ON ENDOMORPHISM RINGS OF AUTOMORPHISM-INVARIANT MODULES

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Abstract. In this paper, we give some results of endomorphism rings of automorphism-invariant modules. It shows that if M is an automorphism-invariant self-generator module with finite Goldie dimension then every maximal left ideal of $End(M)$ has the form A_u for some right uniform element u in $End(M)$. We also show that a ring R is right self-injective if and only if R is right automorphism-invariant and R_R is weakly injective. Finally, we obtain a decomposition of finite Goldie dimension right automorphism-invariant rings.

1. Introduction

The study of generalizations of quasi-injective modules and injective modules have made a significant contribution to ring and module theory. In 1961, Johnson and Wong ([8]) studied quasi-injective modules, a module M is called *quasi-injective* if any homomorphism from a submodule of M to M extends to an endomorphism of M; the authors showed that a module M is quasi-injective if and only if M is invariant under any endomorphism of its injective envelope. This class of modules is known as the class of endomorphism-invariant modules. In 2013, Lee and Zhou ([10]) studied automorphism-invariant modules, a module M is called *automorphism-invariant* if M is invariant under all automorphisms of its injective envelope. In ([3, Theorem 16]), the authors Er,

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Singh and Srivastava proved that a module M is automorphism-invariant if and only if M is a pseudo-injective module, i.e., for any submodule X of M, every monomorphism $X \to M$ can be extended to an endomorphism of M. In 2016, the authors Koşan, Quynh, and Srivastava ([9]) studied rings whose every right ideal is automorphism-invariant (they are called *right a-rings*). In 2021, the authors Quynh, Abyzov and Trang generalized a-rings to fa-rings (rings all of whose finitely generated ideals are automorphism-invariant) ([13]). The authors showed that a right fa-ring is isomorphic to a formal triangular matrix ring of the form $\begin{pmatrix} S & 0 \\ M & T \end{pmatrix}$, where S is a square-full von Neumann regular self-injective ring, T is a right square-free ring and M is a T-S-bimodule. For endomorphism rings of automorphism-invariant modules, the authors Asensio and Srivastava showed that: if M is an automorphism-invariant module, then $\mathrm{End}(M)/J(\mathrm{End}(M))$ is von Neumann regular and idempotents lift modulo $J(\mathrm{End}(M))$; moreover, $\mathrm{End}(M)$ is a clean ring ([4]). They also proved that if $\mathrm{End}(M)$ has no homomorphic images isomorphic to $\mathbb{Z}_2$, then M is quasi-injective (see [14]).

In this paper, we continue to study some properties of endomorphism rings of automorphism-invariant modules and automorphism-invariant rings. In particular, we show that if M is an automorphism-invariant self-generator module with finite Goldie dimension, $S = \mathrm{End}(M_R)$ and I is a maximal left ideal of S, then $I = A_u$ for some right uniform element $u \in S$. Moreover, we obtain that if R is right automorphism-invariant and has finite Goldie dimension, then R has a ring decomposition $R = R_1 \times R_2$, where R_1 is semisimple artinian and every simple right ideal of R_2 is nilpotent.

Throughout this article all rings are associative rings with identity and all modules are right unital unless stated otherwise. For a submodule N of M, we use $N \leq M$ to mean that N is a submodule of M, and we write $N \leq^e M$ to indicate that N is an essential submodule of M, i.e., $N \cap L \neq 0$ for all $0 \neq L \leq M$. If X is a subset of R, we write $r_R(X)$ (resp., $l_R(X)$) for the right (resp., left) annihilator of X in ring R. We also write $E(M)$ to indicate the injective envelope of M. For any term not defined here the reader is referred to [1], [7] and [11].

2. Endomorphism rings of automorphism-invariant modules

Let $S = \mathrm{End}_R(M)$ be the endomorphism ring of a right R-module M. Following [12], an element $u \in S$ is called a *right uniform element* of S if $u \neq 0$ and $\mathrm{Im}(u)$ is a uniform submodule of M. An element $u \in R$ is called *uniform* if uR is a uniform right ideal. Let $u \in S$ be a right uniform element of S,

we denote
$$A_u = \{s \in S | \operatorname{Ker}(s) \cap \operatorname{Im}(u) \neq 0\}.$$

In this section, we carry over some results of Nicholson and Yousif about the principally injectivity of modules to automorphism-invariant modules. First, we need the following lemmas:

Lemma 1. *Let M be an automorphism-invariant module with $S = \operatorname{End}(M)$. If φ and ψ are endomorphisms of M such that $\operatorname{Ker}(\varphi) = \operatorname{Ker}(\psi)$, then $S\varphi = S\psi$.*

Proof. We consider the mapping
$$\begin{aligned} \phi : \varphi(M) &\longrightarrow \psi(M) \\ \varphi(m) &\longmapsto \psi(m). \end{aligned}$$

One can check that ϕ is an isomorphism. We have that M is an automorphism-invariant module and obtain that there is an endomorphism $\bar{\phi}$ of M such that $\bar{\phi}$ is an extension of ϕ. Then, for all $m \in M$ we have
$$\psi(m) = \phi(\varphi(m)) = \bar{\phi}(\varphi(m)) = (\bar{\phi}\varphi)(m).$$

It shows that $\psi = \bar{\phi}\varphi \in S\varphi$. Similarly, we also have $\varphi \in S\psi$. We deduce that $S\varphi = S\psi$. ■

Lemma 2. *Let M be an automorphism-invariant module and $S = \operatorname{End}(M)$. Then, for any right uniform element u of S, the set*
$$A_u = \{s \in S \mid \operatorname{Ker}(s) \cap \operatorname{Im}(u) \neq 0\}$$
is the unique maximal left ideal of S containing $l_S(\operatorname{Im}(u))$.

Proof. One can check that $l_S(\operatorname{Im}(u)) \leq A_u$ and $A_u \neq S$ (because $1 \notin A_u$). Now, we claim that A_u is a maximal left ideal of S. In fact, for any $s \in S \setminus A_u$, we have $\operatorname{Im}(u) \cap \operatorname{Ker}(s) = 0$. It is easy to see that $\operatorname{Ker}(u) \leq \operatorname{Ker}(su)$. For any $m \in \operatorname{Ker}(su)$, we have $sum = 0$ or $um \in \operatorname{Ker}(s) \cap \operatorname{Im}(u) = 0$. It means that $m \in \operatorname{Ker}(u)$. This shows that $\operatorname{Ker}(u) = \operatorname{Ker}(su)$. From Lemma 1, it immediately infers that $Su = Ssu$. Then, we have $S = l_S(u) + Ss$. We deduce that A_u is a maximal left ideal of S. It remains to show that A_u is unique. In fact, assume that there is another maximal left ideal L of S containing $l_S(\operatorname{Im} u)$ and $L \neq A_u$. Take $s \in L \setminus A_u$, and by the same process as above, we have $S = l_S(u) + S_S \subseteq L$, which implies that $S = L$, a contradiction. ■

Corollary 1. *Let M be a quasi-injective module and $S = \operatorname{End}(M)$. Then, for any right uniform element u of S, A_u is the unique maximal left ideal of S containing $l_S(\operatorname{Im}(u))$.*

A ring R is called *right automorphism-invariant* if R_R is an automorphism-invariant module.

Corollary 2. *Let R be a right automorphism-invariant ring. If $u \in R$ is a uniform element of R, A_u is the unique maximal left ideal which contains $l_R(u)$.*

Lemma 3. *Let M be an automorphism-invariant module, $S = \mathrm{End}(M)$ and $W = \bigoplus_{i=1}^{n} u_i(M)$ be a direct sum of uniform submodules $u_i(M)$ of M. If $A \leq S$ is a maximal left ideal which is not of the form A_u for some right uniform element u of S, then there is $\psi \in A$ such that $\mathrm{Ker}(1-\psi) \cap W$ is essential in W.*

Proof. From the hypothesis $A \neq A_{u_1}$, we can take $k \in A \setminus A_{u_1}$. Then $\mathrm{Im}\, u_1 \cap \mathrm{Ker}\, k = 0$, and so $\mathrm{Ker}(u_1) = \mathrm{Ker}(ku_1)$. By Lemma 1, we have $S(ku_1) = = Su_1$. Consequently we have $u_1 = \alpha_1(ku_1)$ for some $\alpha_1 \in S$. Let $\varphi_1 = \alpha_1 k \in \in SA = A$. Then $(1-\varphi_1)u_1 = 0$. This shows that $\mathrm{Ker}(1-\varphi_1) \cap u_1(M) = = u_1(M) \neq 0$. If $\mathrm{Ker}(1-\varphi_1) \cap u_i(M) \neq 0$ for all $i \geq 2$, then we are done and in this case

$$\bigoplus_{i=1}^{n} (\mathrm{Ker}(1-\varphi_1) \cap u_i(M)) \leq^e W.$$

Without loss of generality, we now assume that $\mathrm{Ker}(1-\varphi_1) \cap u_2(M) = 0$. It follows that $(1-\varphi_1)(u_2(M)) \simeq u_2(M)$ is uniform. Since $A \neq A_{(1-\varphi_1)u_2}$, we can take any $h \in A \setminus A_{(1-\varphi_1)u_2}$. By using the above argument, there exists $\alpha_2 \in S$ such that $(1-\varphi_1)u_2 = \alpha_2 h(1-\varphi_1)u_2$. It follows that

$$(1-(\alpha_2 h + \varphi_1 - \alpha_2 h\varphi_1))u_2 = 0.$$

Let $\varphi_2 = \alpha_2 h + \varphi_1 - \alpha_2 h \varphi_1$. Then $(1-\varphi_2)u_i = 0$ for $i = 1, 2$. Continuing this way, we eventually obtain a $\psi \in A$ such that $\mathrm{Ker}(1-\psi) \cap u_i(M) \neq 0$ for all $i = 1, \ldots, n$. In other words, we have shown that $\mathrm{Ker}(1-\psi) \cap W$ is essential in W as required. ■

A module M has *finite Goldie dimension* (or has *finite rank*) if $E(M)$ is a finite direct sum of indecomposable submodules. A module N is called M-*generated* (or M generates N) if there is an epimorphism $M^{(I)} \to N$ for some index set I. A module M is called *self-generator* if it generates all its submodules (see [15]). The following theorem describes some properties of endomorphism rings of automorphism-invariant modules.

Theorem 1. *Let M be an automorphism-invariant self-generator module with finite Goldie dimension and $S = \mathrm{End}(M)$. If I is a maximal left ideal of S, then $I = A_u$ for some right uniform element $u \in S$.*

Proof. Since M is a self-generator module which has finite Goldie dimension, there exist elements $u_1, ..., u_n$ of S such that

$$W = u_1(M) \oplus u_2(M) \oplus \cdots \oplus u_n(M)$$

is essential in M, where each $u_i(M)$ is uniform. Moreover, M is an automorphism-invariant module, we have

$$J(S) = \{s \in S|\ \mathrm{Ker}(s) \text{ is essential in } M\}$$

by Proposition 1 in [4].

Suppose on the contrary that I is not of the form A_u for some right uniform element of $u \in S$. Then, by Lemma 3, there exists a $\varphi \in I$ such that $\mathrm{Ker}(1 - \varphi) \cap W$ is essential in W. It follows that $1 - \varphi \in J(S) \leq I$, a contradiction. Hence $I = A_u$ for some right uniform element $u \in S$. ■

As a consequence, we immediately get the following result for right automorphism-invariant rings.

Corollary 3. *Let R be a right automorphism-invariant ring which has right finite Goldie dimension. If $I \leq R$ is a maximal left ideal, then $I = A_u$ for some right uniform element $u \in R$.*

3. On automorphism-invariant rings

In this section, we consider some properties of automorphism-invariant rings. The structure of rings via automorphism-invariance are studied.

Theorem 2. *Assume that R is a right automorphism-invariant ring and $x, y \in R$.*

(1) *If xR embeds in yR then Rx is an image of Ry.*

(2) *If $xR \cong yR$ then $Rx \cong Ry$.*

Proof. (1) Assume that $\alpha : xR \to yR$ is a monomorphism. Call $\alpha(x) = yu$ for some $u \in R$. Since R is right automorphism-invariant, there is an element z of R such that $\alpha(r) = zr$ for all $r \in xR$. Then, we have $yu = \alpha(x) = zx$.

We consider the mapping

$$\begin{aligned} \phi : Ry &\longrightarrow Rx \\ ry &\longmapsto r(yu) = r(zx). \end{aligned}$$

One can check that θ is an R-homomorphism. Next, we show that ϕ is an S-epimorphism. In fact, we have α is a monomorphism and obtain that $r_R(x) = r_R(\alpha(x))$. Then, the mapping $\beta : \alpha(x)R \to xR$ is defined by $\beta(\alpha(x)r) = xr$ for all $r \in R$, is an R-monomorphism. Thus, there is an element t of R such that $\beta(\alpha(x)r) = t\alpha(x)r$ for all $r \in R$. It follows that

$$x = \beta(\alpha(x)) = t\alpha(x) = t(zx) \in \mathrm{Im}(\phi).$$

(2) Call $\alpha : xR \to yR$ an isomorphism. By the proof of (1), there are R-homomorphisms

$$\begin{aligned} \phi : Ry &\longrightarrow Rx \\ ry &\longmapsto r(zx) \end{aligned}$$

with $yu = \alpha(x) = zx$ for some $u, z \in R$, and

$$\begin{aligned} \phi' : Rx &\longrightarrow Ry \\ f(x) &\longmapsto f\bar{\alpha}'(y), \end{aligned}$$

where $xv = \alpha^{-1}(y) = ty$ for some $v, t \in R$.

One can check that $\phi \circ \phi' = 1$ and $\phi' \circ \phi = 1$. We deduce that ϕ is an R-isomorphism. ■

Corollary 4. *Let R be a right self-injective ring and $x, y \in R$.*

(1) *If xR embeds in yR then Rx is an image of Ry.*

(2) *If $xR \cong yR$ then $Rx \cong Ry$.*

A module M is called *weakly injective* if, for every finitely generated submodule N of $E(M)$, we have $N \leq X \leq E(M)$ for some $X \cong M$.

Theorem 3. *A ring R is right self-injective if and only if R is right automorphism-invariant and R_R is weakly injective.*

Proof. It easy to see that if R is right self-injective then R is right automorphism-invariant and R_R is weakly injective. Conversely, we show that $R = E(R)$. In fact, let x be an arbitrary element of $E(R)$. Then, we have $R+xR \leq X \leq E(R)$ for some $X \cong R$. Note that R is right automorphism-invariant and so X is an automorphism-invariant module. It follows that R is a direct summand of X. Since R is essential in $E(R)$, $R = X$. It shows that $x \in R$. We deduce that $R = E(R)$. ■

Theorem 4. *If R is a right automorphism-invariant ring and has finite Goldie dimension, then R has a ring decomposition $R = R_1 \times R_2$, where R_1 is semisimple artinian and every simple right ideal of R_2 is nilpotent.*

Proof. Assume that R is right automorphism-invariant and has finite Goldie dimension. Then, R is a semiperfect ring. Let $1 = e_1 + e_2 + \cdots + e_n$, where e_i are orthogonal primitive idempotents. We can assume that e_iR is simple for all $1 \leq i \leq m$ and not simple if $n > i > m$. Next, we show that $e_iRe_j = 0 = e_jRe_i$ for all $1 \leq i \leq m < j \leq n$. If there is a homomorphism $f : e_jR \to e_iR$ for some $1 \leq i \leq m < j \leq n$, then f is an epimorphism. But, e_jR is indecomposable. It follows that $f = 0$. Suppose that e_jRe_i is nonzero. Then, there is a nonzero homomorphism $g : e_iR \to e_jR$. Since e_iR is simple, g is a monomorphism. It follows that $e_iR \cong g(e_iR)$ is a direct summand of e_jR, and so $e_iR \cong g(e_iR) = = e_jR$, a contradiction.

Take $e = e_1 + e_2 + \cdots + e_m$. Then, we can check that e is a central idempotent of R. Put, $R_1 = eR$ and $R_2 = (1 - e)R$. It follows that R_1 is semisimple artinian. Next, we show that every simple right ideal of R_2 is nilpotent. Assume that K is a simple right ideal of R_2. Suppose that K is not nilpotent. Then, $K = fR$ for some $f^2 = f \in R_2$. There exists $j > m$ such that $fRe_j \neq 0$. Thus, $fR \cong e_jR$ as before, a contradiction. ■

Corollary 5. *A semiprime semiperfect right automorphism-invariant ring is semisimple artinian.*

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