

ALMOST BALCOBALANCING NUMBERS

Ahmet Tekcan and Meryem Yıldız

(Bursa, Türkiye)

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Abstract. In this work, we determined the general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and second type in terms of balancing and Lucas-balancing numbers.

1. Introduction

A positive integer n is called a balancing number ([2]) if the Diophantine equation

$$(1.1) \quad 1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$$

holds for some positive integer r which is called balancer corresponding to n . If n is a balancing number with balancer r , then from (1.1)

$$(1.2) \quad r = \frac{-2n - 1 + \sqrt{8n^2 + 1}}{2}.$$

Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (1.2), $8(0)^2 + 1 = 1$ and $8(1)^2 + 1 = 3^2$ are perfect squares. So Behera and Panda accepted 0 and 1 to be balancing numbers.

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Later Panda and Ray ([12]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$(1.3) \quad 1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r)$$

holds for some positive integer r which is called cobalancer corresponding to n . If n is a cobalancing number with cobalancer r , then from (1.3)

$$(1.4) \quad r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}.$$

From (1.4), $8(0)^2 + 8(0) + 1 = 1$ is a perfect square. So they accepted 0 to be a cobalancing number, just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then from (1.2), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and from (1.4), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus

$$C_n = \sqrt{8B_n^2 + 1} \quad \text{and} \quad c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

are integers which are called Lucas-balancing number and Lucas-cobalancing number, respectively (see also [10, 11, 15, 16]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [7], Liptai proved that there is no Fibonacci balancing number except 1 and in [8] he proved that there is no Lucas-balancing number. In [18], Szalay considered the same problem and obtained some nice results by a different method. In [5], Kovács, Liptai, Olajos extended the concept of balancing numbers to the (a, b) -balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + r) + b)$$

for some positive integers n and r , then $an + b$ is an (a, b) -balancing number. The sequence of (a, b) -balancing numbers is denoted by $B_m^{(a, b)}$ for $m \geq 1$. In [6], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l) -power numerical center for y if

$$1^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l) -power numerical centers. For positive integers k, x , let $\Pi_k(x) = x(x + 1) \cdots (x + k - 1)$. Then it was proved

in [5] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [25] Tengely, considered the case $k = 5$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [14], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers and in [17], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [3], Dash, Ota, Dash considered the t -balancing numbers for an integer $t \geq 1$. They called that n is a t -balancing number if the Diophantine equation

$$1 + 2 + \cdots + n - 1 = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t)$$

holds for some positive integer r which is called t -balancer. Similarly a positive integer n is called a t -cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t)$$

holds for some positive integer r which is called t -cobalancer. In [24], Tekcan and Aydm determined the general terms of t -balancing and Lucas t -balancing numbers, and in [21], Tekcan and Erdem determined the general terms of t -cobalancing and Lucas t -cobalancing numbers in terms of balancing and Lucas-balancing numbers. In [13], G.K. Panda and A.K. Panda defined that a positive integer n is called an almost balancing number if the Diophantine equation

$$(1.5) \quad [(n + 1) + (n + 2) + \cdots + (n + r)] - [1 + 2 + \cdots + (n - 1)] = 1$$

holds for some positive integer r , which is called the almost balancer. By (1.5), they have two cases: If $[(n + 1) + (n + 2) + \cdots + (n + r)] - [1 + 2 + \cdots + (n - 1)] = 1$, then n is called an almost balancing number of first type and r is called an almost balancer of first type and in this case

$$(1.6) \quad r = \frac{-2n - 1 + \sqrt{8n^2 + 9}}{2},$$

and if $[(n + 1) + (n + 2) + \cdots + (n + r)] - [1 + 2 + \cdots + (n - 1)] = -1$, then n is called an almost balancing number of second type and r is called an almost balancer of second type and in this case

$$(1.7) \quad r = \frac{-2n - 1 + \sqrt{8n^2 - 7}}{2}.$$

(From (1.7), they noted that $8(1)^2 - 7 = 1^2$ and $8(2)^2 - 7 = 5^2$ are perfect squares. So they accepted 1 and 2 to be almost balancing numbers of second type). Let B_n^* denote the almost balancing number of first type and let B_n^{**} denote the almost balancing number of second type. Then from (1.6), B_n^* is an almost balancing number of first type if and only if $8(B_n^*)^2 + 9$ is a perfect

square and from (1.7), B_n^{**} is an almost balancing number of second type if and only if $8(B_n^{**})^2 - 7$ is a perfect square. So

$$C_n^* = \sqrt{8(B_n^*)^2 + 9} \quad \text{and} \quad C_n^{**} = \sqrt{8(B_n^{**})^2 - 7}$$

are integers which are called almost Lucas-balancing number of first type and of second type, respectively. In [19], Tekcan derived some results on almost balancing numbers, triangular numbers and square triangular numbers and in [20], he considered the sums and spectral norms of all almost balancing numbers.

2. Almost balcobalancing numbers

In [22], Tekcan and Yıldız defined balcobalancing numbers, balcobalancers and Lucas-balcobalancing numbers and determined the general terms of them. They sum both sides of (1.1) and (1.3), and they get the Diophantine equation

$$(2.1) \quad 1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n = 2[(n+1)+(n+2)+\cdots+(n+r)]$$

and called a positive integer n is called a balcobalancing number if the Diophantine equation in (2.1) verified for some positive integer r which is called balcobalancer. From (2.1), they get

$$(2.2) \quad r = \frac{-2n - 1 + \sqrt{8n^2 + 4n + 1}}{2}.$$

Let B_n^{bc} denote the balcobalancing number. Then from (2.2), B_n^{bc} is a balcobalancing number if and only if $8(B_n^{bc})^2 + 4B_n^{bc} + 1$ is a perfect square. Thus

$$C_n^{bc} = \sqrt{8(B_n^{bc})^2 + 4B_n^{bc} + 1}$$

is an integer which are called Lucas-balcobalancing number (see also [23]).

As in (1.5), from (2.1), we say that a positive integer n is called an almost balcobalancing number if the Diophantine equation

$$(2.3) \quad \left| \begin{array}{c} 1 + 2 + \cdots + (n - 1) + 1 + 2 + \cdots + (n - 1) + n \\ -2[(n + 1) + (n + 2) + \cdots + (n + r)] \end{array} \right| = 1$$

holds for some positive integer r which is called almost balcobalancer.

From (2.3), we have two cases: If $[1 + 2 + \dots + (n - 1) + 1 + 2 + \dots + (n - 1) + n] - 2[(n + 1) + (n + 2) + \dots + (n + r)] = 1$, then n is called an almost balcobalancing number of first type, r is called an almost balcobalancer of first type and in this case

$$(2.4) \quad r = \frac{-2n - 1 + \sqrt{8n^2 + 4n - 3}}{2}.$$

For example 3, 39, 109 are almost balcobalancing numbers of first type with almost balcobalancers of first type 1, 16, 45. (From (2.4), we note that $8(1)^2 + 4(1) - 3 = 3^2$ is a perfect square, but in this case $r = 0$. Nevertheless we accept that 1 is an almost balcobalancing number of first type, just like Behera and Panda accepted 0 and 1 to be balancing numbers). If $[1 + 2 + \dots + (n - 1) + 1 + 2 + \dots + (n - 1) + n] - 2[(n + 1) + (n + 2) + \dots + (n + r)] = -1$, then n is called an almost balcobalancing number of second type, r is called an almost balcobalancer of second type and in this case

$$(2.5) \quad r = \frac{-2n - 1 + \sqrt{8n^2 + 4n + 5}}{2}.$$

For example 5, 179, 6089 are almost balcobalancing numbers of second type with almost balcobalancers of second type 2, 74, 2522.

Let B_n^{bc*} denote the almost balcobalancing number of first type and let B_n^{bc**} denote the almost balcobalancing number of second type. Then by (2.4), B_n^{bc*} is an almost balcobalancing number of first type if and only if $8(B_n^{bc*})^2 + 4B_n^{bc*} - 3$ is a perfect square. So

$$(2.6) \quad C_n^{bc*} = \sqrt{8(B_n^{bc*})^2 + 4B_n^{bc*} - 3}$$

is an integer which is called almost Lucas-balcobalancing number of first type. Similarly from (2.5), B_n^{bc**} is an almost balcobalancing number of second type if and only if $8(B_n^{bc**})^2 + 4B_n^{bc**} + 5$ is a perfect square. So

$$(2.7) \quad C_n^{bc**} = \sqrt{8(B_n^{bc**})^2 + 4B_n^{bc**} + 5}$$

is an integer which is called almost Lucas-balcobalancing number of second type (We denote the almost balcobalancer of first type by R_n^{bc*} and denote almost balcobalancer of second type by R_n^{bc**}).

In this paper, we try to determine the general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first and second type in terms of balancing and Lucas-balancing numbers.

2.1. Almost balcobalancing numbers of first type

From (2.4), we notice that B_n^{bc*} is an almost balcobalancing number of first type if and only if $8(B_n^{bc*})^2 + 4B_n^{bc*} - 3$ is a perfect square. So we set

$$8(B_n^{bc*})^2 + 4B_n^{bc*} - 3 = y^2$$

for some positive integer y . If we multiply both sides of the last equation by 2, then we get

$$16(B_n^{bc*})^2 + 8B_n^{bc*} - 6 = 2y^2$$

and hence

$$(4B_n^{bc*} + 1)^2 - 7 = 2y^2.$$

Taking $x = 4B_n^{bc*} + 1$, we get the Pell equation (see [1, 9])

$$(2.8) \quad x^2 - 2y^2 = 7.$$

Let Ω^{bc*} denotes the set of all (positive) integer solutions of (2.8), that is,

$$\Omega^{bc*} = \{(x, y) : x^2 - 2y^2 = 7\}.$$

In order to determine the set of all integer solutions of (2.8), we need some notations. Let Δ be a non-square discriminant. Then the Δ -order O_Δ is defined to be the ring $O_\Delta = \{x + y\rho_\Delta : x, y \in \mathbb{Z}\}$, where $\rho_\Delta = \sqrt{\frac{\Delta}{4}}$ if $\Delta \equiv 0 \pmod{4}$ or $\frac{1+\sqrt{\Delta}}{2}$ if $\Delta \equiv 1 \pmod{4}$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta}) = \{x + y\sqrt{\Delta} : x, y \in \mathbb{Q}\}$. The unit group O_Δ^u is defined to be the group of units of the ring O_Δ .

Let $F(x, y) = ax^2 + bxy + cy^2$ be an indefinite integral quadratic form ([4]) of discriminant $\Delta = b^2 - 4ac$. Then we can rewrite

$$F(x, y) = \frac{(xa + y\frac{b+\sqrt{\Delta}}{2})(xa + y\frac{b-\sqrt{\Delta}}{2})}{a}.$$

So the module M_F of F is

$$M_F = \{xa + y\frac{b + \sqrt{\Delta}}{2} : x, y \in \mathbb{Z}\} \subset \mathbb{Q}(\sqrt{\Delta}).$$

Therefore we get $(u + v\rho_\Delta)(xa + y\frac{b+\sqrt{\Delta}}{2}) = x'a + y'\frac{b+\sqrt{\Delta}}{2}$, where

$$(2.9) \quad [x' \ y'] = \begin{cases} [x \ y] \begin{bmatrix} u - \frac{b}{2}v & av \\ -cv & u + \frac{b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 0 \pmod{4} \\ [x \ y] \begin{bmatrix} u + \frac{1-b}{2}v & av \\ -cv & u + \frac{1+b}{2}v \end{bmatrix} & \text{if } \Delta \equiv 1 \pmod{4}. \end{cases}$$

Let m be any integer and let Ω denote the set of all integer solutions of $F(x, y) = ax^2 + bxy + cy^2 = m$. Then there is a bijection

$$\Psi : \Omega \rightarrow \{\gamma \in M_F : N(\gamma) = am\}.$$

The action of $O_{\Delta,1}^u = \{\alpha \in O_\Delta^u : N(\alpha) = 1\}$ on Ω is most interesting when Δ is a positive non-square since $O_{\Delta,1}^u$ is infinite. Therefore the orbit of each solution will be infinite and so Ω is either empty or infinite. Since $O_{\Delta,1}^u$ can be explicitly determined, the set Ω is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let ε_Δ be the smallest unit of O_Δ that is greater than 1 and let $\tau_\Delta = \varepsilon_\Delta$ if $N(\varepsilon_\Delta) = 1$ or ε_Δ^2 if $N(\varepsilon_\Delta) = -1$.

Then every $O_{\Delta,1}^u$ orbit of integral solutions of $F(x, y) = m$ contains a solution $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq y \leq U$, where

$$U = \begin{cases} \left| \frac{am\tau_{\Delta}}{\Delta} \right|^{\frac{1}{2}} \left(1 - \frac{1}{\tau_{\Delta}}\right) & \text{if } am > 0 \\ \left| \frac{am\tau_{\Delta}}{\Delta} \right|^{\frac{1}{2}} \left(1 + \frac{1}{\tau_{\Delta}}\right) & \text{if } am < 0. \end{cases}$$

So for finding a set of representatives of the $O_{\Delta,1}^u$ orbits of integral solutions of $F(x, y) = m$, we must find for each y_0 in the range $0 \leq y_0 \leq U$, whether $\Delta y_0^2 + 4am$ is a perfect square or not since

$$(2.10) \quad ax_0^2 + bx_0y_0 + cy_0^2 = m \Leftrightarrow \Delta y_0^2 + 4am = (2ax_0 + by_0)^2.$$

If $\Delta y_0^2 + 4am$ is a perfect square, then from (2.10)

$$x_0 = \frac{-by_0 \pm \sqrt{\Delta y_0^2 + 4am}}{2a}.$$

So there is a set of representatives $\text{Rep} = \{[x_0 \ y_0]\}$. For the matrix M derived from (2.9), the set of all integer solutions of $F(x, y) = m$ is

$$\{\pm(x, y) : [x \ y] = [x_0 \ y_0]M^n, n \in \mathbb{Z}\}.$$

If $\Delta y_0^2 + 4am$ is not a perfect square, then there is no integer solution.

Now we can give the following theorem.

Theorem 2.1. *The set of all integer solutions of (2.8) is*

$$\Omega^{bc*} = \{(x_{2n-1}, y_{2n-1}), (x_{2n}, y_{2n}) : n \geq 1\},$$

where

$$x_{2n-1} = 4B_{n-1} + 3C_{n-1}, \quad y_{2n-1} = 6B_{n-1} + C_{n-1},$$

$$x_{2n} = -4B_n + 3C_n, \quad y_{2n} = 6B_n - C_n.$$

Proof. For the Pell equation in (2.8), the indefinite form is $F = (1, 0, -2)$ of discriminant $\Delta = 8$. So $\tau_{\Delta} = 3 + 2\sqrt{2}$. Therefore the set of representatives is

$\text{Rep} = \{[\pm 3 \ 1]\}$ and $M = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ by (2.9). Here we notice that

1. $[3 \ 1]M^{n-1}$ generates all integer solutions (x_{2n-1}, y_{2n-1}) for $n \geq 1$
2. $[3 \ -1]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$.

It can be easily seen that the n^{th} power of M is

$$M^n = \begin{bmatrix} C_n & 2B_n \\ 4B_n & C_n \end{bmatrix}$$

for $n \geq 1$. So

$$\begin{aligned} [x_{2n-1} \quad y_{2n-1}] &= [3 \quad 1]M^{n-1} = [4B_{n-1} + 3C_{n-1} \quad 6B_{n-1} + C_{n-1}] \\ [x_{2n} \quad y_{2n}] &= [3 \quad -1]M^n = [-4B_n + 3C_n \quad 6B_n - C_n]. \end{aligned}$$

Thus the set of all integer solutions is $\Omega^{bc*} = \{(4B_{n-1} + 3C_{n-1}, 6B_{n-1} + C_{n-1}), (-4B_n + 3C_n, 6B_n - C_n) : n \geq 1\}$. \blacksquare

From Theorem 2.1, we can give the following result.

Theorem 2.2. *The general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type are*

$$\begin{aligned} B_{2n-1}^{bc*} &= \frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4}, \\ B_{2n}^{bc*} &= \frac{4B_{2n-1} + 3C_{2n-1} - 1}{4}, \\ C_{2n-1}^{bc*} &= 6B_{2n-1} - C_{2n-1}, \\ C_{2n}^{bc*} &= 6B_{2n-1} + C_{2n-1}, \\ R_{2n-1}^{bc*} &= \frac{16B_{2n-1} - 5C_{2n-1} - 1}{4}, \\ R_{2n}^{bc*} &= \frac{8B_{2n-1} - C_{2n-1} - 1}{4} \end{aligned}$$

for $n \geq 1$.

Proof. We proved in Theorem 2.1 that the set of all integer solutions of (2.8) is $\Omega^{bc*} = \{(4B_{n-1} + 3C_{n-1}, 6B_{n-1} + C_{n-1}), (-4B_n + 3C_n, 6B_n - C_n) : n \geq 1\}$. Since $x = 4B_n^{bc*} + 1$, we get

$$B_{2n-1}^{bc*} = \frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4}$$

for $n \geq 1$. Thus from (2.6), we obtain

$$\begin{aligned} C_{2n-1}^{bc*} &= \sqrt{8(B_{2n-1}^{bc*})^2 + 4B_{2n-1}^{bc*} - 3} = \\ &= \sqrt{8\left(\frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4}\right)^2 + 4\left(\frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4}\right) - 3} = \\ &= \sqrt{8B_{2n-1}^2 - 12B_{2n-1}C_{2n-1} + \frac{9C_{2n-1}^2}{2} - \frac{7}{2}} = \\ &= \sqrt{8B_{2n-1}^2 - 12B_{2n-1}C_{2n-1} + \frac{9}{2}(8B_{2n-1}^2 + 1) - \frac{7}{2}} = \\ &= \sqrt{36B_{2n-1}^2 - 12B_{2n-1}C_{2n-1} + 8B_{2n-1}^2 + 1} = \\ &= \sqrt{36B_{2n-1}^2 - 12B_{2n-1}C_{2n-1} + C_{2n-1}^2} = \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{(6B_{2n-1} - C_{2n-1})^2} = \\
 &= 6B_{2n-1} - C_{2n-1}.
 \end{aligned}$$

Finally from (2.4), we deduce that

$$\begin{aligned}
 R_{2n-1}^{bc*} &= \frac{-2B_{2n-1}^{bc*} - 1 + C_{2n-1}^{bc*}}{2} = \\
 &= \frac{-2\left(\frac{-4B_{2n-1} + 3C_{2n-1} - 1}{4}\right) - 1 + 6B_{2n-1} - C_{2n-1}}{2} = \\
 &= \frac{16B_{2n-1} - 5C_{2n-1} - 1}{4}.
 \end{aligned}$$

Similarly it can be shown that $B_{2n}^{bc*} = \frac{4B_{2n-1} + 3C_{2n-1} - 1}{4}$, $C_{2n}^{bc*} = 6B_{2n-1} + C_{2n-1}$ and $R_{2n}^{bc*} = \frac{8B_{2n-1} - C_{2n-1} - 1}{4}$. ■

In **Table 2.1**, the first ten terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of first type is given.

i	B_i^{bc*}	C_i^{bc*}	R_i^{bc*}
1	1	3	0
2	3	9	1
3	39	111	16
4	109	309	45
5	1333	3771	552
6	3711	10497	1537
7	45291	128103	18760
8	126073	356589	52221
9	1538569	4351731	637296
10	4282779	12113529	1773985

Table 2.1 Almost balcobalancing numbers of first type

2.2. Almost balcobalancing numbers of second type

From (2.5), we notice that B_n^{bc**} is an almost balcobalancing number of second type if and only if $8(B_n^{bc**})^2 + 4B_n^{bc**} + 5$ is a perfect square. So we set

$$8(B_n^{bc**})^2 + 4B_n^{bc**} + 5 = y^2$$

for some positive integer y . If we multiply both sides of the last equation by 2, then we get

$$16(B_n^{bc**})^2 + 8B_n^{bc**} + 10 = 2y^2$$

and hence

$$(4B_n^{bc**} + 1)^2 + 9 = 2y^2.$$

Taking $x = 4B_n^{bc**} + 1$, we get the Pell equation

$$(2.11) \quad x^2 - 2y^2 = -9.$$

Let Ω^{bc**} denotes the set of all (positive) integer solutions of (2.11), that is,

$$\Omega^{bc**} = \{(x, y) : x^2 - 2y^2 = -9\}.$$

Then we can give the following theorem.

Theorem 2.3. *The set of all integer solutions of (2.11) is*

$$\Omega^{bc**} = \{(x_n, y_n) : n \geq 1\},$$

where

$$x_n = 12B_{n-1} + 3C_{n-1} \text{ and } y_n = 6B_{n-1} + 3C_{n-1}.$$

Proof. For the Pell equation in (2.11), the indefinite form is $F = (1, 0, -2)$ of discriminant $\Delta = 8$. So $\tau_\Delta = 3 + 2\sqrt{2}$. In this case, the set of representatives is $\text{Rep} = \{[\pm 3 \quad 3]\}$ and $M = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$. Here $[3 \quad 3]M^{n-1}$ generates all integer solutions (x_n, y_n) for $n \geq 1$. Thus the set of all integer solutions is $\Omega^{bc**} = \{(12B_{n-1} + 3C_{n-1}, 6B_{n-1} + 3C_{n-1}) : n \geq 1\}$. ■

From Theorem 2.3, we can give the following result.

Theorem 2.4. *The general terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of second type are*

$$\begin{aligned} B_n^{bc**} &= \frac{12B_{2n-1} + 3C_{2n-1} - 1}{4}, \\ C_n^{bc**} &= 6B_{2n-1} + 3C_{2n-1}, \\ R_n^{bc**} &= \frac{3C_{2n-1} - 1}{4} \end{aligned}$$

for $n \geq 1$.

Proof. Note that $\Omega^{bc**} = \{(12B_{n-1} + 3C_{n-1}, 6B_{n-1} + 3C_{n-1}) : n \geq 1\}$ by Theorem 2.3. So we get

$$B_n^{bc**} = \frac{12B_{2n-1} + 3C_{2n-1} - 1}{4}$$

for $n \geq 1$. Thus from (2.7), we obtain

$$\begin{aligned}
 C_n^{bc**} &= \sqrt{8(B_n^{bc**})^2 + 4B_n^{bc**} + 5} = \\
 &= \sqrt{8\left(\frac{12B_{2n-1} + 3C_{2n-1} - 1}{4}\right)^2 + 4\left(\frac{12B_{2n-1} + 3C_{2n-1} - 1}{4}\right) + 5} = \\
 &= \sqrt{72B_{2n-1}^2 + 36B_{2n-1}C_{2n-1} + \frac{9C_{2n-1}^2}{2} + \frac{9}{2}} = \\
 &= \sqrt{36B_{2n-1}^2 + 36B_{2n-1}C_{2n-1} + 9C_{2n-1}^2} = \\
 &= \sqrt{(6B_{2n-1} + 3C_{2n-1})^2} = \\
 &= 6B_{2n-1} + 3C_{2n-1}.
 \end{aligned}$$

From (2.5), we deduce that $R_n^{bc**} = \frac{3C_{2n-1}-1}{4}$. ■

In **Table 2.2**, the first ten terms of almost balcobalancing numbers, almost Lucas-balcobalancing numbers and almost balcobalancers of second type is given.

i	B_i^{bc**}	C_i^{bc**}	R_i^{bc**}
1	5	15	2
2	179	507	74
3	6089	17223	2522
4	206855	585075	85682
5	7026989	19875327	2910674
6	238710779	675176043	98877242
7	8109139505	22936110135	3358915562
8	275472032399	779152568547	114104251874
9	9357939962069	26468251220463	3876185648162
10	317894486677955	899141388927195	131676207785642

Table 2.2 Almost balcobalancing numbers of second type

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A. Tekcan and M. Yıldız

Bursa Uludag University

Faculty of Science

Department of Mathematics

Bursa, Türkiye

tekcan@uludag.edu.tr, 502011002@ogr.uludag.edu.tr

