}$, $F_2 \in \mathcal{A}$ such that

$$f_1(n) = U \log n + F_1(n), \qquad f_2(n) = U \log n + F_2(n)$$

and

$$F_1(an+b) - F_2(n) - D + U \log a = 0$$

hold for all positive integers $n$. In B. M. Phong [29] the same result has been proved, if the relation

$$\sum_{n\le x} \frac{1}{n} \Big| f_1(an+b) - f_2(n) - D \Big| = o(\log x) \quad \text{as} \quad x \to \infty$$

is satisfied.

With a little modification of [28], [29] the second author improved in [32] some results mentioned above. It was proved that if $a, b, c \in \mathbb{N}$, $D \in \mathbb{R}$ and $f_1, f_2 \in \mathcal{A}$ satisfy the condition

$$\liminf_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{n} \Big| f_1(an+b) - f_2(cn) - D \Big| = 0,$$

then there are a real constant $U$ and functions $F_1 \in \mathcal{A}$, $F_2 \in \mathcal{A}$ such that

$$f_1(n) = U \log n + F_1(n), \qquad f_2(n) = U \log n + F_2(n)$$

and

$$F_1(an+b) - F_2(cn) - D + U \log\left(\frac{a}{c}\right) = 0$$

hold for all positive integers $n$.

## 2. Sets of uniqueness

**Conjecture 2.** (I. Kátai, [19]) *If $f \in \mathcal{A}^*$ and $f(p+1) = 0$ for every $p \in \mathcal{P}$, then $f(n) = 0$ for every $n \in \mathbb{N}$.*

**Definition 1.** (Set of uniqueness for completely additive functions) We say that $A \subseteq \mathbb{N}$ is *a set of uniqueness for completely additive functions* if $f \in \mathcal{A}^*$, $f(a) = 0$ for every $a \in A$ implies that $f(n) = 0$ for all $n \in \mathbb{N}$.

**Definition 2.** (Set of uniqueness for completely additive functions $\pmod 1$) We say that $B \subseteq \mathbb{N}$ is *a set of uniqueness for completely additive functions* $\pmod 1$ if $f \in \mathcal{A}^*$, $f(b) \equiv 0 \pmod 1$ for all $b \in B$ implies that $f(n) \equiv 0 \pmod 1$ for every $n \in \mathbb{N}$.

D. Wolke [40] proved that $A$ is a set of uniqueness if and only if every $n \in \mathbb{N}$ can be written as $n = \prod_{i=1}^{k} a_i^{r_i}$, where $a_i \in A$ and $r_i \in \mathbb{Q}$.

K.-H. Indlekofer [15, 16], P. Hoffman [14], F. Dress and B. Volkmann [2] proved independently that $B$ is a set of uniqueness for completely additive functions $\pmod 1$ if and only if every $n \in \mathbb{N}$ can be written as

$$n = \prod_{j=1}^{k} b_j^{\ell_j}, \text{where } \ \ell_j \in \mathbb{Z}, \ b_j \in B.$$

I. Kátai [18, 19] formulated the conjecture in 1969 that

$$\mathcal{P}_{+1} = \{p+1 | p \in \mathcal{P}\}$$

is a set of uniqueness for additive functions, and proved that there exists such a finite set $Q$ of primes for which $\mathcal{P}_{+1} \cup Q$ is a set of uniqueness $\pmod 1$. P.D.T.A. Elliott [3] proved that $Q = \{p | p \leq 10^{387}, p \in \mathcal{P}\}$ is an appropriate choice, that is every $n \in \mathbb{N}$ can be written as

$$n = t \cdot \prod_{i=1}^{k} (p_i + 1)^{\epsilon_i}, \quad \epsilon_i \in \{-1, 1\},$$

and $t$ is such a rational number the largest prime factor of which does not exceed $10^{387}$.

Furthermore, in [6] P.D.T.A. Elliott proved that for every rational number $r$, one can find such primes $p_1, \ldots, p_k$ and $\epsilon_j \in \{-1, 1\}$ $(j = 1, 2, \ldots, k)$, for which

$$r^g = \prod_{i=1}^{k} (p_i + 1)^{\epsilon_i}.$$

Here $g$ is a constant, $g \in \{1, 2, 3\}$.

A direct consequence of this assertion is the following result: *Let* $f \in \mathcal{M}^*$, $f(p+1) = p+1 \quad (\forall p \in \mathcal{P})$. *Then*

$$f(n) = nH(n), \quad H \in \mathcal{M}^*, \quad H(n)^g = 1 \quad \textit{for every} \quad n \in \mathbb{N}.$$

*Especially, if* $f(n)$ *is a positive real number for every* $n \in \mathbb{N}$, *then*

$$f(n) = n \quad \textit{for every} \quad n \in \mathbb{N}.$$

T. Csajbók, A. Járai and J. Kasza [1] proved that every integer $n \in [2, 10^{14}]$ can be written as $\frac{p+1}{q+1}$ $(p, q \in \mathcal{P})$.

In [34] E. Wirsing proved:

(1) *There are constants* $A, B$ *such that for any additive function* $f$ *and all* $n \in \mathbb{N}$ *the inequality*

$$|f(n)| \leq A \max_{n \leq p+1 \leq n^B} |f(p+1)|$$

*holds.*

(2) *There are constants* $c_1, c_2$ *such that for every* $n \in \mathbb{N}$ *there is a representation*

$$n^a = \prod_{i=1}^{b} (p_i + 1)^{\epsilon_i}, \quad p_i \in \mathcal{P},$$

*where* $a \leq c_1$, $b \leq c_1$, $\epsilon_i \in \{-1, 1\}$, $n \leq p_i + 1 \leq n^{c_2}$.

As a corollary he mentioned that:

•*If* $f \in \mathcal{A}^*$ *and* $f(p+1) = 0$ *for every prime* $p$, *then* $f(n) = 0$ *for every* $n \in \mathbb{N}$.

•*If* $f \in \mathcal{A}^*$ *and* $f(p+1) = o(\log(p+1))$ *for every* $p \in \mathcal{P}$, *then* $f(n) = o(\log n)$, *consequently* $f(n) = 0$ *for every* $n \in \mathbb{N}$.

J. Mehta, G.K. Viswanadham [27] proved that the set $\mathcal{P}[i] + 1$ is a set of uniqueness for additive functions over the set of Gaussian integers, where $\mathcal{P}[i]$ denotes the set of all Gaussian primes.

## 3. Some conjectures concerning multiplicative functions

A function $g$ is said to be unimodular if $g$ satisfies the condition $|g(n)| = 1$ for all positive integers $n$. In the following we shall denote by $\mathcal{M}_1$ and ${\mathcal{M}^*}_1$ the class of all unimodular functions $g \in \mathcal{M}$ and $g \in \mathcal{M}^*$, respectively.

The classes $\mathcal{M}_1$ and ${\mathcal{M}^*}_1$ are very important subclasses in $\mathcal{M}$ and $\mathcal{M}^*$. For each real-valued additive function $f$ the function $g(n) = e^{2\pi i f(n)}$ $(n \in \mathbb{N})$

belongs to $\mathcal{M}_1$, and so the results for unimodular multiplicative functions can be used to obtain information for the distribution of additive functions.

The functions of the form $g(n) = n^s$ $(n \in \mathbb{N})$ belong to $\mathcal{M}^*$ for all fixed complex numbers $s$. These functions play a similar exceptional role among multiplicative functions as the functions $U \log$ among additive functions. This raises the question: Can one characterize the functions of the type $g(n) = n^s$ as multiplicative functions by imposing suitable regularity conditions on $g$? It turns out that this leads to problems that are much more difficult than those arising in the case of additive functions.

More than 41 years ago I. Kátai [22] stated conjectures concerning multiplicative functions. For $\mathcal{M}_1$, the conjectures of I. Kátai are:

**Conjecture 3.** (I. Kátai, [22]) *If $g \in \mathcal{M}_1$ and*

$$g(n+1) - g(n) = o(1) \quad \textit{as} \quad n \to \infty,$$

*then $g(n) = n^{i\tau}$ $(\forall n \in \mathbb{N})$ for some real number $\tau$.*

I. Kátai mentioned this conjecture in his talk in Ooty (India) in a conference celebrating the 75th anniversary of Professor P. Erdős. Some weeks later E. Wirsing wrote him a letter, giving the proof of this conjecture. Several years later Tang Yuansheng and Shao Pintsung proved independently the conjecture. The three authors published a joint paper [38] containing two proofs. The proof of Conjecture 3 also was given in [33, 39].

It is not hard to deduce from this result that if $f,\ g \in \mathcal{M}_1$, $g(n+1) - f(n) = o(1)$ as $n \to \infty$, then $f(n) = g(n) = n^{i\tau}$ $(n \in \mathbb{N})$.

**Conjecture 4.** (I. Kátai [22]) *If $g \in \mathcal{M}_1$ satisfies*

$$(*) \qquad \sum_{n \le x} |g(n+1) - g(n)| = o(1) \quad \textit{as} \quad x \to \infty,$$

*or if*

$$(**) \qquad \frac{1}{\log x} \sum_{n \le x} \frac{|g(n+1) - g(n)|}{n} = o(1) \quad \textit{as} \quad x \to \infty,$$

*then $g(n) = n^{i\tau}$ $(\forall n \in \mathbb{N})$ for some real number $\tau$.*

I. Kátai considered functions $g \in \mathcal{M}$ under the conditions that $g(n+1) - g(n)$ tends to zero in some sense. For example, it follows from Theorem 3 of I. Kátai [23] that a function $g \in \mathcal{M}_1$ satisfying the condition

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n+1) - g(n)| < \infty$$

must be of the form $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

Finally, O. Klurman [26] proved that Conjecture 4 is true.

Hence we can deduce the following

**Theorem 1.** *Assume that $f \in \mathcal{A}$, $\Delta f(n) = f(n+1) - f(n)$, and that for every $\varepsilon > 0$*

$$\frac{1}{x}\sharp\{n \le x| \quad \|\Delta f(n)\| > \varepsilon\} \to 0 \quad as \quad x \to \infty \tag{3.1}$$

*or if*

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ \|\Delta f(n)\| > \varepsilon}} \frac{1}{n} \to 0 \quad as \quad x \to \infty, \tag{3.2}$$

*where $\|x\|$ is the distance of the real number $x$ from a nearest integer.*

*Then $f(n) = \lambda \log n + E(n)$, where $\lambda \in \mathbb{R}, E \in \mathcal{A}, E(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$.*

**Proof.** Let $e(n) := e^{2\pi i n}$ and let $g(n) := e(f(n))$. Then $g(n+1)\overline{g}(n) = = e(\Delta f(n))$.

Consequently, if $\{\Delta f(n)\} < c \cdot \varepsilon$, then $|g(n+1)\overline{g}(n) - 1| \asymp \varepsilon$, while if $1 - c \cdot \varepsilon < \{\Delta f(n)\}$, then the same is true.

From the condition (3.1) or (3.2) we obtain that $(*)$ or $(**)$ holds, consequently

$$g(n) = e^{i\tau \log n} = e\Big(\frac{\tau}{2\pi} \log n\Big) = e(f(n)).$$

Let $E(n) := f(n) - \frac{\tau}{2\pi} \log n$. Then $E \in \mathcal{A}$ and $E(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$.

The proof of Theorem 1 is thus complete. ■

**Theorem 2.** *Assume that $f \in \mathcal{A}$ and that for every $\varepsilon > 0$ either*

$$\frac{1}{x}\sharp\{n \le x| \quad |\Delta f(n)| > \varepsilon\} \to 0 \quad as \quad x \to \infty \tag{3.3}$$

*or*

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ |\Delta f(n)| > \varepsilon}} \frac{1}{n} \to 0 \quad as \quad x \to \infty. \tag{3.4}$$

*Then $f(n) = c \log n$ for some real number $c$.*

**Proof.** From Theorem 1 we obtain that if (3.3) or (3.4) is satisfied, then

$$f(n) = c \log n + E(n)$$

and that

$$\frac{1}{x}\sharp\{n \le x|\ \ E(n) \ne E(n+1)\} \to 0 \quad \text{as} \quad x \to \infty \tag{3.5}$$

or

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ E(n) \ne E(n+1)}} \frac{1}{n} \to 0 \quad \text{as} \quad x \to \infty. \tag{3.6}$$

Assume that $E(n) \ne 0$ for at least one $n$. Let

$$g_\lambda(n) := e^{2\pi i \lambda E(n)},$$

where $\lambda$ is an irrational number. Since

$$g_\lambda(n+1)\overline{g_\lambda}(n) = 1 \quad \text{if} \quad E(n+1) = E(n)$$

and so for some exceptional set of $n$ we have

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ E(n) \ne E(n+1)}} \frac{1}{n} \to 0 \quad \text{as} \quad x \to \infty,$$

we obtain that

$$\lambda E(n) = \tau(\lambda)\log n + V(n), V(n) \in \mathbb{Z} \quad \text{for every} \quad n \in \mathbb{N}.$$

If $\tau(\lambda) = 0$ and there is a number $M \in \mathbb{N}$ for which $E(M) \ne 0$, then $\lambda = \frac{V(M)}{E(M)}$ would be a rational number, which contradicts our assumption.

Assume that $\tau(\lambda) \ne 0$. Since

$$\lambda \Delta E(n) - \Delta V(n) = \tau(\lambda)\log(1 + \tfrac{1}{n}),$$

and $\Delta E(n) = 0$, $\Delta V(n) = 0$ for almost all $n \in \mathbb{N}$, the left-hand side of the above equality is 0, the right-hand side being $\tau(\lambda)\log(1 + \frac{1}{n}) \ne 0$. This is a contradiction.

The proof of Theorem 2 is thus complete. ■

## 4. A conjecture of Kátai and Subbarao

For any function $f \in \mathcal{M}^*{}_1$ let the quotient operator $\mathcal{S}_f$ be defined by

$$\mathcal{S}_f(n) := f(n+1)\overline{f}(n)$$

and let $\mathcal{A}_f$ be the set of limit points of $\mathcal{S}_f(n)$. Let $\mathcal{U}_k$ be the set of all $k$-th roots of unity, i.e. $\mathcal{U}_k = \{\omega \in \mathbb{C}|\ \ \omega^k = 1\}$.

I. Kátai and M. V. Subbarao in their papers [24] and [25] formulated three conjectures, two of which are:

**Conjecture 5.** *Let $f \in \mathcal{M}^*{}_1$ and $\mathcal{A}_f = \{\alpha_1, \cdots, \alpha_k\}$. Then $\mathcal{A}_f = \mathcal{U}_k$. and $f(n) = n^{i\tau} F(n)$ with some $\tau \in \mathbb{R}$, where $F(n)^k = 1$ for all $n \in \mathbb{N}$.*

**Conjecture 6.** *If $f \in \mathcal{M}^*{}_1$ and $F(\mathbb{N}) = \mathcal{U}_k$, then $\mathcal{A}_F = \mathcal{U}_k$. In other words: $\mathcal{S}_F(n)$ attains every $\omega \in \mathcal{U}_k$ infinitely often.*

We note that Conjecture 5 for $\mathcal{A}_f = \{1\}$ has been proposed by the first author, and solved by E. Wirsing in 1984.

In [24], I. Kátai and M. V. Subbarao proved Conjecture 5 for $k \in \{1, 2, 3\}$. For the case $k = 4$, they proved in [25] the following

**Theorem A.** (I. Kátai and M. V. Subbarao [25]) *If $f \in \mathcal{M}^*{}_1$ and $|\mathcal{A}_f| = 4$, then there is some constant $\tau \in \mathbb{R}$ such that $f(n) = n^{i\tau} F(n)$ and either $\mathcal{A}_f = \mathcal{U}_4, F(n) \in \mathcal{U}_4$ for every $n \in \mathbb{N}$ or $\mathcal{A}_f = \mathcal{U}_5, F(n) \in \mathcal{U}_5$ for every $n \in \mathbb{N}$ and $F(n+1)\overline{F}(n) \in \mathcal{A}_f$ for every large $n \in \mathbb{N}$.*

Recently E. Wirsing [37], using the results and methods of [35], [38] and [39], proved the following two results:

**Theorem B.** (E. Wirsing [37]) *If $f \in \mathcal{M}^*{}_1$ and $\mathcal{A}_f$ is finite, then there are constants $\tau \in \mathbb{R}$ and $\ell \in \mathbb{N}$ such that $f^\ell(n) = n^{i\tau}$.*

**Theorem C.** (E. Wirsing [37]) *Conjecture 6 implies Conjecture 5.*

Theorem B is a weak version of Conjecture 5.

For each function $F \in \mathcal{M}^*{}_1$ let

$$\mathcal{B}_F = \{F(n+1)\overline{F}(n) |\ n \in \mathbb{N}\}.$$

A weaker form of Conjecture 6 can be formulated as follows.

**Conjecture 7.** *If $F \in \mathcal{M}^*{}_1$ and $F(\mathbb{N}) = \mathcal{U}_\ell$, then $\mathcal{B}_F = \mathcal{U}_\ell$.*

The second author proved in [30] the following result:

**Theorem D.** (B. M. Phong [30]) *Conjecture 7 is true for $\ell \leq 5$, i.e. if $F \in \mathcal{M}^*{}_1$ and $F(\mathbb{N}) = \mathcal{U}_\ell$, then $\mathcal{B}_F = \mathcal{U}_\ell$ for every $\ell \leq 5$.*

Now we prove that if $f \in \mathcal{M}^*{}_1$ and $|\mathcal{A}_f| < \infty$, then $1 \in \mathcal{A}_f$.

**Theorem 3.** *If $f \in \mathcal{M}^*{}_1$ and $|\mathcal{A}_f| < \infty$, then $1 \in \mathcal{A}_f$.*

**Proof.** Assume $f \in \mathcal{M}^*{}_1$ and $\mathcal{A}_f = \{\alpha_1, \cdots, \alpha_k\}$, where $k \in \mathbb{N}$. Let $\delta = \min_{i \neq j} |\alpha_i - \alpha_j|$. For every large $n$ ($n > N_0$, say), there is exactly one

$\alpha \in \mathcal{A}_f$ for which $|f(n+1)\overline{f}(n) - \alpha| < \delta$. Let $C(n) := \alpha$, i.e. $C(n)$ is that element of $\mathcal{A}_f$ which is the closest to $f(n+1)\overline{f}(n)$.

First we infer from the fact $f \in \mathcal{M}^*{}_1$ that

$$\text{(4.1)} \qquad \text{If} \quad k|n(n+1) \ \text{ and } \ n, k \in \mathbb{N}, \ n > N_0, \ \text{ then } \ C(n)\overline{C}(n+k) \in \mathcal{A}_f.$$

Indeed, if positive integers $k, n \in \mathbb{N}$, $n > N_0$ satisfy the condition $k|n(n+1)$, then $k|n(n+k+1)$ and

$$\frac{f(n+1)}{f(n)} \frac{f(n+k)}{f(n+k+1)} = \frac{f\Big[(n+1)(n+k)\Big]}{f\Big[n(n+k+1)\Big]} = \frac{f\Big[\frac{n(n+k+1)}{k} + 1\Big]}{f\Big[\frac{n(n+k+1)}{k}\Big]},$$

which proves that

$$C(n)\overline{C}(n+k) = C\Big[\frac{n(n+k+1)}{k}\Big] \in \mathcal{A}_f.$$

Consequently, the relation (4.1) is true.

We note from (4.1) that

$$\text{(4.2)} \qquad \text{If} \quad (m-n)|n(n+1) \ \text{ and } \ m > n > N_0, \ \text{ then } \ C(n)\overline{C}(m) \in \mathcal{A}_f.$$

A key element in the proof of Theorem 3 is played by (4.2) and by the sets $\{n_1 < n_2 < \cdots < n_r\}$ of positive integers having the property

$$\text{(4.3)} \qquad n_j - n_i = (n_i, n_j) \quad \text{for every} \ \ 1 \le i < j \le r$$

The existence of such integers for every $r \ge 2$ has been first proved by Heath-Brown [9], a simple construction is given in [10]. For generalizations and applications of such sets we refer to works of Hildebrand [12] and [13].

Now we prove that $1 \in \mathcal{A}_f$. Assume by contradiction that $1 \notin \mathcal{A}_f$. Then we choose a positive integer $r > |\mathcal{A}_f|$ and a sequence $n_1 < n_2 < \cdots < n_r$ of positive integers satisfying (4.3) and $n_1 > N_0$. Hence

$$n_j - n_i = (n_i, n_j) \ \text{ and } \ (n_j - n_i)|n_i(n_j+1) \quad \text{for every} \ \ 1 \le i < j \le r,$$

consequently we get from (4.2) that $C(n_i)\overline{C}(n_j) \in \mathcal{A}_f$, which implies

$$C(n_i)\overline{C}(n_j) \ne 1, \text{i. e.} \ \ C(n_i) \ne C(n_j) \quad \text{for every} \ \ 1 \le i < j \le r.$$

This is impossible, because $C(n_i) \in \mathcal{A}_f \ \ (i = 1, 2, \cdots, r)$ and $|\mathcal{A}_f| < r$.

Hence $1 \in \mathcal{A}_f$ and the proof of Theorem 3 is complete. ■

For an arbitrary, multiplicatively written commutative group $\mathbb{G}$ let $\mathcal{M}(\mathbb{G})$, resp. $\mathcal{M}^*(\mathbb{G})$ denote the classes of multiplicative, resp. completely multiplicative functions. A function $f : \mathbb{N} \to \mathbb{G}$ belongs to $\mathcal{M}(\mathbb{G})$ if $f(mn) = f(m)f(n)$ holds for each pair of coprime $m, n$, and it belongs to $\mathcal{M}^*(\mathbb{G})$ if the above equation holds for all $m, n \in \mathbb{N}$.

In [31] the second author proved the following results.

**Theorem E.** (B. M. Phong [31]) *Assume that $\mathbb{G}$ is any commutative multiplicative group and $F_1, F_2 \in \mathcal{M}^*(\mathbb{G})$. If*

$$\mathcal{B}(F_1, F_2, A, B) := \{F_1(An + B)(F_2(An))^{-1} \mid n \in \mathbb{N}\}$$

*is a finite set, then there are finite subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ of $\mathbb{G}$ such that $\mathbb{G}_2 \subseteq \mathbb{G}_1$ and $F_i(\mathbb{N})$ is a subgroup of $\mathbb{G}_i$ $(i = 1, 2)$.*

The proof of Theorem E is based on the similar result concerning the case when $F_1 = F_2$.

**Theorem F.** (B. M. Phong [31]) *Assume that $\mathbb{G}$ is any commutative multiplicative group and $F \in \mathcal{M}^*(\mathbb{G})$. If*

$$\mathcal{B}(F, F, A, B) := \{F(An + B)(F(An))^{-1} \mid n \in \mathbb{N}\}$$

*is a finite set, then there is a finite subgroup $\mathbb{G}_0$ of $\mathbb{G}$ such that $F(\mathbb{N})$ is a subgroup of $\mathbb{G}_0$.*

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