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ON THE EQUATION $f(n^2+nm+m^2)=f(n)^2+f(n)f(m)+f(m)^2$

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Abstract. We give all solutions of the equation

$$f\big(n^2+nm+m^2\big)=f(n)^2+f(n)f(m)+f(m)^2,$$

where f is an arbitrary complex valued function defined on $\mathbb{N}$.

1. Introduction

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{C}$ be the set of primes, positive integers and complex numbers, respectively.

A function $f:\mathbb{N}\to\mathbb{C}$ is multiplicative if

$$f(nm)=f(n)f(m)\quad\text{for every}\quad n,m\in\mathbb{N},(n,m)=1.$$

Let $\mathcal{M}$ be the set of complex-valued multiplicative functions.

Let $Q(x,y)\in\mathbb{Z}[x,y]$ be a polynomial with two variable and let A,B be subsets of $\mathbb{N}$. We are interested in finding solutions $f:\mathbb{N}\to\mathbb{C}$ to an equation of the form

$$f\big(Q(a,b)\big)=Q\big(f(a),f(b)\big)\quad\text{for every}\quad a\in A,b\in B. \tag{1.1}$$

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If $Q(x,y)=x+y$, $A=B=\mathbb{N}$, then it can be shown that there is a single family of solutions, namely $f(n)=cn$, where $c=f(1)\in\mathbb{C}$ is an arbitrary number.

In 1992, C. Spiro [16] considered the equation (1.1) in the case when $Q(x,y)=x+y$ and $A=B=\mathcal{P}$. She proved that if a real-valued multiplicative function f satisfies

$$f(p+q)=f(p)+f(q)\quad(\forall p,q\in\mathcal{P})\quad\text{and}\quad f(p_0)\neq 0\quad\text{for some}\quad p_0\in\mathcal{P},$$

then $f(n)$ is the identity function.

In 1997 J.-M. De Koninck, I. Kátai and B. M. Phong [10] proved that if a function $f\in\mathcal{M}$ satisfy the condition

$$f\big(p+m^2\big)=f(p)+f\big(m^2\big)\quad\text{for every}\quad p\in\mathcal{P},m\in\mathbb{N},$$

then $f(n)=n$ for all $n\in\mathbb{N}$.

For some generalizations of the above results, we refer to the other works of P. V. Chung and B. M. Phong [2], K. K. Chen and Y. G. Chen [3], A. Dubickas and P. Sarka [4], J-H. Fang [5], B. M. Phong [11]–[14].

In 2014 B. Bojan [1] considered the case $Q(x,y)=x^2+y^2$, $A=B=\mathbb{N}$. He determined all solutions of those $f:\mathbb{N}\to\mathbb{C}$ for which

$$f\big(n^2+m^2\big)=f^2(n)+f^2(m)\quad\text{for every}\quad n,m\in\mathbb{N}.$$

It is proved that all solutions $f(n)$ of this equation satisfy

$$f(n)\in\Big\{0,\ \frac{\varepsilon(n)}{2},\ \varepsilon(n)\,n\Big\},$$

where $\varepsilon(n)\in\{-1,1\}$ and $\varepsilon(n^2+m^2)=1$ for every $n,m\in\mathbb{N}$.

I. Kátai and B. M. Phong posed the following conjecture for the cases when $Q(x,y)=x^2+Dy^2$ and $A=B=\mathbb{N}$:

Conjecture 1. (I. Kátai and B. M. Phong [6]) *Assume that the number $D\in\mathbb{N}$ and the arithmetical function $f:\mathbb{N}\to\mathbb{C}$ satisfy the equation*

$$f\big(n^2+Dm^2\big)=f^2(n)+Df^2(m)\quad\textit{for every}\quad n,m\in\mathbb{N}.$$

Then one of the following assertions holds:

- $f(n)=0$ *for every* $n\in\mathbb{N}$,
- $f(n)=\dfrac{\epsilon(n)}{D+1}$ *for every* $n\in\mathbb{N}$,
- $f(n)=\epsilon(n)\,n$ *for every* $n\in\mathbb{N}$,

where $\epsilon(n)\in\{-1,1\}$ and $\epsilon\big(n^2+Dm^2\big)=1$ for every $n,m\in\mathbb{N}$.

B. M. M. Khanh in [7] and [8] proved this conjecture. In [9] she given all functions $f:\mathbb{N}\to\mathbb{C}$ which satisfy the equation

$$f\big(n^2+m^2+k\big)=f^2(n)+f^2(m)+K \quad \text{for every} \quad n,m\in\mathbb{N},$$

where $k\in\mathbb{N}_0$, $K\in\mathbb{C}$.

Recently Poo Sung Park [15] proved the following results

Theorem A. *If $f\in\mathcal{M}$ and*

$$f(n^2+nm+m^2)=f^2(n)+f(n)f(m)+f^2(m) \quad \textit{for every} \quad n,m\in\mathbb{N},$$

then f is the identity function.

Our general purpose is to determine all those $f:\mathbb{N}\to\mathbb{C}$ for which

$$f\big(n^2+Dnm+m^2\big)=f^2(n)+Df(n)f(m)+f^2(m) \quad \text{for every} \quad n,m\in\mathbb{N},$$

where $D\in\mathbb{Z}$ is a fixed number.

In this paper we will consider the case $D=1$. It seems to us that the cases of other D need some new ideas.

Some of the lemmas used in the proof are formulated in more general form, containing D if it true in this form.

Theorem 1. *Assume that $f:\mathbb{N}\to\mathbb{C}$ satisfies*

$$f\big(n^2+nm+m^2\big)=f^2(n)+f(n)f(m)+f^2(m) \quad \textit{for every} \quad n,m\in\mathbb{N}.$$

Then the following assertions holds:

(a) *If* $f(1)=0$, *then* $f(n)=0$ *for every* $n\in\mathbb{N}$.

(b) *If* $f(1)=\frac{1}{3}$, *then* $f(n)=\frac{1}{3}$ *for every* $n\in\mathbb{N}$.

(c) *If* $f(1)=1$, *then* $f(n)=n$ *for every* $n\in\mathbb{N}$.

We note that Theorem A is a direct consequence of Theorem 1.

2. Lemmas

In this section we prove some lemmas.

Lemma 1. *Let $D\in\mathbb{N}$ be fixed and $f:\mathbb{N}\to\mathbb{C}$ satisfies*

$$f\big(n^2+Dnm+m^2\big)=f^2(n)+Df(n)f(m)+f^2(m) \quad \textit{for every} \quad n,m\in\mathbb{N}. \tag{2.1}$$

If one of the following assertions holds:

$$\text{(a)} \quad f(n)=0 \quad \textit{for every} \quad n\le 8D+20,$$

$$\text{(b)} \quad f(n)=\frac{1}{D+2} \quad \textit{for every} \quad n\le 8D+20,$$

$$\text{(c)} \quad f(n)=n \quad \textit{for every} \quad n\le 8D+20,$$

then the corresponding one of (a), (b), (c) *holds for all* $n\in\mathbb{N}$.

Proof. We shall prove the case (c). The proof of (a) and (b) is similar way.

Assume that the conditions (2.1) and (c) are satisfied. Assume that $f(n)= = n$ for all $n<N$, where $N\ge 8D+21$. We will prove that $f(N)=N$.

If $N=2k+1$, then $N=2k+1\ge 8D+21$ and $k\ge 4D+10$. Consequently

$$1\le k-D-2<N,\quad 1\le 2k-2D-1<N,$$
$$1\le k-3D-7<N \quad\text{and}\quad \le 2k-6D-5<N,$$

which with our assumptions imply

$$f(k-D-2)=k-D-2,\ f(2k-2D-1)=2k-2D-1,$$
$$f(k-3D-7)=k-3D-7,\ f(2k-6D-5)=2k-6D-5.$$

By using (2.1), the above relations with the following relations

$$(2k+1)^2+D(2k+1)(k-D-2)+(k-D-2)^2=$$
$$=(2k-2D-1)^2+D(2k-2D-1)(k+2)+(k+2)^2$$

and

$$(2k+1)^2+D(2k+1)(k-3D-7)+(k-3D-7)^2=$$
$$=(2k-6D-5)^2+D(2k-6D-5)(k+5)+(k+5)^2$$

show that

$$f(2k+1)^2+Df(2k+1)(k-D-2)+(k-D-2)^2=$$
$$=(2k-2D-1)^2+D(2k-2D-1)(k+2)+(k+2)^2$$

and

$$f(2k+1)^2+Df(2k+1)(k-3D-7)+(k-3D-7)^2=$$
$$=(2k-6D-5)^2+D(2k-6D-5)(k+5)+(k+5)^2.$$

It is easily shown from the last two relations that $f(2k+1)=2k+1$, $f(N)=N$.

In the same way, if $N=2k$, then $N=2k\ge 8D+21$ and $k\ge 4D+11$. Consequently

$$1\le k-2D-5<N,\quad 1\le 2k-4D-4<N,$$
$$1\le k-4D-10<N \quad\text{and}\quad 1\le 2k-8D-8<N,$$

which with our assumptions imply

$$f(k-2D-5)=k-2D-5,\ f(2k-4D-4)=2k-4D-4,$$
$$f(k-4D-10)=k-4D-10,\ f(2k-8D-8)=2k-8D-8.$$

By using (2.1), the above relations with the following relations

$$(2k)^2+D(2k)(k-2D-5)+(k-2D-5)^2=$$
$$=(2k-4D-4)^2+D(2k-4D-4)(k+3)+(k+3)^2$$

and

$$(2k)^2+D(2k)(k-4D-10)+(k-4D-10)^2=$$
$$(2k-8D-8)^2+D(2k-8D-8)(k+6)+(k+6)^2$$

show that

$$f(2k)^2+Df(2k)(k-2D-5)+(k-2D-5)^2=$$
$$=(2k-4D-4)^2+D(2k-4D-4)(k+3)+(k+3)^2$$

and

$$f(2k)^2+Df(2k)(k-4D-10)+(k-4D-10)^2=$$
$$(2k-8D-8)^2+D(2k-8D-8)(k+6)+(k+6)^2.$$

It is easily shown from the last two relations that $f(2k)=2k,\ f(N)=N$.

Lemma 1 is proved. ■

In the following we assume that $D=1$ and $f:\mathbb{N}\to\mathbb{C}$ satisfies (2.1), i.e.

$$f\big(n^2+nm+m^2\big)=f(n)^2+f(n)f(m)+f^2(m)\quad \text{for every}\quad n,m\in\mathbb{N}. \tag{2.2}$$

Let

$$E:=\{N\in\mathbb{N}\ |\ N=n^2+nm+m^2=u^2+uv+v^2\quad\text{with}\quad n>u, m\le n, v\le u\}.$$

It is obvious that if $N\in E$, then we obtain from (2.2) that

$$f(N)=f^2(n)+f(n)f(m)+f^2(m)\quad\text{and}\quad f(N)=f^2(u)+f(u)f(v)+f^2(v)$$

which show that

$$E(N)=f^2(n)+f(n)f(m)+f^2(m)-f^2(u)-f(u)f(v)-f^2(v)=0 \tag{2.3}$$

for every $N \in E$. First we note from (2.2) that

$$(2.4)\qquad \begin{cases} f(3) &= f^2(1)+f(1)f(1)+f^2(1),\\ f(7) &= f^2(2)+f(2)f(1)+f^2(1),\\ f(12) &= f^2(2)+f(2)f(2)+f^2(2),\\ f(13) &= f^2(3)+f(3)f(1)+f^2(1),\\ f(19) &= f^2(3)+f(3)f(2)+f^2(2),\\ f(21) &= f^2(4)+f(4)f(1)+f^2(1),\\ f(27) &= 3f^2(3),\\ f(28) &= f^2(4)+f(4)f(2)+f^2(2). \end{cases}$$

Lemma 2. *Assume that $f:\mathbb{N}\to\mathbb{C}$ satisfies (2.2). If*

$$f(1)=0,\quad f(2)=0\quad \textit{and}\quad f(4)=0,$$

then $f(n)=0$ for every $n\in\mathbb{N}$.

Proof. Let

$$\mathcal{S}_0 := \big\{n\in\mathbb{N}\ \big|\ f(n)=0\big\}.$$

Since $D=1$, by using Lemma 1, we need to prove that

$$\{1,\ 2,\ \dots,\ 28\}\subset\mathcal{S}_0.$$

We obtain from $f(1)=f(2)=f(4)=0$ and (2.4) that

$$\{3,\ 7,\ 12,\ 13,\ 19,\ 21,\ 27,\ 28\}\subset\mathcal{S}_0,$$

consequently

$$\{1,\ 2,\ 3,\ 4,\ 7,\ 12,\ 13,\ 19,\ 21,\ 27,\ 28\}\subset\mathcal{S}_0$$

and

$$\begin{aligned} f^2(11) &= f^2(11)+f(11)f(2)+f^2(2)-3f^2(7)=E(147)=0,\\ -f^2(17) &= f^2(21)+f(21)f(7)+f^2(7)-f^2(17)-f(17)f(12)-f^2(12)=\\ &= E(637)=0,\\ f^2(20) &= f^2(20)+f(20)f(7)-f^2(13)-f(13)f(12)-f^2(12)=E(469)=0,\\ f^2(22) &= f^2(22)+f(22)f(1)+f^2(1)-3f^2(13)=E(507)=0,\\ f^2(23) &= f^2(23)+f(23)f(4)+f^2(4)-f^2(21)-f(21)f(7)-f^2(7)=\\ &= E(637)=0, \end{aligned}$$

which gives

$$\{1,\ 2,\ 3,\ 4,\ 7,\ 11,\ 12,\ 13,\ 17,\ 19,\ 20,\ 21,\ 22,\ 23,\ 27,\ 28\}\subset\mathcal{S}_0.$$

Thus we have

$$\begin{aligned}
f^2(5) &= f^2(28)+f(28)f(5)+f^2(5)-f^2(23)-f(23)f(12)-f^2(12)=\\
&= E(949)=0,\\
-f^2(6) &= f^2(19)+f(19)f(3)+f^2(3)-f^2(17)-f(17)f(6)-f^2(6)=\\
&= E(427)=0,\\
f^2(8) &= f^2(23)+f(23)f(8)+f^2(8)-f^2(19)-f(19)f(13)-f^2(13)=\\
&= E(777)=0,\\
-f(9)^2 &= f(11)^2+f(11)f(1)+f(1)^2-f(9)^2-f(9)f(4)-f(4)^2=\\
&= E(133)=0,\\
-f(10)^2 &= f(22)^2+f(22)f(3)+f(3)^2-f(17)^2-f(17)f(10)-f(10)^2=\\
&= E(559)=0,\\
-3f(14)^2 &= f(22)^2+f(22)f(4)+f(4)^2-3f(14)^2=E(588)=0,\\
-f(15)^2 &= f(20)^2+f(20)f(7)+f(7)^2-f(15)^2-f(15)f(13)-f(13)^2=\\
&= E(589)=0,\\
-f(16)^2 &= f(23)^2+f(23)f(1)+f(1)^2-f(16)^2-f(16)f(11)-f(11)^2=\\
&= E(553)=0,\\
f(24)^2 &= f(24)^2+f(24)f(7)+f(7)^2-f(21)^2-f(21)f(11)-f(11)^2=\\
&= E(793)=0,\\
f(25)^2 &= f(25)^2+f(25)f(2)+f(2)^2-f(17)^2-f(17)f(13)-f(13)^2=\\
&= E(679)=0,\\
f(26)^2 &= f(26)^2+f(26)f(11)+f(11)^2-3f(19)^2=E(1083)=0,
\end{aligned}$$

which implies

$$\begin{gathered}
\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\\
15,\ 16,\ 17,\ 19,\ 20,\ 21,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27,\ 28\}\subset\mathcal{S}_0.
\end{gathered}$$

Finally, we have

$$f(18)^2=f(18)^2+f(18)f(1)+f(1)^2-f(14)^2-f(14)f(7)-f(7)^2=E(343)=0,$$

and so

$$18\in\mathcal{S}_0.$$

Thus, Lemma 2 is proved. ■

Lemma 3. *Assume that $f : \mathbb{N} \to \mathbb{C}$ satisfies (2.2). If*

$$f(1) = \frac{1}{3}, \quad f(2) = \frac{1}{3} \quad \textit{and} \quad f(4) = \frac{1}{3},$$

then $f(n) = \frac{1}{3}$ for every $n \in \mathbb{N}$.

Proof. Let

$$\mathcal{S}_1 = \Big\{ n \in \mathbb{N} \ \ f(n) = \frac{1}{3} \Big\}.$$

Since $D = 1$, by using Lemma 1, we need to prove that

$$\{1, \ 2, \ \ldots \ 28\} \subset \mathcal{S}_1.$$

It is obvious from $f(1) = \frac{1}{3}$, $f(2) = \frac{1}{3}$, $f(4) = \frac{1}{3}$ and (2.4) that

$$f(3) = f(7) = f(12) = f(13) = f(19) = f(21) = f(27) = f(28) = \frac{1}{3},$$

consequently

$$\{1, \ 2, \ 3, \ 4, \ 7, \ 12, \ 13, \ 19, \ 21, \ 27, \ 28\} \subset \mathcal{S}_1. \tag{2.5}$$

We will prove in the following that

$$\{5, \ 6, \ 8, \ 9, \ 10, \ 11, \ 14, \ 15, \ 16, \ 17, \ 18, \ 20, \ 22, \ 23, \ 24, \ 25, \ 26\} \in \mathcal{S}_1.$$

Let

$$I := \Big\{ n \in \mathbb{N} \ \Big| \ f(n) \in \Big\{ -\frac{2}{3}, \ \frac{1}{3} \Big\} \Big\}.$$

Since

$$f\big(3n^2\big) = f\big(n^2 + n \cdot n + n^2\big) = 3f(n)^2,$$

we have

$$\text{If} \quad n \in I \quad \text{and} \quad 3n^2 \in I, \quad \text{then} \quad n \in \mathcal{S}_1. \tag{2.6}$$

We write $N = [m, n]$ if $N = m^2 + mn + n^2$. It is enough to show

$$\text{If} \ \ m \in I \quad \text{and} \quad n \in \mathcal{S}_1, \quad \text{then} \quad N = [m, n] \in \mathcal{S}_1. \tag{2.7}$$

It is clear that if $m \in \mathcal{S}_1$, then (2.7) is true. Assume that $f(m) = -\frac{2}{3}$ and $f(n) = \frac{1}{3}$. Then

$$f(N) = f(m)^2 + f(m)f(n) + f(n)^2 = \frac{4}{9} - \frac{2}{9} + \frac{1}{9} = \frac{1}{3},$$

which completes the proof of (2.7).

We prove that $5\in\mathcal{S}_1$. By using (2.6), we need to prove that $5\in I$, $75\in I$. By using (2.7), we have

$$\begin{cases} 37=[4,3]\in\mathcal{S}_1, \quad 48=[4,4]\in\mathcal{S}_1, \quad 79=[7,3]\in\mathcal{S}_1, \\ f(31)=f(5)^2+f(5)f(1)+f(1)^2=f(5)^2+\frac{1}{3}f(5)+\frac{1}{9}, \\ f(61)=f(5)^2+f(5)f(4)+f(4)^2=f(5)^2+\frac{1}{3}f(5)+\frac{1}{9}, \end{cases} \tag{2.8}$$

therefore we infer from (2.5) and (2.8) that

$$\begin{cases} E(1533)=f(37)^2+f(37)f(4)+f(4)^2-f(31)^2-f(31)f(13)-f(13)^2= \\ \qquad =-\dfrac{1}{81}\big(3f(5)+2\big)\big(3f(5)-1\big)\big(9f(5)^2+3f(5)+7\big)=0, \\ E(6573)=f(79)^2+f(79)f(4)+f(4)^2-f(61)^2-f(61)f(31)-f(31)^2= \\ \qquad =-\dfrac{1}{27}\big(3f(5)+2\big)\big(3f(5)-1\big)\big(9f(5)^2+3f(5)+4\big)=0. \end{cases}$$

These imply that $\big(3f(5)+2\big)\big(3f(5)-1\big)=0$, and so $5\in I$. We also have

$$\begin{cases} E(2587)=f(47)^2+f(47)f(7)+f(7)^2-f(37)^2-f(37)f(21)-f(21)^2= \\ \qquad =\dfrac{1}{9}\big(3f(47)+2\big)\big(3f(47)-1\big)=0, \\ E(6769)=f(75)^2+f(75)f(13)+f(13)^2-f(48)^2-f(48)f(47)-f(47)^2= \\ \qquad =\dfrac{1}{3}\big(3f(75)+1+3f(47)\big)\big(f(75)-f(47)\big)=0, \end{cases}$$

which show that $47\in I$, and so we infer from the last relation that $75\in I$. By using (2.6), we infer from facts $5\in I$ and $3.5^2\in I$ that $5\in\mathcal{S}_1$.

Thus, we can assume that

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 7,\ 12,\ 13,\ 19,\ 21,\ 27,\ 28\}\subset\mathcal{S}_1$$

By using (2.7), one can check that $31=[5,1]\in\mathcal{S}_1$, $49=[5,3]\in\mathcal{S}_1$, $147=[7,7]$, $172=[12,2]\in\mathcal{S}_1$, consequently

$$\begin{cases} E(31213)=f(172)^2+f(172)f(9)+f(9)^2-f(147)^2-f(147)f(49)- \\ \qquad -f(49)^2=\dfrac{1}{9}\big(3f(9)+2\big)\big(3f(9)-1\big)=0, \\ E(147)=f(11)^2+f(11)f(2)+f(2)^2-3f(7)^2= \\ \qquad =\dfrac{1}{9}\big(3f(11)+2\big)\big(3f(11)-1\big)=0, \\ E(259)=f(15)^2+f(15)f(2)+f(2)^2-f(13)^2-f(13)f(5)-f(5)^2= \\ \qquad =\dfrac{1}{9}\big(3f(15)+2\big)\big(3f(15)-1\big)=0, \end{cases}$$

$$\begin{cases} E(637) = f(21)^2 + f(21)f(7) + f(7)^2 - f(17)^2 - f(17)f(12) - f(12)^2 = \\ \qquad = -\frac{1}{9}\big(3f(17)+2\big)\big(3f(17)-1\big) = 0, \\ E(1027) = f(31)^2 + f(31)f(2) + f(2)^2 - f(19)^2 - f(19)f(18) - f(18)^2 = \\ \qquad = -\frac{1}{9}\big(3f(18)+2\big)\big(3f(18)-1\big) = 0, \\ E(469) = f(20)^2 + f(20)f(3) + f(3)^2 - f(13)^2 - f(13)f(12) - f(12)^2 = \\ \qquad = \frac{1}{9}\big(3f(20)+2\big)\big(3f(20)-1\big) = 0, \\ E(507) = f(22)^2 + f(22)f(1) + f(1)^2 - 3f(13)^2 = \\ \qquad = \frac{1}{9}\big(3f(22)+2\big)\big(3f(22)-1\big) = 0, \\ E(637) = f(23)^2 + f(23)f(4) + f(4)^2 - f(21)^2 - f(21)f(7) - f(7)^2 = \\ \qquad = \frac{1}{9}\big(3f(23)+2\big)\big(3f(23)-1\big) = 0, \end{cases}$$

we have $\{9,\ 11,\ 15,\ 17,\ 18,\ 20,\ 22,\ 23\} \subset I$, and so

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 11,\ 12,\ 13,\ 15,\ 17,\ 18,\ 19,\ 20,\ 21,\ 22,\ 23,\ 27,\ 28\} \subset I.$$

In the following we use the simple fact:

(2.9) If $a \in I$ and $\big(3f(a)+1+3f(n)\big)\big(f(a)-f(n)\big) = 0$, then $n \in I$.

It is obvious if $f(a) - f(n) = 0$. If $f(a) - f(n) \neq 0$, then

$$3f(a)+1+3f(n) = 0 \quad \text{and} \quad f(n) = -\frac{3f(a)+1}{3} = \begin{cases} -\frac{2}{3} & \text{if } f(a) = \frac{1}{3} \\ \frac{1}{3} & \text{if } f(a) = -\frac{2}{3}. \end{cases}$$

It is clear that $37 = [4,3] \in \mathcal{S}_1$ and $39 = [5,2] \in \mathcal{S}_1$, which with Maple, we obtain that

$$\begin{aligned} E(91) &= f(9)^2 + f(9)f(1) + f(1)^2 - f(6)^2 - f(6)f(5) - f(5)^2 = \\ &= \frac{1}{3}\big(3f(9)+1+3f(6)\big)\big(f(9)-f(6)\big) = 0, \\ E(1897) &= f(39)^2 + f(39)f(8) + f(8)^2 - f(37)^2 - f(37)f(11) - f(11)^2 = \\ &= -\frac{1}{3}\big(3f(8)+1+3f(11)\big)\big(f(11)-f(8)\big) = 0, \\ E(399) &= f(17)^2 + f(17)f(5) + f(5)^2 - f(13)^2 - f(13)f(10) - f(10)^2 = \\ &= \frac{1}{3}\big(3f(17)+1+3f(10)\big)\big(f(17)-f(10)\big) = 0, \end{aligned}$$

$$E(247)=f(14)^2+f(14)f(3)+f(3)^2-f(11)^2-f(11)f(7)-f(7)^2=$$
$$=-\frac{1}{3}\big(3f(11)+1+3f(14)\big)\big(f(11)-f(14)\big)=0,$$
$$E(793)=f(24)^2+f(24)f(7)+f(7)^2-f(21)^2-f(21)f(11)-f(11)^2=$$
$$=-\frac{1}{3}\big(3f(11)+1+3f(24)\big)\big(f(11)-f(24)\big)=0,$$
$$E(679)=f(25)^2+f(25)f(2)+f(2)^2-f(17)^2-f(17)f(13)-f(13)^2=$$
$$=-\frac{1}{3}\big(3f(17)+1+3f(25)\big)\big(f(17)-f(25)\big)=0,$$
$$E(1183)=f(31)^2+f(31)f(6)+f(6)^2-f(26)^2-f(26)f(13)-f(13)^2=$$
$$=\frac{1}{3}\big(3f(26)+1+3f(6)\big)\big(f(6)-f(26)\big)=0,$$

which with (2.9) imply that the numbers 6, 8, 10, 14, 24, 25, 26 belong to I. Finally, we prove that $16\in I$. We infer from the relations

$$E(5833)=f(56)^2+f(56)f(31)+f(31)^2-f(49)^2-f(49)f(39)-f(39)^2=$$
$$=\frac{1}{9}\big(3f(56)+2\big)\big(3f(56)-1\big)=0,$$
$$E(3441)=f(56)^2+f(56)f(5)+f(5)^2-f(49)^2-f(49)f(16)-f(16)^2=$$
$$=\frac{1}{3}\big(3f(56)+1+3f(16)\big)\big(f(56)-f(16)\big)=0,$$

that $56\in I$ and $16\in I$.

Thus we have proved

$$A:=\{1,\ 2,\ 3,\ 4,\ 5,\ 7,\ 12,\ 13,\ 19,\ 21,\ 27,\ 28\}\subset\mathcal{S}_1$$

and

$$B:=\{6,\ 8,\ 9,\ 10,\ 11,\ 14,\ 15,\ 16,\ 17,\ 18,\ 20,\ 22,\ 23,\ 24,\ 25,\ 26\}\subset I.$$

Now we prove that

$$3n^2\in I\quad\text{for every}\quad n\in B. \tag{2.10}$$

We note from (2.7) that the numbers $52=[6,2]$, $67=[7,2]$, $75=[5,5]$, $84=[8,2]$, $93=[7,4]$, $117=[9,3]$, $124=[10,2]$, $172=[12,2]$, $201=[11,5]$, $237=[13,4]$, $237=[13,4]$, $309=[13,7]$, $333=[9,12]$, $337=[8,13]$, $543==[19,7]$ are in $\mathcal{S}_1$. Thus, we have

$$E(12229)=f(108)^2+f(108)f(5)+f(5)^2-f(75)^2-f(75)f(52)-f(52)^2=$$
$$=\frac{1}{9}\big(3f(108)+2\big)\big(3f(108)-1\big)=0,$$

$$E(56889) = f(237)^2 + f(237)f(3) + f(3)^2 - f(192)^2 - f(192)f(75) - $$
$$- f(75)^2 = -\frac{1}{9}\big(3f(192)+2\big)\big(3f(192)-2\big) = 0,$$
$$E(104557) = f(243)^2 + f(243)f(124) + f(124)^2 - f(201)^2 - f(201)f(172) - $$
$$- f(172)^2 = \frac{1}{9}\big(3f(243)-1\big)\big(3f(243)+2\big) = 0,$$
$$E(114589) = f(337)^2 + f(337)f(3) + f(3)^2 - f(300)^2 - f(300)f(67) - $$
$$- f(67)^2 = -\frac{1}{9}\big(3f(300)+2\big)\big(3f(300)-1\big) = 0,$$
$$E(132867) = f(363)^2 + f(363)f(3) + f(3)^2 - f(309)^2 - f(309)f(93) - $$
$$- f(93)^2 = \frac{1}{9}\big(3f(363)+2\big)\big(3f(363)-1\big) = 0,$$
$$E(347517) = f(588)^2 + f(588)f(3) + f(3)^2 - f(543)^2 - f(543)f(84) - $$
$$- f(84)^2 = \frac{1}{9}\big(3f(588)+2\big)\big(3f(588)-1\big) = 0,$$

which imply that $108 = 3\cdot 6^2 \in \mathcal{S}_1$, $192 = 3\cdot 8^2 \in \mathcal{S}_1$, $243 = 3\cdot 9^2 \in \mathcal{S}_1$, $300 = 3\cdot 10^2 \in \mathcal{S}_1$, $363 = 3\cdot 11^2 \in \mathcal{S}_1$, $588 = 3\cdot 14^2 \in \mathcal{S}_1$. Thus we have proved

$$C := \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 19,\ 21,\ 27,\ 28\} \subset \mathcal{S}_1.$$

Similar to the above proof, by using (2.7), the numbers $39 = [5,2]$, $73 = [8,1]$, $97 = [8,3]$, $333 = [9,12]$, $363 = [11,11]$, $381 = [19,1]$, $417 = [16,7]$, $471 = [14,11]$, $487 = [21,2]$, $633 = [16,13]$, $793 = [21,11]$ are in $\mathcal{S}_1$, thus we have

$$E(510229) = f(675)^2 + f(675)f(73) + f(73)^2 - f(487)^2 - f(487)f(333) - $$
$$- f(333)^2 = \frac{1}{9}\big(3f(675)+2\big)\big(3f(675)-1\big) = 0,$$
$$E(592137) = f(768)^2 + f(768)f(3) + f(3)^2 - f(471)^2 - f(471)f(417) - $$
$$- f(417)^2 = \frac{1}{9}\big(3f(768)+2\big)\big(3f(768)-1\big) = 0,$$
$$E(787023) = f(867)^2 + f(867)f(39) + f(39)^2 - f(633)^2 - f(633)f(381) - $$
$$- f(381)^2 = \frac{1}{9}\big(3f(867)+2\big)\big(3f(867)-1\big) = 0,$$
$$E(1048477) = f(972)^2 + f(972)f(97) + f(97)^2 - f(793)^2 - f(793)f(363) - $$
$$- f(363)^2 = \frac{1}{9}\big(3f(972)+2\big)\big(3f(972)-1\big) = 0,$$

consequently

$$3\cdot 15^2 = 675 \in I, \quad 3\cdot 16^2 = 768 \in I, \quad 3\cdot 17^2 = 867 \in I, \quad 3\cdot 18^2 = 972 \in I$$

and so

$$15 \in \mathcal{S}_1, \quad 16 \in \mathcal{S}_1, \quad 17 \in \mathcal{S}_1, \quad 18 \in \mathcal{S}_1.$$

For the proof of $3\cdot 20^2=1200\in I$ and $3\cdot 22^2=1452\in I$, we use (2.7) to get that the numbers $57=[7,1]$, $199=[13,2]$, $208=[12,4]$, $351=[15,6]$, $489==[17,8]$, $516=[16,10]$, $601=[24,1]$, $613=[19,9]$, $687=[22,7]$, $711=[21,9]$, $727=[18,13]$, $739=[23,7]$, $768=[16,16]$ belong to $\mathcal{S}_1$. Therefore, we have

$$\begin{aligned}
E(566257)&=f(752)^2+f(752)f(1)+f(1)^2-f(633)^2-f(633)f(199)-\\
&\quad -f(199)^2=\frac{1}{9}\big(3f(752)+2\big)\big(3f(752)-1\big)=0,\\
E(1732864)&=f(1200)^2+f(1200)f(208)+f(208)^2-f(768)^2-\\
&\quad -f(768)f(752)-f(752)^2=-\frac{1}{3}\big(3f(752)+1+f(1200)\big)\cdot\\
&\quad \cdot\big(f(752)-f(1200)\big)=0,\\
E(1169917)&=f(1052)^2+f(1052)f(57)+f(57)^2-f(727)^2-f(727)f(516)-\\
&\quad -f(516)^2=\frac{1}{9}\big(3f(1052)+2\big)\big(3f(1052)-1\big)=0,\\
E(1732864)&=f(1452)^2+f(1452)f(13)+f(13)^2-f(1052)^2-\\
&\quad -f(1052)f(613)-f(613)^2=-\frac{1}{3}\big(3f(1052)+1+3f(1452)\big)\cdot\\
&\quad \cdot\big(f(1052)-f(1452)\big)=0.
\end{aligned}$$

Thus we have proved that $20\in\mathcal{S}_1$ and $22\in\mathcal{S}_1$.

Now we prove that

$$3\cdot 23^2\in I,\quad 3\cdot 24^2\in I,\quad 3\cdot 25^2\in I,\quad 3\cdot 26^2\in I\quad\text{and}\quad \{23,\ 24,\ 25,\ 26\}\subset\mathcal{S}_1.$$

Since $711\in\mathcal{S}_1,\ 601\in\mathcal{S}_1,\ 687\in\mathcal{S}_1$, we have

$$\begin{aligned}
E(1294033)&=f(1131)^2+f(1131)f(13)+f(13)^2-f(711)^2-f(711)f(601)-\\
&\quad -f(601)^2=\frac{1}{9}\big(3f(1131)+2\big)\big(3f(1131)-1\big)=0,\\
E(2528127)&=f(1587)^2+f(1587)f(6)+f(6)^2-f(1131)^2-f(1131)f(687)-\\
&\quad -f(687)^2=\frac{1}{3}\big(3f(1131)+3f(1587)+1\big)\cdot\\
&\quad \cdot\big(f(1131)-f(1587)\big)=0,
\end{aligned}$$

that $1131\in I$ and $1587=3\cdot 23^2\in I$, and so $23\in\mathcal{S}_1$.

For the proof of $3\cdot 24^2\in I$, we use the facts that $351\in\mathcal{S}_1$, $489\in\mathcal{S}_1$, $727\in\mathcal{S}_1$, $739\in\mathcal{S}_1$ to get the following 3 relations

$$\begin{aligned}
E(533961)&=f(720)^2+f(720)f(21)+f(21)^2-f(489)^2-f(489)f(351)-\\
&\quad -f(351)^2=\frac{1}{9}\big(3f(720)+2\big)\big(f(720)-1\big)=0,
\end{aligned}$$

$$E(1596601) = f(1256)^2 + f(1256)f(15) + f(15)^2 - f(739)^2 - f(739)f(720) - f(720)^2 = -\frac{1}{3}\big(3f(720) + 3f(1256) + 1\big)\cdot\big(f(720) - f(1256)\big) = 0$$

$$E(3019177) = f(1728)^2 + f(1728)f(19) + f(19)^2 - f(1256)^2 - f(1256)f(727) - f(727)^2 = -\frac{1}{3}\big(3f(1256) + 3f(1728) + 1\big)\cdot\big(f(1256) - f(1728)\big) = 0,$$

which imply $720 \in I$, $1256 \in I$ and $1728 = 3.24^2 \in I$. Therefore $24 \in \mathcal{S}_1$.

For the proof of $3.25^2 \in I$, we infer from (2.7) that $37 = [4,3]$, $43 = [6,1]$, $52 = [6,2]$, $403 = [14,9]$, $652 = [22,6]$, $723 = [17,14]$, $777 = [19,13]$ are in $\mathcal{S}_1$, consequently we have the following relations

$$E(850269) = f(895)^2 + f(895)f(52) + f(52)^2 - f(652)^2 - f(652)f(403) - f(403)^2 = \frac{1}{9}\big(3f(895) + 2\big)\big(3f(895) - 1\big) = 0,$$

$$E(1970839) = f(1385)^2 + f(1385)f(37) + f(37)^2 - f(895)^2 - f(895)f(723) - f(723)^2 = -\frac{1}{3}\big(3f(895) + 3f(1385) + 1\big)\cdot(f(895) - f(1385)) = 0,$$

$$E(3598099) = f(1875)^2 + f(1875)f(43) + f(43)^2 - f(1385)^2 - f(1385)f(777) - f(777)^2 = -\frac{1}{3}\big(3f(1385) + 3f(1875) + 1\big)\cdot\big(f(1385) - f(1875)\big) = 0,$$

that $895 \in I$, $1385 \in I$ and $1875 = 3 \cdot 25^2 \in I$, $25 \in \mathcal{S}_1$.

Finally, we prove that $3 \cdot 26^2 \in I$. Since $163 = [11,3] \in \mathcal{S}_1$, $541 = [21,4] \in \mathcal{S}_1$, $721 = [24,5] \in \mathcal{S}_1$, $756 = [24,6] \in \mathcal{S}_1$, $793 = [21,11] \in \mathcal{S}_1$, we have

$$E(1202583) = f(1006)^2 + f(1006)f(163) + f(163)^2 - f(721)^2 - f(721)f(541) - f(541)^2 = \frac{1}{9}(3f(1006) + 2)\cdot(3f(1006) - 1) = 0,$$

$$E(2344108) = f(1524)^2 + f(1524)f(14) + f(14)^2 - f(1006)^2 - f(1006)f(756) - f(756)^2 = -\frac{1}{3}(3f(1006) + 3f(1524) + 1)\cdot(f(1006) - f(1524)) = 0,$$

and

$$E(4159957)=f(2028)^2+f(2028)f(23)+f(23)^2-f(1524)^2- \\ -f(1524)f(793)-f(793)^2=-\frac{1}{3}(3f(1524)+3f(2028)+1)\cdot \\ (f(1524)-f(2028))=0,$$

that $1006\in I$, $1524\in I$ and $2028=3\cdot 26^2\in I$. Therefore we proved that $26\in\mathcal{S}_1$.

The proof of Lemma 3 is completes. ■

Lemma 4. *Assume that* $D=1$, $f:\mathbb{N}\to\mathbb{C}$ *satisfies* (2.3). *If*

$$f(1)=1,\quad f(2)=2\quad\textit{and}\quad f(4)=4,$$

then $f(n)=n$ *for every* $n\in\mathbb{N}$.

Proof. Let

$$\mathcal{E}:=\{n\in\mathbb{N}\mid f(n)=n\}.$$

In order to prove Lemma 4, by using Lemma 1, we need to show that

$$\{1,\ 2,\ \ldots,\ 28\}\subset\mathcal{E}. \tag{2.11}$$

Since $1,\ 2,\ 4\in\mathcal{E}$, we have

$$\{3,\ 7,\ 12,\ 13,\ 19,\ 21,\ 27,\ 28\}\subset\mathcal{E},$$

because $f(3)=f(1)^2+f(1)f(1)+f(1)^2=3$, $f(7)=f(2)^2+f(2)f(1)+f(1)^2==7$, $f(12)=f(2)^2+f(2)f(2)+f(2)^2=12$, $f(13)=f(3)^2+f(3)f(1)+f(1)^2==13$, $f(19)=f(3)^2+f(3)f(2)+f(2)^2=19$, $f(21)=f(4)^2+f(4)f(1)+f(1)^2==21$, $f(27)=3f(3)^2=27$ and $f(28)=f(4)^2+f(4)f(2)+f(2)^2=28$.

First we use the following relations

$$\begin{aligned}E(1533)&=f(37)^2+f(37)f(4)+f(4)^2-f(31)^2-f(31)f(13)-f(13)^2=\\&=-\big(f(5)+6\big)\big(f(5)-5\big)\big(f(5)^2+f(5)+45\big)=0,\\E(1603)&=f(39)^2+f(39)f(2)+f(2)^2-f(27)^2-f(27)f(19)-f(19)^2=\\&=\big(f(5)-5\big)\big(f(5)+7\big)\big(f(5)^2+2f(5)+45\big)=0,\\E(637)&=f(23)^2+f(23)f(4)+f(4)^2-f(21)^2-f(21)f(7)-f(7)^2=\\&=\big(f(23)+27\big)\big(f(23)-23\big)=0,\\E(1957)&=f(37)^2+f(37)f(12)+f(12)^2-f(28)^2-f(28)f(23)-f(23)^2=\\&=-\big(f(23)+51\big)\big(f(23)-23\big)=0\end{aligned}$$

to obtain that $5\in\mathcal{E}$ and $23\in\mathcal{E}$.

Next, by using the facts

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 7,\ 12,\ 13,\ 19,\ 21,\ 23,\ 27,\ 28\} \subset \mathcal{E},$$

we have

$$\begin{aligned}
E(4237) &= f(63)^2 + f(63)f(4) + f(4)^2 - f(52)^2 - f(52)f(21) - f(21)^2 = \\
&= 2\big(f(6)-6\big)\big(f(6)^2 + 5f(6) + 34\big) = 0,\\
E(6357) &= f(76)^2 + f(76)f(7) + f(7)^2 - f(49)^2 - f(49)f(43) - f(43)^2 = \\
&= 3\big(f(6)-6\big)\big(2f(6)^2 + 13f(6) + 113\big) = 0,\\
E(777) &= f(23)^2 + f(23)f(8) + f(8)^2 - f(19)^2 - f(19)f(13) - f(13)^2 = \\
&= \big(f(8)+31\big)\big(f(8)-8\big) = 0,\\
E(5089) &= f(67)^2 + f(67)f(8) + f(8)^2 - f(57)^2 - f(57)f(23) - f(23)^2 = \\
&= \big(f(8)+75\big)\big(f(8)-8\big) = 0,\\
E(259) &= f(15)^2 + f(15)f(2) + f(2)^2 - f(13)^2 - f(13)f(5) - f(5)^2 = \\
&= \big(f(15)+17\big)\big(f(15)-15\big) = 0,\\
E(4329) &= f(57)^2 + f(57)f(15) + f(15)^2 - f(48)^2 - f(48)f(27) - f(27)^2 = \\
&= \big(f(15)+72\big)\big(f(15)-15\big) = 0,\\
E(637) &= f(21)^2 + f(21)f(7) + f(7)^2 - f(17)^2 - f(17)f(12) - f(12)^2 = \\
&= -\big(f(17)+29\big)\big(f(17)-17\big) = 0,\\
E(1477) &= f(31)^2 + f(31)f(12) + f(12)^2 - f(27)^2 - f(27)f(17) - f(17)^2 = \\
&= -\big(f(17)+44\big)\big(f(17)-17\big) = 0,\\
E(1027) &= f(31)^2 + f(31)f(2) + f(2)^2 - f(19)^2 - f(19)f(18) - f(18)^2 = \\
&= -\big(f(18)+37\big)\big(f(18)-18\big) = 0,\\
E(1843) &= f(39)^2 + f(39)f(7) + f(7)^2 - f(31)^2 - f(31)f(18) - f(18)^2 = \\
&= -\big(f(18)+49\big)\big(f(18)-18\big) = 0,\\
E(469) &= f(20)^2 + f(20)f(3) + f(3)^2 - f(13)^2 - f(13)f(12) - f(12)^2 = \\
&= \big(f(20)+23\big)\big(f(20)-20\big) = 0,\\
E(1141) &= f(31)^2 + f(31)f(5) + f(5)^2 - f(20)^2 - f(20)f(19) - f(19)^2 = \\
&= -\big(f(20)+39\big)\big(f(20)-20\big) = 0.
\end{aligned}$$

These imply that $\{6,\ 8,\ 15,\ 17,\ 18,\ 20\} \subset \mathcal{E}$. In the following we will prove that

$$\{9,\ 10,\ 11,\ 14,\ 22,\ 25,\ 26\} \subset \mathcal{E}.$$

We have

$$\begin{aligned}
E(91) &= f(9)^2+f(9)f(1)+f(1)^2-f(6)^2-f(6)f(5)-f(5)^2= \\
&= \big(f(9)+10\big)\big(f(9)-9\big)=0, \\
E(217) &= f(13)^2+f(13)f(3)+f(3)^2-f(9)^2-f(9)f(8)-f(8)^2= \\
&= -\big(f(9)+17\big)\big(f(9)-9\big)=0, \\
E(399) &= f(17)^2+f(17)f(5)+f(5)^2-f(13)^2-f(13)f(10)-f(10)^2= \\
&= -\big(f(10)+23\big)\big(f(10)-10\big)=0, \\
E(364) &= f(18)^2+f(18)f(2)+f(2)^2-f(12)^2-f(12)f(10)-f(10)^2= \\
&= -\big(f(10)+22\big)\big(f(10)-10\big)=0, \\
E(147) &= f(11)^2+f(11)f(2)+f(2)^2-f(7)^2-f(7)f(7)-f(7)^2= \\
&= \big(f(11)+13\big)\big(f(11)-11\big)=0, \\
E(511) &= f(19)^2+f(19)f(6)+f(6)^2-f(15)^2-f(15)f(11)-f(11)^2= \\
&= -\big(f(11)+26\big)\big(f(11)-11\big)=0, \\
E(343) &= f(18)^2+f(18)f(1)+f(1)^2-f(14)^2-f(14)f(7)-f(7)^2= \\
&= -\big(f(14)+21\big)\big(f(14)-14\big)=0, \\
E(5047) &= f(63)^2+f(63)f(14)+f(14)^2-f(43)^2-f(43)f(39)-f(39)^2= \\
&= \big(f(14)+77\big)\big(f(14)-14\big)=0, \\
E(507) &= f(22)^2+f(22)f(1)+f(1)^2-3f(13)^2= \\
&= \big(f(22)+23\big)\big(f(22)-22\big)=0, \\
E(532) &= f(22)^2+f(22)f(2)+f(2)^2-f(18)^2-f(18)f(8)-f(8)^2= \\
&= \big(f(22)+24\big)\big(f(22)-22\big)=0, \\
E(679) &= f(25)^2+f(25)f(2)+f(2)^2-f(17)^2-f(17)f(13)-f(13)^2= \\
&= \big(f(25)+27\big)\big(f(25)-25\big)=0, \\
E(889) &= f(27)^2+f(27)f(5)+f(5)^2-f(25)^2-f(25)f(8)-f(8)^2= \\
&= -\big(f(25)+33\big)\big(f(25)-25\big)=0, \\
E(703) &= f(26)^2+f(26)f(1)+f(1)^2-f(23)^2-f(23)f(6)-f(6)^2= \\
&= \big(f(26)+27\big)\big(f(26)-26\big)=0, \\
E(1183) &= f(31)^2+f(31)f(6)+f(6)^2-f(26)^2-f(26)f(13)-f(13)^2= \\
&= -\big(f(26)+39\big)\big(f(26)-26\big)=0,
\end{aligned}$$

These imply that $\{9,\ 10,\ 11,\ 14,\ 22,\ 25,\ 26\}\subset\mathcal{E}$.

Finally, we prove that $16 \in \mathcal{E}$ and $24 \in \mathcal{E}$. Indeed, we infer from

$$\begin{aligned} E(273) &= f(16)^2 + f(16)f(1) + f(1)^2 - f(11)^2 - f(11)f(8) - f(8)^2 = \\ &= \big(f(16)+17\big)\big(f(16)-16\big) = 0, \\ E(481) &= f(19)^2 + f(19)f(5) + f(5)^2 - f(16)^2 - f(16)f(9) - f(9)^2 = \\ &= -\big(f(16)+25\big)\big(f(16)-16\big) = 0, \\ E(793) &= f(24)^2 + f(24)f(7) + f(7)^2 - f(21)^2 - f(21)f(11) - f(11)^2 = \\ &= \big(f(24)+31\big)\big(f(24)-24\big) = 0, \\ E(793) &= f(31)^2 + f(31)f(8) + f(8)^2 - f(24)^2 - f(24)f(17) - f(17)^2 = \\ &= -\big(f(24)+41\big)\big(f(24)-24\big) = 0, \end{aligned}$$

that $16 \in \mathcal{E}$ and $24 \in \mathcal{E}$.

Consequently, (2.11) and Lemma 4 are proved. ■

3. Proof of Theorem 1

Assume that $f : \mathbb{N} \to \mathbb{C}$ satisfies

$$(3.1)\quad f\big(n^2+nm+m^2\big) = f(n)^2 + f(n)f(m) + f(m)^2 \quad \text{for every} \quad n, m \in \mathbb{N}.$$

In the following, let $f(1) = x$, $f(2) = y$ and $f(4) = z$. By using the computer and Maple program, we have

$$\begin{aligned} f(3) &= f(1)^2 + f(1)f(1) + f(1)^2 = 3f(1)^2 = 3x^2, \\ f(7) &= f(2)^2 + f(2)f(1) + f(1)^2 = x^2 + xy + y^2, \\ f(12) &= f(2)^2 + f(2)f(2) + f(2)^2 = 3f(2)^2 = 3y^2, \\ f(13) &= f(3)^2 + f(3)f(1) + f(1)^2 = 9x^4 + 3x^3 + x^2, \\ f(19) &= f(3)^2 + f(3)f(2) + f(2)^2 = 9x^4 + 3x^2y + y^2, \\ f(21) &= f(4)^2 + f(4)f(1) + f(1)^2 = x^2 + xz + z^2, \\ f(27) &= 3f(3)^2 = 27x^4, \\ f(28) &= f(4)^2 + f(4)f(2) + f(2)^2 = y^2 + yz + z^2, \\ f(37) &= f(4)^2 + f(4)f(3) + f(3)^2 = 9x^4 + 3x^2z + z^2, \\ f(48) &= 3f(4)^2 = 3z^2, \\ f(147) &= 3f(7)^2 = 3(x^2+xy+y^2)^2, \\ f(157) &= f(12)^2 + f(12)f(1) + f(1)^2 = x^2 + 3xy^2 + 9y^4, \\ f(237) &= f(13)^2 + f(13)f(4) + f(4)^2 = \\ &= 81x^8 + 54x^7 + 27x^6 + 6x^5 + 9x^4z + x^4 + 3x^3z + x^2z + z^2, \end{aligned}$$

$$f(309)=f(13)^2+f(13)f(7)+f(7)^2=81x^8+54x^7+36x^6+9x^5y+9x^4y^2+$$
$$+9x^5+3x^4y+3x^3y^2+3x^4+3x^3y+4x^2y^2+2xy^3+y^4,$$
$$f(463)=f(21)^2+f(21)f(1)+f(1)^2=x^4+2x^3z+3x^2z^2+2xz^3+z^4+$$
$$+x^3+x^2z+xz^2+x^2,$$
$$f(487)=f(21)^2+f(21)f(2)+f(2)^2=x^4+2x^3z+3x^2z^2+2xz^3+z^4+$$
$$+x^2y+xyz+yz^2+y^2,$$
$$f(733)=f(19)^2+f(19)f(12)+f(12)^2=$$
$$=81x^8+54x^6y+54x^4y^2+15x^2y^3+13y^4.$$

We can find with the computer 6 numbers $n\in E$ such that $\max(x,y,u,v)\leq$ ≤ 800 and $E(n)$ can written in terms of x,y and z. In the following we shall use only 4 such number n, namely

$$\{1677,\ 3097,\ 677019,\ 112617\}\subset E.$$

For these numbers, we have

$$E(1677)=f(37)^2+f(37)f(7)+f(7)^2-f(28)^2-f(28)f(19)-f(19)^2=$$
$$=-54x^6y+54x^6z+9x^6+9x^5y-27x^4y^2-9x^4yz+18x^4z^2+$$
$$+3x^4z+x^2z^2+3x^3yz-9x^2y^3-3x^2yz^2+6x^2z^3+x^4+$$
$$+2x^3y+3x^2y^2+2xy^3+xyz^2-2y^4-3y^3z-3y^2z^2-2yz^3=0,$$
$$E(3097)=f(48)^2+f(48)f(13)+f(13)^2-f(37)^2-f(37)f(27)-f(27)^2=$$
$$=-972x^8+54x^7-135x^6z+27x^6-27x^4z^2+6x^5+9x^3z^2-$$
$$-6x^2z^3+x^4+3x^2z^2+8z^4=0,$$
$$E(677019)=f(733)^2+f(733)f(157)+f(157)^2-f(487)^2-f(487)f(463)-$$
$$-f(463)^2=6561x^{16}+8748x^{14}y+11664x^{12}y^2+8262x^{10}y^3+$$
$$+7371x^8y^4+243x^9y^2+3510x^6y^5+81x^{10}+162x^7y^3+$$
$$+2115x^4y^6+54x^8y+162x^5y^4+525x^2y^7-3x^8-12x^7z+$$
$$+54x^6y^2-30x^6z^2-48x^5z^3-57x^4z^4+45x^3y^5-48x^3z^5-$$
$$-30x^2z^6-12xz^7+367y^8-3z^8-3x^7-3x^6y-9x^6z-9x^5yz-$$
$$-18x^5z^2+15x^4y^3-18x^4yz^2-21x^4z^3-21x^3yz^3-18x^3z^4-$$
$$-18x^2yz^4-9x^2z^5+93xy^6-9xyz^5-3xz^6-3yz^6-4x^6-$$
$$-x^5y-8x^5z-4x^4y^2-2x^4yz-12x^4z^2-8x^3y^2z-3x^3yz^2-$$
$$-8x^3z^3+40x^2y^4-12x^2y^2z^2-2x^2yz^3-4x^2z^4-8xy^2z^3-$$
$$-xyz^4-4y^2z^4-2x^5-x^4y-2x^4z+5x^3y^2-x^3yz-2x^3z^2-$$
$$-2x^2y^3-x^2y^2z-x^2yz^2-2xy^3z-xy^2z^2-2y^3z^2-x^2y^2-y^4$$

and

$$\begin{aligned}E(112617) = &f(309)^2 + f(309)f(48) + f(48)^2 - f(237)^2 - f(237)f(147)-\\ &- f(147)^2 = (x^2+xy+y^2-z)(1458x^{12}+1458x^{11}+972x^{10}+\\ &+378x^9+126x^8+36x^7y+54x^6y^2+36x^5y^3+18x^4y^4+24x^7+\\ &+12x^6y-9x^6z+18x^5y^2-9x^5yz+12x^4y^3-9x^4y^2z+6x^3y^4-\\ &-4x^6-20x^5y-3x^5z-42x^4y^2-3x^4yz+18x^4z^2-52x^3y^3-\\ &-3x^3y^2z-46x^2y^4-24xy^5-8y^6-9x^4z-17x^3yz+6x^3z^2-\\ &-25x^2y^2z-16xy^3z-8y^4z-6x^2z^2-8xyz^2-8y^2z^2-8z^3)=\\ =&\ 0.\end{aligned}$$

In the following we use a notation $rem\big(a(x,y,z),\, b(x,y,z),\, x\big)$ as follows: If $a(x,y,z)$, $b(x,y,z)$ are polynomials in $(\mathbb{Q}[y,z])[x]$ with $b(x,y,z)\neq 0$, then there exist unique polynomials $q(x,y,z)$ and $r(x,y,z)$ in $(\mathbb{Q}[y,z])[x]$ with

$$a(x,y,z) = q(x,y,z)b(x,y,z) + r(x,y,z)$$

and such that the degree of $r(x,y,z)$ is smaller than the degree of $b(x,y,z)$ in x. The polynomials $q(x,y,z)$ and $r(x,y,z)$ are uniquely determined by $a(x,y,z)$ and $b(x,y,z)$. We denote by $rem\big(a(x,y,z),\, b(x,y,z),\, x\big)$ the $r(x,y,z)$. Similar way, we can define $rem\big(a(x,y,z),\, b(x,y,z),\, y\big)$ and $rem\big(a(x,y,z),\, b(x,y,z),\, z\big)$.

Proof of (a). Assume that $x=f(1)=0$. Then $E(3097)=8z^4=0$ and so $z=f(4)=0$. Therefore, we have $E(1677)=-2y^4=0$, which implies that $y=f(2)=0$. Consequently, Lemma 2 implies that (a) of Theorem 1 holds.

Proof of (b). Assume that $x=f(1)=\frac{1}{3}$. Then

$$E(3097) = \frac{1}{81}(3z-1)(6z+1)\big(36z^2+3z+4\big)=0.$$

We have three cases:

(i) $6z+1=0$,

(ii) $36z^2+3z+4=0$,

(iii) $3z-1=0$.

Case (i). Assume that $x=f(1)=\frac{1}{3}$ and $z=-\frac{1}{6}$. Then

$$0 = 324E(1677) = -(3y-1)(6y+1)(36y^2+3y+4)$$

and

$$\begin{gathered}0 = 26244E(112617) =\\ = -(3y+2)(3y-1)\big(18y^2+6y+5\big)\big(36y^2+30y+13\big)\big(36y^2-6y+7\big).\end{gathered}$$

But
$$0 = gcd\left(324E(1677),\, 26244E(112617)\right) = 3y-1,$$
which proves that $y=\frac{1}{3}$. On the other hand, if $x=\frac{1}{3}$, $z=-\frac{1}{6}$ and $y=\frac{1}{3}$, then $E(677019)=\frac{69}{256}\neq 0$, which is a contradiction. Thus, the case (i) does not occur.

Case (ii). Now assume that $x=f(1)=\frac{1}{3}$ and $Q(z):=36z^2+3z+4=0$. Let
$$Q_1 := rem\left(648E(1677),\, Q,\, z\right)$$
and
$$Q_2 := rem\Big(\frac{9}{8}(12y+1)^2Q,\, Q1,\, z\Big).$$
Then
$$Q2 = \left(36y^2+21y+7\right)\left(72y^2-3y+2\right).$$
Similar way, let
$$Q_3 := rem\left(11943936\cdot E(677019),\, Q,\, z\right)$$
and
$$Q_4 := rem\Big(\frac{6561}{16384}\left(221184y^3+27648y^2+9360y+6541\right)^2Q,\, Q3,\, z\Big).$$
Then with help of computer, we have
$$\begin{aligned}Q_4 = {} & 379972472325341184y^{16}+120790522537574400y^{15}+\\ & +127859283595689984y^{14}+32218119998668800y^{13}+\\ & +22173993529638912y^{12}+4681113817227264y^{11}+\\ & +2460889496613888y^{10}+435555863548272y^{9}+\\ & +187636070089827y^{8}+27301842493713y^{7}+\\ & +10017117436986y^{6}+1170978598953y^{5}+\\ & +360330894360y^{4}+35176239549y^{3}+8145469377y^{2}+\\ & +550910895y+136484728.\end{aligned}$$
But
$$gcd\left(Q2, Q4\right) = 1,$$
which is impossible. Thus, the case (ii) does not occur.

Case (iii). Now assume that $x=f(1)=\frac{1}{3}$ and $z=\frac{1}{3}$. Then
$$E(1677) = -\frac{1}{8}(3y+2)(3y-1)\left(18y^2+6y+5\right) = 0$$

and

$$E(677019) = \frac{1}{6561}(3y-1)\big(802629y^7 + 395118y^6 + 256608y^5 + \\ + 99711y^4 + 44685y^3 + 14310y^2 + 4035y + 962\big) = 0.$$

But

$$0 = \big(8E(1677),\, 6561E(677019)\big) = 3y - 1,$$

which implies that $y = \frac{1}{3}$. Thus, we have proved that $x = y = z = \frac{1}{3}$, which with Lemma 2 implies that (b) of Theorem 1 holds.

Proof of (c). Assume that $f(1) = 1$. Then

$$E(3097) = (z-4)\big(8z^3 + 26z^2 + 89z + 221\big) = 0,$$

and so in the following we distinguish two cases:

(A) $8z^3 + 26z^2 + 89z + 221 = 0$,

(B) $z = 4$.

First we consider the case (A). We will prove that this case does not occur. Let $x = 1$ and $P := 8z^3 + 26z^2 + 89z + 221 = 0$. Let

$$P_1 = rem\,\big(4E(1677),\, P,\, z\big),$$
$$P_2 = rem\,\Big(\frac{1}{2}\big(6y^2 - 9y + 1\big)^2 P,\, P_1,\, z\Big),$$

and

$$P_3 = rem\Big(\tfrac{1}{2}\big(48y^6 - 660y^5 + 80y^4 + 847y^3 - 5510y^2 + 8145y + 66\big)^2 P_1,\, P_2,\, z\Big).$$

With the help of Maple and computer, we have

$$P3 = -(1024y^{12} + 5760y^{11} + 47232y^{10} + 132360y^9 + 778008y^8 + 951066y^7 + \\ + 3260222y^6 + 16475731y^5 + 16651871y^4 + 84676957y^3 + \\ + 161713315y^2 + 127018494y + 389604312) \cdot (6y^2 - 9y + 1)^2.$$

On other hand, let

$$P_4 := rem\big(2E(112617),\, P,\, z\big), \quad P_4 = 0,$$

$$P_5 := rem\Big(\big(39y^2 + 39y + 119\big)^2 P,\, P_4, z\Big), \quad P_5 = 0,$$

and

$$P_6 = rem\Big(\big(2496y^{10} + 12480y^9 + 46616y^8 + 111584y^7 + 226986y^6 + \\ + 342830y^5 + 1788437y^4 + 3110712y^3 + 5747670y^2 + \\ + 4234599y + 141179756\big)^2 P_4,\, P_5,\, z\Big), \quad P_6 = 0.$$

With the help of Maple and computer, we have

$$\begin{aligned}P_6 = -2\Big(&8y^6+24y^5+74y^4+108y^3+215y^2+165y+344\Big)\cdot\\ &\cdot\Big(4096y^{18}+36864y^{17}+131072y^{16}+212992y^{15}-28800y^{14}-\\ &-1004416y^{13}-8792640y^{12}-40869888y^{11}-78484864y^{10}-\\ &-37231680y^9+201232250y^8+611161704y^7+4099068715y^6+\\ &+10417529621y^5+4618398473y^4-7546004973y^3-47172429173y^2-\\ &-41744731095y-672668750764\Big)\big(39y^2+39y+119\big)^2.\end{aligned}$$

Consequently, P_3 and P_6 are impossible, because

$$P_3=0, \quad P_6=0 \quad \text{and} \quad gcd(P_3,P_6)=1.$$

Thus, we have proved that the case (A) does not occur.

Now we consider the case (B). Let $x=1$ and $z=4$. Then

$$E(1677)=-(y-2)\big(2y^3+23y^2+118y+463\big)=0$$

and

$$\begin{aligned}E(677019)=(y-2)\big(&367y^7+1259y^6+4726y^5+13007y^4+33586y^3+\\ &+75569y^2+161319y+303195)=0.\end{aligned}$$

Consequently

$$0=gcd\left(E(1677),\,E(677019)\right)=y-2,$$

which gives $y=2$. Thus we have proved that $x=1$, $y=2$, $z=4$, and so Lemma 2 implies that (c) of Theorem 1 holds.

Theorem 1 is proved. ■

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