# ON ENDOMORPHISM RINGS OF AUTOMORPHISM-INVARIANT MODULES

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Abstract. In this paper, we give some results of endomorphism rings of automorphism-invariant modules. It shows that if M is an automorphisminvariant self-generator module with finite Goldie dimension then every maximal left ideal of End(M) has the form  $A_u$  for some right uniform element u in End(M). We also show that a ring R is right self-injective if and only if R is right automorphism-invariant and  $R_R$  is weakly injective. Finally, we obtain a decomposition of finite Goldie dimension right automorphism-invariant rings.

# 1. Introduction

The study of generalizations of quasi-injective modules and injective modules have made a significant contribution to ring and module theory. In 1961, Johnson and Wong ([8]) studied quasi-injective modules, a module M is called quasi-injective if any homomorphism from a submodule of M to M extends to an endomorphism of M; the authors showed that a module M is quasi-injective if and only if M is invariant under any endomorphism of its injective envelope. This class of modules is known as the class of endomorphism-invariant modules. In 2013, Lee and Zhou ([10]) studied automorphism-invariant modules, a module M is called *automorphism-invariant* if M is invariant under all automorphisms of its injective envelope. In ([3, Theorem 16]), the authors Er,

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Singh and Srivastava proved that a module M is automorphism-invariant if and only if M is a pseudo-injective module, i.e., for any submodule X of M, every monomorphism  $X \to M$  can be extended to an endomorphism of M. In 2016, the authors Koşan, Quynh, and Srivastava ([9]) studied rings whose every right ideal is automorphism-invariant (they are called *right a-rings*). In 2021. the authors Quynh, Abyzov and Trang generalized a-rings to fa-rings (rings all of whose finitely generated ideals are automorphism-invariant) ([13]). The authors showed that a right fa-ring is isomorphic to a formal triangular matrix ring of the form  $\begin{pmatrix} S & 0 \\ M & T \end{pmatrix}$ , where S is a square-full von Neumann regular self-injective ring, T is a right square-free ring and M is a T-S-bimodule. For endomorphism rings of automorphism-invariant modules, the authors Asensio and Srivastava showed that: if M is an automorphism-invariant module, then  $\operatorname{End}(M)/J(\operatorname{End}(M))$  is von Neumann regular and idempotents lift modulo J(End(M)); moreover, End(M) is a clean ring ([4]). They also proved that if  $\operatorname{End}(M)$  has no homomorphic images isomorphic to  $\mathbb{Z}_2$ , then M is quasiinjective (see [14]).

In this paper, we continue to study some properties of endomorphism rings of automorphism-invariant modules and automorphism-invariant rings. In particular, we show that if M is an automorphism-invariant self-generator module with finite Goldie dimension,  $S = \text{End}(M_R)$  and I is a maximal left ideal of S, then  $I = A_u$  for some right uniform element  $u \in S$ . Moreover, we obtain that if R is right automorphism-invariant and has finite Goldie dimension, then Rhas a ring decomposition  $R = R_1 \times R_2$ , where  $R_1$  is semisimple artinian and every simple right ideal of  $R_2$  is nilpotent.

Throughout this article all rings are associative rings with identity and all modules are right unital unless stated otherwise. For a submodule N of M, we use  $N \leq M$  to mean that N is a submodule of M, and we write  $N \leq^e M$  to indicate that N is an essential submodule of M, i.e.,  $N \cap L \neq 0$  for all  $0 \neq L \leq M$ . If X is a subset of R, we write  $r_R(X)$  (resp.,  $l_R(X)$ ) for the right (resp., left) annihilator of X in ring R. We also write E(M) to indicate the injective envelope of M. For any term not defined here the reader is referred to [1], [7] and [11].

#### 2. Endomorphism rings of automorphism-invariant modules

Let  $S = \text{End}_R(M)$  be the endomorphism ring of a right *R*-module *M*. Following [12], an element  $u \in S$  is called a *right uniform element* of *S* if  $u \neq 0$ and Im(u) is a uniform submodule of *M*. An element  $u \in R$  is called *uniform* if uR is a uniform right ideal. Let  $u \in S$  be a right uniform element of *S*, we denote

$$A_u = \{s \in S | \operatorname{Ker}(s) \cap \operatorname{Im}(u) \neq 0\}.$$

In this section, we carry over some results of Nicholson and Yousif about the principally injectivity of modules to automorphism-invariant modules. First, we need the following lemmas:

**Lemma 1.** Let M be an automorphism-invariant module with S = End(M). If  $\varphi$  and  $\psi$  are endomorphisms of M such that  $\text{Ker}(\varphi) = \text{Ker}(\psi)$ , then  $S\varphi = S\psi$ .

**Proof.** We consider the mapping

$$\phi:\varphi(M)\longrightarrow\psi(M)$$
$$\varphi(m)\longmapsto\psi(m).$$

One can check that  $\phi$  is an isomorphism. We have that M is an automorphisminvariant module and obtain that there is an endomorphism  $\overline{\phi}$  of M such that  $\overline{\phi}$  is an extension of  $\phi$ . Then, for all  $m \in M$  we have

$$\psi(m) = \phi(\varphi(m)) = \bar{\phi}(\varphi(m)) = (\bar{\phi}\varphi)(m).$$

It shows that  $\psi = \overline{\phi}\varphi \in S\varphi$ . Similarly, we also have  $\varphi \in S\psi$ . We deduce that  $S\varphi = S\psi$ .

**Lemma 2.** Let M be an automorphism-invariant module and S = End(M). Then, for any right uniform element u of S, the set

$$A_u = \{ s \in S \mid \operatorname{Ker}(s) \cap \operatorname{Im}(u) \neq 0 \}$$

is the unique maximal left ideal of S containing  $l_S(\operatorname{Im}(u))$ .

**Proof.** One can check that  $l_S(\operatorname{Im}(u)) \leq A_u$  and  $A_u \neq S$  (because  $1 \notin A_u$ ). Now, we claim that  $A_u$  is a maximal left ideal of S. In fact, for any  $s \in S \setminus A_u$ , we have  $\operatorname{Im}(u) \cap \operatorname{Ker}(s) = 0$ . It is easy to see that  $\operatorname{Ker}(u) \leq \operatorname{Ker}(su)$ . For any  $m \in \operatorname{Ker}(su)$ , we have sum = 0 or  $um \in \operatorname{Ker}(s) \cap \operatorname{Im}(u) = 0$ . It means that  $m \in \operatorname{Ker}(u)$ . This shows that  $\operatorname{Ker}(u) = \operatorname{Ker}(su)$ . From Lemma 1, it immediately infers that Su = Ssu. Then, we have  $S = l_S(u) + Ss$ . We deduce that  $A_u$  is a maximal left ideal of S. It remains to show that  $A_u$  is unique. In fact, assume that there is another maximal left ideal L of S containing  $l_S(\operatorname{Im} u)$  and  $L \neq A_u$ . Take  $s \in L \setminus A_u$ , and by the same process as above, we have  $S = l_S(u) + S_S \subseteq L$ , which implies that S = L, a contradiction.

**Corollary 1.** Let M be a quasi-injective module and S = End(M). Then, for any right uniform element u of S,  $A_u$  is the unique maximal left ideal of Scontaining  $l_S(\text{Im}(u))$ . A ring R is called *right automorphism-invariant* if  $R_R$  is an automorphism-invariant module.

**Corollary 2.** Let R be a right automorphism-invariant ring. If  $u \in R$  is a uniform element of R,  $A_u$  is the unique maximal left ideal which contains  $l_R(u)$ .

**Lemma 3.** Let M be an automorphism-invariant module, S = End(M) and  $W = \bigoplus_{i=1}^{n} u_i(M)$  be a direct sum of uniform submodules  $u_i(M)$  of M. If  $A \leq S$  is a maximal left ideal which is not of the form  $A_u$  for some right uniform element u of S, then there is  $\psi \in A$  such that  $\text{Ker}(1-\psi) \cap W$  is essential in W.

**Proof.** From the hypothesis  $A \neq A_{u_1}$ , we can take  $k \in A \setminus A_{u_1}$ . Then Im  $u_1 \cap \text{Ker } k = 0$ , and so  $\text{Ker}(u_1) = \text{Ker}(ku_1)$ . By Lemma 1, we have  $S(ku_1) =$  $= Su_1$ . Consequently we have  $u_1 = \alpha_1(ku_1)$  for some  $\alpha_1 \in S$ . Let  $\varphi_1 = \alpha_1 k \in$  $\in SA = A$ . Then  $(1 - \varphi_1)u_1 = 0$ . This shows that  $\text{Ker}(1 - \varphi_1) \cap u_1(M) =$  $= u_1(M) \neq 0$ . If  $\text{Ker}(1 - \varphi_1) \cap u_i(M) \neq 0$  for all  $i \geq 2$ , then we are done and in this case

$$\bigoplus_{i=1}^{n} (\operatorname{Ker}(1-\varphi_1) \cap u_i(M)) \leq^{e} W.$$

Without loss of generality, we now assume that  $\operatorname{Ker}(1-\varphi_1) \cap u_2(M) = 0$ . It follows that  $(1-\varphi_1)(u_2(M)) \simeq u_2(M)$  is uniform. Since  $A \neq A_{(1-\varphi_1)u_2}$ , we can take any  $h \in A \setminus A_{(1-\varphi_1)u_2}$ . By using the above argument, there exists  $\alpha_2 \in S$  such that  $(1-\varphi_1)u_2 = \alpha_2h(1-\varphi_1)u_2$ . It follows that

$$(1 - (\alpha_2 h + \varphi_1 - \alpha_2 h \varphi_1))u_2 = 0.$$

Let  $\varphi_2 = \alpha_2 h + \varphi_1 - \alpha_2 h \varphi_1$ . Then  $(1 - \varphi_2)u_i = 0$  for i = 1, 2. Continuing this way, we eventually obtain a  $\psi \in A$  such that  $\operatorname{Ker}(1 - \psi) \cap u_i(M) \neq 0$  for all  $i = 1, \ldots, n$ . In other words, we have shown that  $\operatorname{Ker}(1 - \psi) \cap W$  is essential in W as required.

A module M has finite Goldie dimension (or has finite rank) if E(M) is a finite direct sum of indecomposable submodules. A module N is called Mgenerated (or M generates N) if there is an epimorphism  $M^{(I)} \to N$  for some index set I. A module M is called self-generator if it generates all its submodules (see [15]). The following theorem describes some properties of endomorphism rings of automorphism-invariant modules.

**Theorem 1.** Let M be an automorphism-invariant self-generator module with finite Goldie dimension and S = End(M). If I is a maximal left ideal of S, then  $I = A_u$  for some right uniform element  $u \in S$ .

**Proof.** Since M is a self-generator module which has finite Goldie dimension, there exist elements  $u_1, ..., u_n$  of S such that

$$W = u_1(M) \oplus u_2(M) \oplus \cdots \oplus u_n(M)$$

is essential in M, where each  $u_i(M)$  is uniform. Moreover, M is an automorphism-invariant module, we have

$$J(S) = \{ s \in S | \text{ Ker}(s) \text{ is essential in } M \}$$

by Proposition 1 in [4].

Suppose on the contrary that I is not of the form  $A_u$  for some right uniform element of  $u \in S$ . Then, by Lemma 3, there exists a  $\varphi \in I$  such that  $\text{Ker}(1 - -\varphi) \cap W$  is essential in W. It follows that  $1 - \varphi \in J(S) \leq I$ , a contradiction. Hence  $I = A_u$  for some right uniform element  $u \in S$ .

As a consequence, we immediately get the following result for right automorphism-invariant rings.

**Corollary 3.** Let R be a right automorphism-invariant ring which has right finite Goldie dimension. If  $I \leq R$  is a maximal left ideal, then  $I = A_u$  for some right uniform element  $u \in R$ .

#### 3. On automorphism-invariant rings

In this section, we consider some properties of automorphism-invariant rings. The structure of rings via automorphism-invariance are studied.

**Theorem 2.** Assume that R is a right automorphism-invariant ring and  $x, y \in R$ .

- (1) If xR embeds in yR then Rx is an image of Ry.
- (2) If  $xR \cong yR$  then  $Rx \cong Ry$ .

**Proof.** (1) Assume that  $\alpha : xR \to yR$  is a monomorphism. Call  $\alpha(x) = yu$  for some  $u \in R$ . Since R is right automorphism-invariant, there is an element z of R such that  $\alpha(r) = zr$  for all  $r \in xR$ . Then, we have  $yu = \alpha(x) = zx$ .

We consider the mapping

$$\phi: Ry \longrightarrow Rx$$
$$ry \longmapsto r(yu) = r(zx).$$

One can check that  $\theta$  is an *R*-homomorphism. Next, we show that  $\phi$  is an *S*-epimorphism. In fact, we have  $\alpha$  is a monomorphism and obtain that  $r_R(x) = r_R(\alpha(x))$ . Then, the mapping  $\beta : \alpha(x)R \to xR$  is defined by  $\beta(\alpha(x)r) = xr$  for all  $r \in R$ , is an *R*-monomorphism. Thus, there is an element t of R such that  $\beta(\alpha(x)r) = t\alpha(x)r$  for all  $r \in R$ . It follows that

$$x = \beta(\alpha(x)) = t\alpha(x) = t(zx) \in \operatorname{Im}(\phi).$$

(2) Call  $\alpha : xR \to yR$  an isomorphism. By the proof of (1), there are *R*-homomorphisms

$$\phi : Ry \longrightarrow Rx$$
$$ry \longmapsto r(zx)$$

with  $yu = \alpha(x) = zx$  for some  $u, z \in R$ , and

$$\phi' : Rx \longrightarrow Ry$$
$$f(x) \longmapsto f\bar{\alpha}'(y)$$

where  $xv = \alpha^{-1}(y) = ty$  for some  $v, t \in R$ .

One can check that  $\phi \circ \phi' = 1$  and  $\phi' \circ \phi = 1$ . We deduce that  $\phi$  is an *R*-isomorphism.

**Corollary 4.** Let R be a right self-injective ring and  $x, y \in R$ .

- (1) If xR embeds in yR then Rx is an image of Ry.
- (2) If  $xR \cong yR$  then  $Rx \cong Ry$ .

A module M is called *weakly injective* if, for every finitely generated submodule N of E(M), we have  $N \leq X \leq E(M)$  for some  $X \cong M$ .

**Theorem 3.** A ring R is right self-injective if and only if R is right automorphism-invariant and  $R_R$  is weakly injective.

**Proof.** It easy to see that if R is right self-injective then R is right automorphism-invariant and  $R_R$  is weakly injective. Conversely, we show that R = E(R). In fact, let x be an arbitrary element of E(R). Then, we have  $R+xR \le \le X \le E(R)$  for some  $X \cong R$ . Note that R is right automorphism-invariant and so X is an automorphism-invariant module. It follows that R is a direct summand of X. Since R is essential in E(R), R = X. It shows that  $x \in R$ . We deduce that R = E(R).

**Theorem 4.** If R is a right automorphism-invariant ring and has finite Goldie dimension, then R has a ring decomposition  $R = R_1 \times R_2$ , where  $R_1$  is semisimple artinian and every simple right ideal of  $R_2$  is nilpotent.

**Proof.** Assume that R is right automorphism-invariant and has finite Goldie dimension. Then, R is a semiperfect ring. Let  $1 = e_1 + e_2 + \cdots + e_n$ , where  $e_i$  are orthogonal primitive idempotents. We can assume that  $e_iR$  is simple for all  $1 \le i \le m$  and not simple if n > i > m. Next, we show that  $e_iRe_j = 0 = e_jRe_i$  for all  $1 \le i \le m < j \le n$ . If there is a homomorphism  $f : e_jR \to e_iR$  for some  $1 \le i \le m < j \le n$ , then f is an epimorphism. But,  $e_jR$  is indecomposable. It

follows that f = 0. Suppose that  $e_j R e_i$  is nonzero. Then, there is a nonzero homomorphism  $g : e_i R \to e_j R$ . Since  $e_i R$  is simple, g is a monomorphism. It follows that  $e_i R \cong g(e_i R)$  is a direct summand of  $e_j R$ , and so  $e_i R \cong g(e_i R) = e_j R$ , a contradiction.

Take  $e = e_1 + e_2 + \cdots + e_m$ . Then, we can check that e is a central idempotent of R. Put,  $R_1 = eR$  and  $R_2 = (1 - e)R$ . It follows that  $R_1$  is semisimple artinian. Next, we show that every simple right ideal of  $R_2$  is nilpotent. Assume that K is a simple right ideal of  $R_2$ . Suppose that K is not nilpotent. Then, K = fR for some  $f^2 = f \in R_2$ . There exists j > m such that  $fRe_i \neq 0$ . Thus,  $fR \cong e_iR$  as before, a contradiction.

**Corollary 5.** A semiprime semiperfect right automorphism-invariant ring is semisimple artinian.

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