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CHARACTERIZATION OF THE IDENTITY FUNCTION WITH AN EQUATION FUNCTION

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Abstract. Let $D \in \{2, 3\}$ and $f : \mathbb{N} \to \mathbb{C}$. We prove that if

$$f(n^2 + Dnm + m^2) = f(n)^2 + Df(n)f(m) + f(m)^2 \quad \text{for every} \quad n, m \in \mathbb{N},$$

then f is the identity function.

1. Introduction

A characterization of the identity was studied by Spiro [5], De Koninck, Kátai and Phong [2] and by others. Recently Poo Sung Park [4] proved the following result:

Theorem A. *If f is a multiplicative function and*

$$f(n^2 + nm + m^2) = f^2(n) + f(n)f(m) + f^2(m) \quad \textit{for every} \quad n, m \in \mathbb{N},$$

then f is the identity function.

Our general purpose is to determine all those $f : \mathbb{N} \to \mathbb{C}$ for which

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N},$$

where $D \in \mathbb{Z}$ is a fixed number.

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B. M. Phong and R. B. Szeidl [3] consider the case $D = 1$, they proved the following theorem:

Theorem B. ([3]) *If $f : \mathbb{N} \to \mathbb{C}$ is an arbitrary function with $f(1) = 1$ and*

$$f(n^2 + nm + m^2) = f^2(n) + f(n)f(m) + f^2(m)$$

holds for every $n, m \in \mathbb{N}$, then f is the identity function.

It is obvious that if f is multiplicative, then $f(1) = 1$, consequently Theorem B improves Theorem A.

We prove the following results:

Theorem 1. *Assume that f is an arithmetical function $\mathbb{N} \to \mathbb{C}$ with $f(1) = 1$ and $D \in \{2, 3\}$. If*

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Then $f(n) = n$ for every $n \in \mathbb{N}$.

We note that $f(1) = 1$ if a function f is multiplicative, therefore the following result follows directly from Theorem 1.

Corollary 1. *Assume that a multiplicative function f and $D \in \{2, 3\}$ satisfy*

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Then $f(n) = n$ for every $n \in \mathbb{N}$.

We think that the following assertion is true:

Conjecture 1. *Assume that $f \in \mathcal{M}$ and $D \in \mathbb{N}$ satisfy the following equation*

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Then one of the following assertions holds:

- $f(n) = 0 \quad$ *for every* $\quad n \in \mathbb{N}$,
- $f(n) = \dfrac{1}{D+2} \quad$ *for every* $\quad n \in \mathbb{N}$,
- $f(n) = n \quad$ *for every* $\quad n \in \mathbb{N}$.

We cannot prove Conjecture 1 for other D. It seems that they need some new ideas.

2. Proof of Theorem 1 for the case $D = 2$

First we prove the following lemma.

Lemma 1. *Assume that a function* $f : \mathbb{N} \to \mathbb{C}$ *satisfies* $f(1) = 1$ *and*

$$f(n^2 + 2nm + m^2) = f^2(n) + 2f(n)f(m) + f^2(m) \tag{2.1}$$

for every $n, m \in \mathbb{N}$. *Then*

$$f(1) = 1,\ f(2) = 2,\ f(3) = 3.$$

Proof. Assume that a function $f : \mathbb{N} \to \mathbb{C}$ satisfies $f(1) = 1$. We infer from (2.1) that

$$f\big((n+m)^2\big) = \big(f(n) + f(m)\big)^2 \quad \text{for every} \quad n, m \in \mathbb{N},$$

consequently

$$\begin{cases} f(4) &= f(2^2) = f((1+1)^2) = (f(1)+f(1))^2 = 4, \\ f(9) &= f(3^2) = f((1+2)^2) = (f(1)+f(2))^2 = (f(2)+1)^2, \\ f(16) &= f(4^2) = f((1+3)^2) = (f(2)+f(2))^2 = 4f(2)^2. \end{cases}$$

Consequently

$$\begin{cases} f(25) &= f((4+1)^2) = (f(4)+f(1))^2 = 25, \\ f(36) &= f((4+2)^2) = (f(4)+f(2))^2 = (f(2)+4)^2, \end{cases}$$

which imply

$$\begin{cases} 4f(2)^2 &= f(16) = f((3+1)^2) = (f(3)+1)^2, \\ 25 &= f(25) = f((3+2)^2) = (f(3)+f(2))^2, \\ (f(2)+4)^2 &= f(36) = f((3+3)^2) = (f(3)+f(3))^2 = 4f(3)^2. \end{cases}$$

It easily follows from these relations that

$$\begin{cases} 4f(3)^2 &= (f(2)+4)^2, \\ 16f(2)^2 &= 4(f(3)+1)^2 \quad = (f(2)+4)^2 + 8f(3) + 4, \\ 100 &= 4(f(3)+f(2))^2 = (f(2)+4)^2 + 8f(2)f(3) + 4f(2)^2. \end{cases}$$

Consequently

$$8f(3) = 15f(2)^2 - 8f(2) - 20 \tag{2.2}$$

and

$$8f(2)f(3) = -5f(2)^2 - 8f(2) + 84. \tag{2.3}$$

We infer from (2.2) that

$$f(3) = \frac{15}{8}f(2)^2 - f(2) - \frac{5}{2}$$

and so we get from (2.2) that

$$8f(2)\Big(\frac{15}{8}f(2)^2 - f(2) - \frac{5}{2}\Big) = -5f(2)^2 - 8f(2) + 84.$$

Consequently

$$\begin{aligned} 0 &= f(2)\big(15f(2)^2 - 8f(2) - 20\big) + 5f(2)^2 + 8f(2) - 84 = \\ &= 3\big(5f(2)^3 - f(2)^2 - 4f(2) - 28\big) = 3\big(f(2)-2\big)\big(5f(2)^2 + 9f(2) + 14\big), \end{aligned}$$

that is

$$\big(f(2)-2\big)\big(5f(2)^2 + 9f(2) + 14\big) = 0. \tag{2.4}$$

On the other hand, we have

$$\begin{aligned} 4f(2)^2 = f(4^2) = \big(f(3)+f(1)\big)^2 &= \Big(\frac{15}{8}f(2)^2 - f(2) - \frac{5}{2} + 1\Big)^2 = \\ &= \Big(\frac{15}{8}f(2)^2 - f(2) - \frac{3}{2}\Big)^2, \end{aligned}$$

which implies that

$$\big(5f(2)+6\big)\big(f(2)-2\big)\big(5f(2)+2\big)\big(3f(2)-2\big) = 0. \tag{2.5}$$

Finally, it is easy to check from (2.4) and (2.5) that

$$5f(2)^2 + 9f(2) + 14 \neq 0 \quad \text{if} \quad f(2) \in \Big\{-\frac{6}{5},\ -\frac{2}{5},\ \frac{2}{3}\Big\},$$

therefore

$$f(2) = 2 \quad \text{and} \quad f(3) = \frac{15}{8}f(2)^2 - f(2) - \frac{5}{2} = 3.$$

Lemma 1 is proved. ■

Now we complete the proof of Theorem 1 by showing the following lemma.

Lemma 2. *Assume that a function* $f:\mathbb{N}\to\mathbb{C}$ *satisfies* $f(1)=1$ *and*

$$f(n^2+2nm+m^2)=f^2(n)+2f(n)f(m)+f^2(m) \tag{2.6}$$

for every $n,m\in\mathbb{N}$. *Then*

$$f(n)=n \quad \textit{for every} \quad n\in\mathbb{N}.$$

Proof. By using Lemma 1 we have $f(1)=1$, $f(2)=2$ and $f(3)=3$. We assume that

$$f(N-1)=N-1 \quad \text{for some} \quad N\geq 4.$$

We will prove that

$$f(N)=N.$$

We infer from (2.1) that

$$f\big((N+1)^2\big)=f\Big(\big((N-1)+2\big)^2\Big)=\big(f(N-1)+f(2)\big)^2=(N+1)^2$$

and

$$f\big((N+2)^2\big)=f\Big(f\big((N-1)+3\big)^2\Big)=\big(f(N-1)+f(3)\big)^2=(N+2)^2.$$

But

$$f\big((N+1)^2\big)=\big(f(N)+f(1)\big)^2$$

and

$$f\big((N+2)^2\big)=\big(f(N)+f(2)\big)^2=\big(f(N)+2\big)^2,$$

consequently

$$\begin{aligned}2f(N)+3&=\big(f(N)+2\big)^2-\big(f(N)+1\big)^2=\\&=f\big((N+2)^2\big)-f\big((N+1)^2\big)=(N+2)^2-(N+1)^2=\\&=2N+3.\end{aligned}$$

This implies that $f(N)=N$. Lemma 2 and so Theorem 1 are proved for the case $D=2$. ■

3. Proof of Theorem 1 for the case $D=3$. Lemmas

First we state the following result.

Lemma 3. (B. M. Phong and R. B. Szeidl [3]) *Let $D \in \mathbb{N}$ be fixed and the function $f : \mathbb{N} \to \mathbb{C}$ satisfies*

$$f(n^2 + Dnm + m^2) = f^2(n) + Df(n)f(m) + f^2(m) \quad \textit{for every} \quad n, m \in \mathbb{N}.$$

If $f(n) = n$ for every $n \le 8D + 20$, then $f(n) = n$ for every $n \in \mathbb{N}$.

Now assume that $D = 3$ and

$$f(n^2 + 3nm + m^2) = f^2(n) + 3f(n)f(m) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{3.1}$$

In order to prove our theorem, by using Lemma 3, we need show that

$$f(n) = n \quad \text{for every} \quad n \le 8D + 20 = 8 \cdot 3 + 20 = 44.$$

First, we prove the following lemma.

Lemma 4. *Assume that $f : \mathbb{N} \to \mathbb{C}$ is arbitrary arithmetical function with $f(1) = 1$ and*

$$f(n^2 + 3nm + m^2) = f(n)^2 + 3f(n)f(m) + f^2(m) \tag{3.2}$$

holds for every $n, m \in \mathbb{N}$. Then

$$f(k) = k \quad \textit{for every} \quad k \le 4.$$

Proof. In the following we shall use the notation $[N, n, m, u, v] \in E$ if

$$N = n^2 + 3nm + m^2 = u^2 + 3uv + v^2 \quad \text{with} \quad n > u, m \le n, v \le u.$$

It is obvious that if $[N, n, m, u, v] \in E$, then we obtain from (3.2) that

$$f(N) = f^2(n) + 3f(n)f(m) + f^2(m) \text{ and } f(N) = f^2(u) + 3f(u)f(v) + f^2(v)$$

which show that

$$E(N) := f^2(n) + 3f(n)f(m) + f^2(m) - f^2(u) - 3f(u)f(v) - f^2(v) = 0. \tag{3.3}$$

We can find 6 numbers $[N, n, m, u, v] \in E$ such that $\max(n, m, u, v) \le 79$ and $E(N)$ can written in terms of $f(2), f(3), f(4)$.

Our method as follows: Let

$$\mathcal{A} := \{n \in \mathbb{N} \mid f(n) = P(f(2), f(3), f(4)) \ (n \le 79) \quad \text{where} \\ P(x, y, y) \in \mathbb{Z}[x, y, z]\}.$$

Then we can show that there are 16 number in $\mathcal{A}$, namely

$$\mathcal{A} = \{ 1, 2, 3, 4, 5, 11, 19, 20, 29, 31, 41, 44, 45, 59, 61, 79 \}.$$

By using the computer and Maple, we find those elements $[N, n, m, u, v] \in E$ such $n, m, u, v \in \mathcal{A}$, and so we get equation

$$E(N) = f^2(n) + 3f(n)f(m) + f^2(m) - f^2(u) - 3f(u)f(v) - f^2(v) = 0.$$

It is obvious that $E(N) = Q(P(f(2), f(3), f(4))$, where $Q(x, y, z) \in \mathbb{Z}[x, y, z]$ and $\max(n, m, u, v) \leq 79$.

We can show that there is 6 such number $[N, n, m, u, v] \in E$, namely

$$[605, 19, 4, 11, 11], [1805, 41, 1, 19, 19], [4805, 44, 19, 31, 31],$$
$$[4205, 59, 4, 29, 29], [7205, 79, 4, 59, 19], [18605, 79, 44, 61, 61].$$

We will consider the following 4 equations $E(605)$, $E(4205)$, $E(7205)$ and $E(1805)$.

Since $f(1) = 1$, we obtain from (3.2) that

$$(3.4) \qquad \begin{cases} f(5) &= f^2(1) + 3f(1)f(1) + f^2(1) = 5, \\ f(11) &= f^2(2) + 3f(2)f(1) + f^2(1) = f(2)^2 + 3f(2) + 1, \\ f(19) &= f^2(3) + 3f(3)f(1) + f^2(1) = f(3)^2 + 3f(3) + 1, \\ f(20) &= 5f(2)^2, \\ f(29) &= f^2(4) + 3f(4)f(1) + f^2(1) = f(4)^2 + 3f(4) + 1, \\ f(31) &= f^2(3) + 3f(3)f(2) + f^2(2), \\ f(41) &= f^2(5) + 3f(5)f(1) + f^2(1) = 41, \\ f(44) &= f^2(4) + 3f(4)f(2) + f^2(2), \\ f(45) &= 5f(3)^2, \\ f(59) &= f^2(5) + 3f(5)f(2) + f^2(2) = f(2)^2 + 15f(2) + 25, \\ f(61) &= f^2(4) + 3f(4)f(3) + f^2(3), \\ f(79) &= f^2(5) + 3f(5)f(3) + f^2(3) = f(3)^2 + 15f(3) + 25, \end{cases}$$

consequently, we have

$$\begin{aligned} E(605) &= f(19)^2 + 3f(19)f(4) + f(4)^2 - 5f(11)^2 = \\ &= \big(f(3)^2 + 3f(3) + 1\big)^2 + 3\big(f(3)^2 + 3f(3) + 1\big)f(4) + f(4)^2 - \\ &\quad - 5\big(f(2)^2 + 3f(2) + 1\big)^2 = \\ &= -5f(2)^4 + f(3)^4 - 30f(2)^3 + 6f(3)^3 + 3f(3)^2f(4) - 55f(2)^2 + \\ &\quad + 11f(3)^2 + +9f(3)f(4) + f(4)^2 - \\ &\quad - 30f(2) + 6f(3) + 3f(4) - 4 = 0, \end{aligned}$$

$$\begin{aligned}E(4205) &= f(59)^2 + 3f(59)f(4) + f(4)^2 - 5f(29)^2 = \\ &= \big(f(2)^2 + 15f(2) + 25\big)^2 + 3\big(f(2)^2 + 15f(2) + 25\big)f(4) + f(4)^2 - \\ &\quad - 5\big(f(4)^2 + 3f(4) + 1\big)^2 = \\ &= f(2)^4 - 5f(4)^4 + 30f(2)^3 + 3f(2)^2f(4) - 30f(4)^3 + 275f(2)^2 + \\ &\quad + 45f(2)f(4) - 54f(4)^2 + 750f(2) + 45f(4) + 620 = 0,\end{aligned}$$

$$\begin{aligned}E(7205) &= f(79)^2 + 3f(79)f(4) + f(4)^2 - f(59)^2 - 3f(59)f(19) - f(19)^2 = \\ &= \big(f(3)^2 + 15f(3) + 25\big)^2 + 3\big(f(3)^2 + 15f(3) + 25\big)f(4) + f(4)^2 - \\ &\quad - \big(f(2)^2 + 15f(2) + 25\big)^2 - \\ &\quad - 3\big(f(2)^2 + 15f(2) + 25\big)\big(f(3)^2 + 3f(3) + 1\big) - \\ &\quad - \big(f(3)^2 + 3f(3) + 1\big)^2 = \\ &= -f(2)^4 - 3f(3)^2f(2)^2 - 30f(2)^3 - 9f(2)^2f(3) - 45f(2)f(3)^2 + \\ &\quad + 24f(3)^3 + 3f(3)^2f(4) - 278f(2)^2 - 135f(2)f(3) + 189f(3)^2 + \\ &\quad + 45f(3)f(4) + f(4)^2 + 519f(3) - 795f(2) + 75f(4)) - 76 = 0\end{aligned}$$

and

$$\begin{aligned}E(1805) = f(41)^2 + 3f(41)f(1) + f(1)^2 - 5f(19)^2 = 41^2 + 3\cdot 41 + 1^2 - \\ - 5\big(f(3)^2 + 3f(3) + 1\big)^2 = \\ = -5\big(f(3) - 3)(f(3) + 6)(f(3)^2 + 3f(3) + 20\big) = 0.\end{aligned}$$

Let

$$\begin{aligned}P_1(x,y,z) &= -5x^4 + y^4 - 30x^3 + 6y^3 + 3y^2z - 55x^2 + \\ &\quad + 11y^2 + 9yz + z^2 - 30x + 6y + 3z - 4, \\ P_2(x,y,z) &= x^4 - 5z^4 + 30x^3 + 3x^2z - 30z^3 + 275x^2 + \\ &\quad + 45xz - 54z^2 + 750x + 45z + 620\end{aligned}$$

and

$$\begin{aligned}P_3(x,y,z) = -x^4 - 3x^2y^2 - 30x^3 - 9x^2y - 45xy^2 + 24y^3 + 3y^2z - 278x^2 - \\ - 135xy + 189y^2 + 45yz + z^2 - 795x + 519y + 75z - 76.\end{aligned}$$

It is clear to see that

$$\begin{cases} P_1\big(f(2),\ f(3),\ f(4)\big) = E(605) = 0, \\ P_2\big(f(2),\ f(3),\ f(4)\big) = E(4205) = 0, \\ P_3\big(f(2),\ f(3),\ f(4)\big) = E(7205) = 0. \end{cases} \tag{3.5}$$

Since

$$E(1805) = -5\big(f(3)-3\big)\big(f(3)+6\big)\big(f(3)^2+3f(3)+20\big) = 0,$$

in the proof of Lemma 4 we shall distinguish three cases, namely

(a) $f(3) = -6$,
(b) $f(3)^2 + 3f(3) + 20 = 0$,
(c) $f(3) = 3$.

We shall prove that the cases (a) and (b) are not occur.

Case (a). Assume that $f(3) = -6$. Then

$$0 = E(7205) - E(605) = -1575f(2) - 1926 - 277f(2)^2 - 144f(4) + 4f(2)^4,$$

which implies

$$f(4) = \frac{-1575f(2) - 1926 - 277f(2)^2 + 4f(2)^4}{144}.$$

Therefore, with the help of Maple and using a computer, we obtain the following values:

$$p := 20736E(605) = \big(f(2)+6\big)\big(16f(2)^7 - 96f(2)^6 - 1640f(2)^5 - 2760f(2)^4 + \\ +7033f(2)^3 + 208272f(2)^2 - 1116099f(2) - 786186\big) = 0$$

and

$$q := 429981696E(4205) = -1280f(2)^{16} + 354560f(2)^{14} + 2016000f(2)^{13} - \\ -34641120f(2)^{12} - 418824000f(2)^{11} + 54891440f(2)^{10} + 26418042000f(2)^9 + \\ +165565745371f(2)^8 + 1154380500f(2)^7 - 5225102615718f(2)^6 - \\ -32941396108980f(2)^5 - 107581783487877f(2)^4 - 209955740732280f(2)^3 - \\ -244519617134808f(2)^2 - 156273690782880f(2) - 42082903696464 = 0.$$

Let

$$P(x) = \big(x+6\big)\big(16x^7 - 96x^6 - 1640x^5 - 2760x^4 + 7033x^3 + \\ +208272x^2 - 1116099x - 786186\big)$$

and

$$Q(x) = -1280x^{16} + 354560x^{14} + 2016000x^{13} - 34641120x^{12} - 418824000x^{11} + \\ +54891440x^{10} + 26418042000x^9 + 165565745371x^8 + 1154380500x^7 - \\ -5225102615718x^6 - 32941396108980x^5 - 107581783487877x^4 - \\ -209955740732280x^3 - 244519617134808x^2 - 156273690782880x - \\ -42082903696464.$$

Then

$$P\big(f(2)\big) = p = 20736 \cdot E(605) = 0$$

and

$$Q\big(f(2)\big) = q = 429981696 \cdot E(4805) = 0.$$

But with help of computer, we have

$$\big(P(x),\, Q(x)\big) = 1,$$

which is impossible. The case (a) is not occur.

Case (b). If $a(x,y,z)$ and $b(x,y,z)$ are polynomials in $\big(\mathbb{Q}[y,z]\big)[x]$ with $b(x,y,z) \neq 0$, then there exist unique polynomials $q(x,y,z)$ and $r(x,y,z)$ in $\big(\mathbb{Q}[y,z]\big)[x]$ with

$$a(x,y,z) = q(x,y,z)b(x,y,z) + r(x,y,z)$$

and such that the degree of $r(x,y,z)$ is smaller than the degree of $b(x,y,z)$ in x. The polynomials $q(x,y,z)$ and $r(x,y,z)$ are uniquely determined by $a(x,y,z)$ and $b(x,y,z)$. We denote by $rem\big(a(x,y,z), b(x,y,z), x\big)$ the $r(x,y,z)$. Similar way, we can define $rem\big(a(x,y,z), b(x,y,z), y\big)$ and $rem\big(a(x,y,z), b(x,y,z), z\big)$.

Now assume that $f(3)^2 + 3f(3) + 20 = 0$. Let

$$Q(x,y,z) := y^2 + 3y + 20.$$

Then

$$Q\big(f(2), f(3), f(4)\big) = 0. \tag{3.6}$$

Let

$$Q_1(x,y,z) = rem\big(P_1(x,y,z), Q(x,y,z), y\big), \quad Q_2(x,y,z) = P_2(x,y,z)$$

and

$$Q_3(x,y,z) = rem\big(-P_3(x,y,z), Q(x,y,z), y\big).$$

Then we have

$$Q_1(x,y,z) = -\big(5x^4 + 30x^3 + 55x^2 - z^2 + 30x + 57z - 356\big),$$

$$Q_2(x,y,z) = x^4 - 5z^4 + 30x^3 + 3x^2z - 30z^3 + 275x^2 + 45xz - 54z^2 + 750x + 45z + 620$$

and

$$Q_3(x,y,z) = -x^4 - 30x^3 - 218x^2 + 36yz + z^2 + 105x - 312y + 15z - 2416.$$

It can check that

$$Q_1(x,y,z) = P_1(x,y,z) - Q(x,y,z)\big(y^2+3y+3z-18\big)$$

and

$$Q_3(x,y,z) = P_3(x,y,z) - 3Q(x,y,z)(x^2+15x-8y-z-39).$$

Thus, we infer from (3.5) and (3.6) that

$$Q_i\big(f(2),f(3),f(4)\big) = 0 \quad (i=1,2,3). \tag{3.7}$$

Next, we define the following polynomials

$$\begin{gathered}Q_4(x,y,z) = rem\big(-Q_2(x,y,z),Q_1(x,y,z),z\big) = \\ = 125x^8+1500x^7+7250x^6+18000x^5+3000x^4z+96369x^4+18000x^3z+ \\ +449940x^3+32997x^2z+798920x^2+17955xz+ \\ +432720x+812868z-5778144\end{gathered}$$

$$\begin{gathered}Q_5(x,y,z) = rem(Q_1(x,y,z),Q_4(x,y,z),y) = \\ = 9\big(1000x^4+6000x^3+10999x^2+5985x+270956\big)^2Q_1(x,y,z)+ \\ +Q_4(x,y,z)\big(125x^8+1500x^7+7250x^6+18000x^5-3000x^4z+267369x^4- \\ -18000x^3z+1475940x^3-32997x^2z+2679749x^2-17955xz+1456155x- \\ -812868z+40555332\big) = \\ = 15625x^{16}+375000x^{15}+4062500x^{14}+26250000x^{13}+107029750x^{12}+ \\ +237342000x^{11}-60155875x^{10}-2383543125x^9-7842885634x^8- \\ -8072184150x^7+22066388781x^6+90163881705x^5+91053921601x^4- \\ -165337109910x^3-458418024684x^2-295729578720x+894012947136.\end{gathered}$$

On the other hand, if we define $Q_6(x,y,z)$ and $Q_7(x,y,z)$ as follows

$$Q_6(x,y,z) = Q_1(x,y,z) - Q_3(x,y,z)$$

and

$$Q_7(x,y,z) = rem\big(144(3z-26)^2Q(x,y,z),Q_6(x,y,z),y\big)$$

then

$$Q_6(x,y,z) = -4x^4+163x^2-36yz-135x+312y-72z+2772,$$

and

$$\begin{gathered}Q_7(x,y,z) = 16x^8-1304x^6+1080x^5+144x^4z+8137x^4-44010x^3- \\ -5868x^2z+769329x^2+4860xz+23328z^2-622080x-481680z+7036272.\end{gathered}$$

Finally, we define $Q_8(x,y,z)$ by equation:

$$Q_8(x,y,z) = rem((1000x^4 + 6000x^3 + 10999x^2 + 5985x + 270956)^2 Q7, Q4, z).$$

One can check with computer that

$$\begin{gathered} Q_8(x,y,z) = 50500000x^{16} + 1056000000x^{15} + 9552474000x^{14} + \\ +56193990000x^{13} + 331085315516x^{12} + 2106409113480x^{11} + \\ +11508205483916x^{10} + 50700142595880x^9 + 197646102014197x^8 + \\ +711483952342020x^7 + 2537656772345138x^6 + 7179892664435100x^5 + \\ +13657348154690953x^4 + 23102991892891800x^3 + 94982689780252416x^2 - \\ -20002804321668480x + 351745639528990464. \end{gathered}$$

By using of definitions of $Q_i(x,y,z)$ $(i = 1, 2, \cdots, 8)$, we obtain from (3.6) and (3.7) that

$$Q_i\big(f(2), f(3), f(4)\big) = 0 \quad (i = 1, 2, \cdots, 8).$$

This is impossible, because we can show with computer that

$$\big(Q_5(x,y,z), Q_8(x,y,z)\big) = 1.$$

The case (b) is not occur.

Case (c). $f(3) = 3$.

Let

$$A_i(x,z) := P_i(x,3,z) \quad (i = 1, 2, 3).$$

Then

$$A_i\big(f(2), f(4)\big) = P_i\big(f(2), f(3), f(4)\big) = 0 \quad (i = 1, 2, 3) \tag{3.8}$$

and

$$A_1(x,z) := -5x^4 - 30x^3 - 55x^2 + z^2 - 30x + 57z + 356,$$

$$A_2(x,z) := x^4 - 5z^4 + 30x^3 + 3x^2z - 30z^3 + 275x^2 + 45xz - 54z^2 + 750x + 45z + 620,$$

$$A_3(x,z) := -x^4 - 30x^3 - 332x^2 + z^2 - 1605x + 237z + 3830,$$

Let $B(x,z) := A_1(x,z) - A_3(x,z)$. Then

$$\begin{cases} B(x,z) & = -4x^4 + 277x^2 + 1575x - 180z - 3474, \\ B(f(2), f(4)) & = A_1\big(f(2), f(4)\big) + A_3\big(f(2), f(4)\big) = 0. \end{cases} \tag{3.9}$$

$$\begin{aligned} B_1(x,z) = factor\,\big(rem\big(32400A1, B, z\big)\big) = \\ = (x-2)(x-3)\big(16x^6 + 80x^5 - 1912x^4 - 22640x^3 - \\ - 200247x^2 - 964845x - 2006694\big), \end{aligned}$$

$$\begin{aligned}B_2(x,z) &= rem\big(209952000A_2(x,z), B(x,z), z\big) = \\ &= -(x-2)\big(256x^{15}+512x^{14}-69888x^{13}-542976x^{12}+7100256x^{11}+ \\ &\quad +97965312x^{10}-76393840x^{9}-6922389680x^{8}-28155424095x^{7}+ \\ &\quad +149268389310x^{6}+1413378129186x^{5}+1734555943872x^{4}- \\ &\quad -11553800720103x^{3}-19538613395406x^{2}+ \\ &\quad +77057131513356x-52323738856488\big).\end{aligned}$$

It follows from (3.8) and (3.9) that

$$B_1\big(f(2), f(4)\big) = 0 \quad \text{and} \quad B_1\big(f(2), f(4)\big) = 0. \tag{3.10}$$

One can prove that

$$\big(B_1(x,z),\ B_2(x,z)\big) = x-2,$$

and so there are suitable polynomials $c_1(x,z) \in \mathbb{Q}[x,z], c_2(x,z) \in \mathbb{Q}[x,z]$ such that

$$x-2 = c_1(x,z)B_1(x,z) + c_2(x,z)B_2(x,z).$$

This with (3.10) show that

$$f(2)-2 = c_1\big(f(2), f(4)\big)B_1\big(f(2), f(4)\big) + c_2\big(f(2), f(4)\big)B_2\big(f(2), f(4)\big) = 0,$$

consequently

$$f(2) = 2.$$

If $f(2) = 2$, then

$$0 = B(f(2), f(4)) = B(2, f(4)) = -180f(4) + 720.$$

Therefore, we have $f(4) = 4$, and so the proof of Lemma 4 is completed. ■

Lemma 5. *Assume that $f : \mathbb{N} \to \mathbb{C}$ is arbitrary arithmetical function with $f(1) = 1$ satisfying (3.2). Then*

$$f(6) = 6 \quad \textit{and} \quad f(7) = 7.$$

Proof. Let

$$I := \{n \in \mathbb{N} \ \mid\ f(n) = n\}.$$

By using Lemma 4, we have $f(1) = 1$, $f(2) = 2$, $f(3) = 3$ and $f(4) = 4$. Consequently, we infer from (3.4) that

$$\{\,1, 2, 3, 4, 5, 11, 19, 20, 29, 31, 41, 44, 45, 59, 61, 79\,\} \subseteq I. \tag{3.11}$$

It is clear to check that

$$\begin{cases} 8405 &= 76^2 + 3.76.11 + 11^2 = 41^2 + 3\cdot 41\cdot 41 + 41^2, \\ 51005 &= 124^2 + 3\cdot 124\cdot 79 + 79^2 = 101^2 + 3\cdot 101\cdot 101 + 101^2, \end{cases}$$

consequently

$$\begin{cases} E(8405) &= f(76)^2+3f(76)f(11)+f(11)^2-5f(41)^2=0, \\ E(51005) &= f(124)^2+3f(124)f(79)+f(79)^2-5f(101)^2=0. \end{cases}$$

But $11, 41, 79 \in I$ and

$$\begin{aligned} f(76) = & \; f(6)^2+3f(6)f(2)+f(2)^2 = f(6)^2+6f(6)+4, \\ f(101) = & \; f(5)^2+3f(5)f(4)+f(4)^2 = 101, \\ f(124) = & \; f(6)^2+3f(6)f(4)+f(4)^2 = f(6)^2+12f(6)+16, \end{aligned}$$

we have

$$\begin{cases} E(8405) &= \big(f(6)^2+6f(6)+4\big)^2+3\cdot 11\cdot\big(f(6)^2+6f(6)+4\big)+ \\ & \quad +11^2-5\cdot 41^2 = \\ &= \big(f(6)-6\big)\big(f(6)+12\big)\big(f(6)^2+6f(6)+113\big)=0, \\ E(51005) &= \big(f(6)^2+12f(6)+16\big)^2+3\cdot 79\cdot\big(f(6)^2+12f(6)+16\big)+ \\ & \quad +79^2-5\cdot 101^2 = \\ &= \big(f(6)-6\big)\big(f(6)+18\big)\big(f(6)^2+12f(6)+377\big)=0. \end{cases}$$

Let

$$a(x) := (x-6)(x+12)(x^2+x+113)$$

and

$$b(x) = (x-6)(x+18)(x^2+12x+377).$$

Then

$$a(f(6)) = 0 \quad \text{and} \quad b(f(6)) = 0$$

and

$$gcd\,\big(a(x), b(x)\big) = x-6,$$

which imply that $f(6) = 6$. Thus, we have proved that $f(6) = 6$.

Now we prove $f(7) = 7$. Since

$$\begin{cases} 14801 &= 95^2+3\cdot 95\cdot 19+19^2 \quad \text{and} \quad 14801 = 80^2+3\cdot 80\cdot 31+31^2, \\ 31205 &= 121^2+3\cdot 121\cdot 41+41^2 \quad \text{and} \quad 31205 = 5\cdot 79^2, \end{cases}$$

we have

$$\begin{cases} E(14801) &= f(95)^2+3f(95)f(19)+f(19)^2-f(80)^2- \\ & \quad -3f(80)f(31)-f(31)^2=0, \\ E(31205) &= f(121)^2+3f(121)f(41)+f(41)^2-5f(79)^2=0. \end{cases}$$

One can check that

$$f(80) = 5f(4)^2 = 80,$$

$$f(95) = f(7)^2 + 3f(7)f(2) + f(2)^2 = f(7)^2 + 12f(7) + 4,$$

$$f(121) = f(7)^2 + 3f(7)f(3) + f(3)^2 = f(7)^2 + 9f(7) + 9.$$

Since $19 \in I$, $31 \in I$, $41 \in I$ and $79 \in I$, we obtain from above relations that

$$\begin{cases} E(14801) & = \big(f(7)^2 + 12f(7) + 4\big)^2 + 57\big(f(7)^2 + 12f(7) + 4\big) + \\ & \quad + 19^2 - 80^2 - 3 \cdot 80 \cdot 31 - 31^2 = \\ & = \big(f(7) - 7\big)\big(f(7) + 13\big)\big(f(7)^2 + 6f(7) + 156\big) = 0, \\ E(31205) & = \big(f(7)^2 + 9f(7) + 9\big)^2 + 123\big(f(7)^2 + 9f(7) + 9\big) + \\ & \quad + 41^2 - 5 \cdot 79^2 = \\ & = \big(f(7) - 7\big)\big(f(7) + 16\big)\big(f(7)^2 + 9f(7) + 253\big) = 0. \end{cases}$$

Similar as above, if we put

$$c(x) := (x - 7)(x + 13)(x^2 + 6x + 156)$$

and

$$d(x) := (x - 7)(x + 16)(x^2 + 9x + 253),$$

then

$$c\big(f(7)\big) = b\big(f(7)\big) = 0 \quad \text{and} \quad gcd\big(c(x), d(x)\big) = x - 7$$

imply that $f(7) = 7$.

Thus we have prove that $f(7) = 7$ and the proof of Lemma 5 is finished. ■

Lemma 6. *Assume that $f : \mathbb{N} \to \mathbb{C}$ is arbitrary arithmetical function with $f(1) = 1$ satisfying (3.2). Then*

$$f(n) = n \quad \textit{for every} \quad n \in \{\,8,\ 15,\ 16,\ 22,\ 23,\ 28,\ 34,\ 35,\ 37\,\}.$$

Proof. By using Lemma 4 and Lemma 5, we have $f(n) = n$ for every $n \le 7$, consequently

$$\{\,1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 11,\ 19,\ 20,\ 29,\ 31,\ 41,\ 44,\ 45,\ 59,\ 61,\ 79\,\} \subseteq I, \tag{3.12}$$

where

$$I = \{n \in \mathbb{N} \ \mid \ f(n) = n\}.$$

Since 1, 2, 5, 6, 7 $\in I$, we have

$$\begin{aligned}
f(55) &= f(6)^2 + 3f(6)f(1) + f(1)^2 = 55,\\
f(71) &= f(7)^2 + 3f(7)f(1) + f(1)^2 = 71,\\
f(76) &= f(6)^2 + 3f(6)f(2) + f(2)^2 = 76,\\
f(116) &= f(8)^2 + 3f(8)f(2) + f(2)^2 = f(8)^2 + 6f(8) + 4,\\
f(121) &= f(7)^2 + 3f(7)f(3) + f(3)^2 = 121.\\
f(125) &= 5f(5)^2 = 125.
\end{aligned}$$

By using (3.12) and these relations, we shall prove

$$f(n) = n \quad \text{for every} \quad n = 8,\ 15,\ 16,\ 22,\ 23,\ 28,\ 34,\ 35,\ 37.$$

In the proof, we need to find two elements $[N_1, x_1, y_1, u_1, v_1] \in E$ and $[N_2, x_2, y_2, u_2, v_2] \in E$ for which $E(N_1)$ and $E(N_1)$ can express as a polynomial of $f(e)$ for each $e \in \{8, 15, 16, 22, 23, 28, 34, 35, 37\}$. We find those elements with help of computer and Maple program.

○ **Proof of $f(8) = 8$.**
Since [589, 116, 1, 76, 31] $\in E$ and [4961, 59, 8, 55, 11] $\in E$, we have

$$\begin{cases}
E(589) &= f(116)^2 + 3f(116)f(1) + f(1)^2 - f(76)^2 -\\
&\quad - 3f(76)f(31) - f(31)^2 =\\
&= \big(f(8)^2 + 6f(8) + 4\big)^2 + 3\cdot\big(f(8)^2 + 6f(8) + 4\big) +\\
&\quad + 1 - 76^2 - 3\cdot 76\cdot 31 - 31^2 =\\
&= \big(f(8) + 14\big)\big(f(8) - 8\big)\big(f(8)^2 + 6f(8) + 123\big),\\
E(4961) &= f(59)^2 + 3f(59)f(8) + f(8)^2 - f(55)^2 -\\
&\quad - 3f(55)f(11) - f(11)^2 =\\
&= 59^2 + 3\cdot 59\cdot f(8) + f(8)^2 - 55^2 - 3\cdot 55\cdot 11 - 11^2 =\\
&= \big(f(8) + 185\big)\big(f(8) - 8\big).
\end{cases}$$

It is obvious that $E(589) \neq 0$ if $f(8) = -185$, consequently $f(8) = 8$.

∘ **Proof of $f(15) = 15$.**
Since [589, 20, 3, 15, 7] $\in E$ and [2581, 45, 4, 31, 15] $\in E$, we have

$$\begin{cases} E(589) & = & f(20)^2 + 3f(20)f(3) + f(3)^2 - f(15)^2 - \\ & & \quad - 3f(15)f(7) - f(7)^2 = \\ & = & 20^2 + 3 \cdot 20 \cdot 3 + 3^2 - f(15)^2 - 3 \cdot 7 \cdot f(15) - 7)^2 = \\ & = & -(f(15) + 36)(f(15) - 15), \\ E(2581) & = & f(45)^2 + 3f(45)f(4) + f(4)^2 - f(31)^2 - \\ & & \quad - 3f(31)f(15) - f(15)^2 = \\ & = & 45^2 + 3 \cdot 45 \cdot 4 + 4^2 - 31^2 - 3 \cdot 31 f(15) - f(15)^2 = \\ & = & -\big(f(15) + 108\big)\big(f(15) - 15\big). \end{cases}$$

This implies that $f(15) = 15$.

∘ **Proof of $f(16) = 16$.**
Since [3905, 61, 1, 41, 16] $\in E$ and [9680, 76, 16, 44, 44] $\in E$, we have

$$\begin{cases} E(3905) & = & f(61)^2 + 3f(61)f(1) + f(1)^2 - f(41)^2 - \\ & & \quad - 3f(41)f(16) - f(16)^2 = \\ & = & 61^2 + 3 \cdot 61 + 1^2 - 41^2 - 3 \cdot 41 f(16) - f(16)^2 = \\ & = & -\big(f(16) + 139\big)\big(f(16) - 16\big), \\ E(9680) & = & f(76)^2 + 3f(76)f(16) + f(16)^2 - f(44)^2 - \\ & & \quad - 3f(44)f(44) - f(44)^2) = \\ & = & 76^2 + 3 \cdot 76 f(16) + f(16)^2 - 5 \cdot 44^2 = \\ & = & \big(f(16) + 244\big)\big(f(16) - 16\big). \end{cases}$$

This implies that $f(16) = 16$.

∘ **Proof of $f(22) = 22$.**
Since [2204, 44, 4, 22, 20] $\in E$ and [23111, 125, 19, 121, 22] $\in E$, we have

$$\begin{cases} E(2204) & = & f(44)^2 + 3f(44)f(2) + f(2)^2 - f(22)^2 - \\ & & \quad - 3f(22)f(20) - f(20)^2 = \\ & = & 44^2 + 3 \cdot 44 \cdot 2 + 2^2 - 22^2 - 3 \cdot 22 \cdot 20 - 20^2 = \\ & = & -\big(f(22) + 82\big)\big(f(22) - 22\big), \\ E(23111) & = & f(125)^2 + 3f(125)f(19) + f(19)^2 - f(121)^2 - \\ & & \quad - 3f(121)f(22) - f(22)^2 = \\ & = & 125^2 + 3 \cdot 125 \cdot 19 + 19^2 - 121^2 - 3 \cdot 121 \cdot f(22) - f(22)^2 = \\ & = & -\big(f(22) + 385\big)\big(f(22) - 22\big). \end{cases}$$

This implies that $f(22) = 22$.

∘ **Proof of $f(23) = 23$.**
Since [671, 23, 2, 19, 5] $\in E$ and [10469, 76, 19, 71, 23] $\in E$, we have

$$\begin{cases} E(671) &= f(23)^2 + 3f(23)f(2) + f(2)^2 - f(19)^2 - \\ & \quad - 3f(19)f(5) - f(5)^2 = \\ &= f(23)^2 + 3\cdot 2f(23) + 4 - 19^2 - 3\cdot 19\cdot 5 - 5^2 = \\ &= \big(f(23)+29\big)\big(f(23)-23\big), \\ E(10469) &= f(76)^2 + 3f(76)f(19) + f(19)^2 - f(71)^2 - \\ & \quad - 3f(71)f(23) - f(23)^2 = \\ &= 76^2 + 3\cdot 76\cdot 19 + 19^2 - f(71)^2 - 3\cdot 71f(23) - f(23)^2 = \\ &= -\big(f(23)+236\big)\big(f(23)-23\big). \end{cases}$$

This implies that $f(23) = 23$.

∘ **Proof of $f(28) = 28$.**
Since [869, 28, 1, 20, 7] $\in E$ and [5909, 71, 4, 41, 28] $\in E$, we have

$$\begin{cases} E(869) &= f(28)^2 + 3f(28)f(1) + f(1)^2 - f(20)^2 - \\ & \quad - 3f(20)f(7) - f(7)^2) = \\ &= f(28)^2 + 3f(28) + 1 - 20^2 - 3\cdot 7\cdot 20 - 7^2) = \\ &= \big(f(28)+31\big)\big(f(28)-28\big), \\ E(5909) &= f(71)^2 + 3f(71)f(4) + f(4)^2 - f(41)^2 - \\ & \quad - 3f(41)f(28) - f(28)^2) = \\ &= 71^2 + 3\cdot 71\cdot 4 + 4^2 - 41^2 - 3\cdot 41f(28) - f(28)^2) = \\ &= -\big(f(28)+151\big)\big(f(28)-28\big). \end{cases}$$

This implies that $f(28) = 28$.

∘ **Proof of $f(34) = 34$.**
Since [1919, 34, 7, 29, 11] $\in E$ and [15455, 79, 34, 71, 41] $\in E$, we have

$$\begin{cases} E(1919) &= f(34)^2 + 3f(34)f(7) + f(7)^2 - f(29)^2 - \\ & \quad - 3f(29)f(11) - f(11)^2 = \\ &= f(34)^2 + 3\cdot 7f(34) + 7^2 - 29^2 - 3\cdot 29\cdot 11 - 11^2 = \\ &= \big(f(34)+55\big)\big(f(34)-34\big), \\ E(15455) &= f(79)^2 + 3f(79)f(34) + f(34)^2 - f(71)^2 - \\ & \quad - 3f(71)f(41) - f(41)^2 = \\ &= 79^2 + 3\cdot 79\cdot f(34) + f(34)^2 - 71^2 - 371\cdot 41 - 41^2 = \\ &= \big(f(34)+271\big)\big(f(34)-34\big). \end{cases}$$

This implies that $f(34) = 34$.

$\circ$ **Proof of** $f(35) = 35$**.**
Since [1661, 35, 4, 31, 7] $\in E$ and [7781, 61, 20, 44, 35] $\in E$, we have

$$\begin{cases} E(1661) & = & f(35)^2 + 3f(35)f(4) + f(4)^2 - f(31)^2 - \\ & & \quad - 3f(31)f(7) - f(7)^2 = \\ & = & f(35)^2 + 3 \cdot 4 \cdot f(35) + 4^2 - 31^2 - 3 \cdot 31 \cdot 7 - 7^2 = \\ & = & \big(f(35) + 47\big)\big(f(35) - 35\big), \\ E(7781) & = & f(61)^2 + 3f(61)f(20) + f(20)^2 - f(44)^2 - \\ & & \quad - 3f(44)f(35) - f(35)^2 = \\ & = & 61^2 + 361 \cdot 20 + 20^2 - 44^2 - 3 \cdot 44 \cdot f(35) - f(35)^2 = \\ & = & -\big(f(35) + 167\big)\big(f(35) - 35\big). \end{cases}$$

This implies that $f(35) = 35$.

$\circ$ **Proof of** $f(37) = 37$**.**
Since [3839, 59, 2, 37, 19] $\in E$ and [16379, 125, 2, 79, 37] $\in E$, we have

$$\begin{cases} E(3839) & = & f(59)^2 + 3f(59)f(2) + f(2)^2 - f(37)^2 - \\ & & \quad - 3f(37)f(19) - f(19)^2 = \\ & = & 59^2 + 3 \cdot 59 \cdot 2 + 2^2 - f(37)^2 - 3 \cdot 19 \cdot f(37) - 19^2 = \\ & = & -\big(f(37) + 94\big)\big(f(37) - 37\big), \\ E(16379) & = & f(125)^2 + 3f(125)f(2) + f(2)^2 - f(79)^2 - \\ & & \quad - 3f(79)f(37) - f(37)^2 = \\ & = & 125^2 + 3 \cdot 125 \cdot 2 + 2)^2 - 79^2 - 3 \cdot 79 \cdot f(37) - f(37)^2 = \\ & = & -\big(f(37) + 274\big)\big(f(37) - 37\big). \end{cases}$$

This implies that $f(37) = 37$.

Lemma 6 is proved. ■

Lemma 7. *Assume that* $f : \mathbb{N} \to \mathbb{C}$ *is arbitrary arithmetical function with* $f(1) = 1$ *satisfying* (3.2). *Then*

$$f(n) = n \quad \textit{for every} \quad n \le 44.$$

Proof. By using Lemma 4 and Lemma 6 we assume that

$$\{1, 2, 3, 4, 5, 6, 7, 8, 11, 15, 16, 19, 20, 22, 23, 28, 29, 31, 34, 35, 37, 41, 44\} \subseteq I.$$

Similar as in the proof of Lemma 6, we shall find those elements of the form $[N, x, y, u, v] \in E$, that is

$$N = x^2 + 3xy + y^2 = u^2 + 3uv + v^2$$

and

$$E(N) = f(x)^2 + 3f(x)f(y) + f(y)^2 - f(u)^2 - 3f(u)f(v) - f(v)^2 \in \mathbb{Z}[n]$$

is a polynomial of $f(n)$ with a suitable number $n \in \mathbb{N}$. We give a solutions in the following table.

Table 1

$[N, x, y, u, v]$	$E(N) \in \mathbb{Z}[n], E(N) = 0$	$f(n) = n$
$[319, 15, 2, 9, 7]$ $[34655, 139, 34, 109, 59]$	$-(f(9) - 9)(f(9) + 30)$ $3(f(9) - 9)(2f(9)^2 + 4f(9) + 77)$	$f(9) = 9$
$[551, 22, 1, 11, 10]$ $[21605, 131, 11, 89, 44]$	$-(f(10) - 10)(f(10) + 43)$ $(f(10) - 10)(f(10) + 13)(f(10)^2 + 3f(10) + 165)$	$f(10) = 10$
$[209, 13, 1, 8, 5]$ $[979, 23, 6, 15, 13]$	$(f(13) + 16)(f(13) - 13)$ $-(f(13) + 58)(f(13) - 13)$	$f(13) = 13$
$[1691, 34, 5, 23, 14]$ $[6479, 61, 14, 37, 35]$	$-(f(14) + 83)(f(14) - 14)$ $(f(14) + 197)(f(14) - 14)$	$f(14) = 14$
$[2501, 35, 11, 28, 17]$ $[4061, 41, 17, 29, 28]$	$-(f(17) + 101)(f(17) - 17)$ $(f(17) + 140)(f(17) - 17)$	$f(17) = 17$
$[1804, 34, 6, 20, 18]$ $[1711, 37, 3, 19, 18]$	$-(f(18) + 78)(f(18) - 18)$ $-(f(18) + 75)(f(18) - 18)$	$f(18) = 18$
$[4141, 44, 15, 37, 21]$ $[11005, 99, 4, 76, 21]$	$-(f(21) + 132)(f(21) - 21)$ $-(f(21) + 249)(f(21) - 21)$	$f(21) = 21$
$[1111, 29, 3, 25, 6]$ $[3509, 44, 11, 28, 25]$	$-(f(25) + 43)(f(25) - 25)$ $-(f(25) + 109)(f(25) - 25)$	$f(25) = 25$
$[2071, 37, 6, 26, 15]$ $[1364, 34, 2, 26, 8]$	$-(f(26) + 71)(f(26) - 26)$ $-(f(26) + 50)(f(26) - 26)$	$f(26) = 26$
$[1529, 32, 5, 19, 16]$ $[2201, 32, 11, 23, 19$	$(f(32) + 47)f(32) - 32)$ $(f(32) + 65)(f(32) - 32)$	$f(32) = 32$
$[3751, 41, 15, 33, 22]$ $[2299, 45, 2, 33, 11]$	$-(f(33) + 99)(f(33) - 33)$ $-(f(33) + 66)(f(33) - 33)$	$f(33) = 33$
$[2489, 40, 7, 29, 16]$ $[5921, 55, 16, 40, 29]$	$(f(40) + 61)(f(40) - 40)$ $-(f(40) + 127)(f(40) - 40)$	$f(40) = 40$
$[4661, 61, 5, 43, 19]$ $[6809, 71, 8, 43, 31]$	$-(f(43) + 100)(f(43) - 43)$ $-(f(43) + 136)(f(43) - 43)$	$f(43) = 43$

Thus we have proved that

$$\begin{aligned}&\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, \\ &21, 22, 23, 25, 26, 28, 29, 31, 32, 33, 34, 35, 37, 40, 41, 43, 44\} \subseteq I\end{aligned} \tag{3.13}$$

and we complete the proof of Lemma 7, it suffices to show

$$\{12, 24, 27, 30, 36, 38, 39, 42\} \subseteq I. \tag{3.14}$$

By using (3.13), we give the solutions of (3.14) in the following Table 2.

Table 2

$[N,x,y,u,v]$	$E(N)\in\mathbb{Z}[n], E(N)=0$	$f(n)=n$
$[781,21,5,13,12]$ $[1045,28,3,17,12$	$-(f(12)+51)(f(12)-12)$ $-(f(12)+63)(f(12)-12)$	$f(12)=12$
$[649,24,1,15,8]$ $[2761,40,9,24,23]$	$(f(24)+27)(f(24)-24)$ $-(f(24)+93)(f(24)-24)$	$f(24)=24$
$[1159,27,5,22,9]$ $[1441,27,8,19,15]$	$(f(27)+42)(f(27)-27)$ $(f(27)+51)(f(27)-27)$	$f(27)=27$
$[1276,30,4,18,14]$ $[2356,40,6,30,14]$	$(f(30)+42)(f(30)-30)$ $-(f(30)+72)(f(30)-30)$	$f(30)=30$
$[2101,36,7,21,20]$ $[2869,41,9,36,13]$	$(f(36)+57)(f(36)-36)$ $-(f(36)+75)(f(36)-36)$	$f(36)=36$

$[N,x,y,u,v]$	$E(N)\in\mathbb{Z}[n], E(N)=0$	$f(n)=n$
$[2291,38,7,26,17]$ $[2420,38,8,22,22]$	$(f(38)+59)(f(38)-38)$ $(f(38)+62)(f(38)-38)$	$f(38)=38$
$[2929,39,11,33,16]$ $[3355,39,14,31,21]$	$(f(39)+72)(f(39)-39)$ $(f(39)+81)(f(39)-39)$	$f(39)=39$
$[1891,42,1,35,6]$ $[2419,42,5,23,21$	$(f(42)+45)(f(42)-42)$ $(f(42)+57)(f(42)-42)$	$f(42)=42$

Thus, (3.13) and Table 2 prove Lemma 7.

Lemma 7 is proved. ■

4. Proof of Theorem 1 for $D = 3$

Assume that $f:\mathbb{N}\to\mathbb{C}$ with $f(1)=1$ and

$$f(n^2+3nm+m^2)=f^2(n)+3f(n)f(m)+f^2(m) \quad \text{for every} \quad n,m\in\mathbb{N}.$$

Applying Lemma 3 with $D=3$, we obtain that $f(n)=n$ for every $n\in\mathbb{N}$ if

$$f(k)=k \quad \text{for every} \quad k\le 8D+20=8.3+20=44.$$

This assertion is proved in Lemma 7. Theorem 1 is proved. ■

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