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ON A CONJECTURE REGARDING MULTIPLICATIVE FUNCTIONS

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Abstract. Let g be a completely multiplicative function, $g(n) \in \mathbb{T}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathbb{A} = \{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{T}$. Assume that

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x \mid |g(n+1)\overline{g}(n) - \alpha_j| < \epsilon \right\} = \rho(\alpha_j) > 0$$

for each $\epsilon > 0$ which is small enough. Assume furthermore that if $\delta \in \mathbb{T} \setminus \mathbb{A}$, then

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x \mid |g(n+1)\overline{g}(n) - \delta| < \epsilon \right\} = 0$$

if ϵ is small enough. We formulate our conjecture on the possible $\mathbb{A}$ and g. We can prove our conjecture for $k = 1$, 2 and for $k = 3$ with exception if $\mathbb{A} = \{1, \beta, \overline{\beta}\}$.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ be the set of positive integers, real and complex numbers, respectively. Let $\mathcal{A}$, $\mathcal{A}^*$, $\mathcal{M}$, $\mathcal{M}^*$ be the set of real-valued additive (completely additive) and the set of complex-valued multiplicative (completely multiplicative) functions. We say that $f \in \mathcal{M}_1$ (resp. $\mathcal{M}_1^*$), if $f \in \mathcal{M}$ (resp. $\mathcal{M}^*$) and $|f(n)| = 1$ for every $n \in \mathbb{N}$. Let $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.

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In 1946 P. Erdős proved the following result.

Theorem A. (P. Erdős [2]) *If* $f \in \mathcal{A}$ *and*

$$f(n+1) - f(n) \to 0 \quad (n \to \infty),$$

then $f(n) = c \log n$ *with some* $c \in \mathbb{R}$*.*

Conjecture 1. (P. Erdős [2]) *If* $f \in \mathcal{A}$ *and*

$$\frac{1}{x} \sum_{n \le x} \left| f(n+1) - f(n) \right| \to 0 \quad (x \to \infty),$$

then $f(n) = c \log n \quad (c \in \mathbb{R})$*.*

Conjecture 1 was proved by I. Kátai [4] and E. Wirsing [14].

We now present the short history of the results initiated by Erdős, and analogous questions for multiplicative functions.

Conjecture 2. (I. Kátai [5]) *If* $g \in \mathcal{M}$ *and*

$$g(n+1) - g(n) \to 0 \quad (n \to \infty),$$

then either $g(n) \to 0$ $(n \to \infty)$*, or* $g(n) = n^s$*,* $\Re s < 1$*.*

It is proved by E. Wirsing. Another proof is given in [16].

Conjecture 3. (I. Kátai and M. V. Subbarao [7]) *Let*

$$\mathbb{A} = \{\alpha_1, \ldots, \alpha_k\} \subseteq \mathbb{T}.$$

If $g \in \mathcal{M}_1$ *and the whole set of limit points of* $g(n+1)\overline{g}(n)$ $(n \in \mathbb{N})$ *is* $\mathbb{A}$*, then*

$$\mathbb{A}^k = \{1\}, \quad i.e \quad \alpha_1^k = \cdots = \alpha_k^k = 1.$$

Kátai and Subbarao proved this conjecture for $k = 1, 2, 3$. E. Wirsing[15] and B. M. Phong [13] proved a somewhat weaker result, namely that if the conditions hold, then there exists such an $\ell \in \mathbb{N}$ for which $\mathbb{A}^\ell = \{1\}$.

Conjecture 4. (I. Kátai [5]) *If* $g \in \mathcal{M}_1$ *and*

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} |g(n+1) - g(n)| = 0, \tag{1.1}$$

then

$$g(n) = n^{i\tau} \quad (n \in \mathbb{N}), \quad \tau \in \mathbb{R}. \tag{1.2}$$

Theorem B. (O. Klurman [8]) *Conjecture* 4 *is true.*

He proved more, namely that if $g \in \mathcal{M}_1$ and

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g(n+1)-g(n)|}{n} = 0, \tag{1.3}$$

then (1.2) is true.

Remark. O. Klurman proved it for $g \in \mathcal{M}_1^*$, but using the method of Mauclaire and Murata [3] one can prove that if (1.3) holds for $g \in \mathcal{M}_1$, then $g \in \mathcal{M}_1^*$.

Further results on these topics can be found in [6], [9], [10], [11], [12].

2. Our conjecture

We state the following conjecture.

Conjecture 4. (I. Kátai and B. M. Phong) *Let*

$$\mathbb{A} = \{\alpha_1, \ldots, \alpha_k\} \subseteq \mathbb{T}.$$

Let $g \in \mathcal{M}_1^*$ *and* $C(n) = g(n+1)\overline{g}(n)$ $(n \in \mathbb{N})$. *Assume that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \min_{\beta\in\mathbb{A}} \left|C(n)-\beta\right| = 0 \tag{2.1}$$

and that for every $\gamma \in \mathbb{A}$,

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x \;\middle|\; \left|C(n)-\gamma\right| = \min_{\beta\in\mathbb{A}} |C(n)-\beta| \right\} = \rho(\gamma) \tag{2.2}$$

exists, and $\rho(\gamma) > 0$.

Then there exists such an $\ell \in \mathbb{N}$ *for which* $\mathbb{A}^\ell = \{1\}$. *Consequently*

$$g^\ell(n) = n^{i\tau}, \quad g(n) = n^{i\tau/\ell} G(n), \quad G \in \mathcal{M}_1^*, \quad G^\ell(n) = 1.$$

We shall prove the following result:

Theorem 1. *Conjecture* 4 *is true if*

a) $k = 1,\ 2,$

b) $k = 3$, *possibly except the case when* $\mathbb{A} = \{\alpha_1, \alpha_2, \alpha_3\} = \{1, \beta, \overline{\beta}\}$.

Proof. Let $B(n)$ be the closest element of $\mathbb{A}$ to $C(n)$. $B(n)$ is well determined for almost all $n \in \mathbb{N}$ and furthermore

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |B(n) - C(n)| = 0. \tag{2.3}$$

Since

$$g(n+1)\overline{g}(n) = g(2n+2)\overline{g}(2n) = g(2n+2)\overline{g}(2n+1)g(2n+1)\overline{g}(2n)$$

and

$$\begin{aligned} g(n+1)\overline{g}(n) &= g(3n+3)\overline{g}(3n) = \\ &= g(3n+3)\overline{g}(3n+2)g(3n+2)\overline{g}(3n+1)g(3n+1)\overline{g}(3n), \end{aligned}$$

thus

$$C(n) = C(2n)C(2n+1), \quad C(n) = C(3n)C(3n+1)C(3n+2),$$

and so

$$B(n) = B(2n)B(2n+1), \quad B(n) = B(3n)B(3n+1)B(3n+2),$$

and in general,

$$B(n) = B(\ell n)B(\ell n+1)\cdots B(\ell n+\ell-1) \tag{2.4}$$

holds for almost all n, for every fixed ℓ.

Case $k = 1$**.** This case is immediate, since $B(n) = B(2n) = B(2n+1) = \alpha$ implies that $\alpha = \alpha^2$. This shows that $\alpha = 1$ and our result follows from Theorem B.

Case $k = 2$**.** If $1 \notin \mathbb{A}$, $B(2n) \neq B(2n+1)$ cannot occur for almost all n, since in the opposite case (2.4) $\ell = 2$ would imply that $1 \in \mathbb{A}$. Then $\alpha = \beta^2$ and $\beta = \alpha^2$, and so $\alpha\beta = 1$, $\alpha = \alpha^4$, $\alpha^3 = \beta^3 = 1$. Consequently (1.1) holds for g^3 instead of g. Thus $g^3(n) = n^{i\tau}$, $g(n) = n^{i\tau/3}G(n)$, $G \in \mathcal{M}^*$, $G(n)^3 = 1$.

We shall prove that this cannot occur. Since

$$B(n) = G(n+1)\overline{G}(n) \in \{\omega, \overline{\omega}\}, \;\; \omega^3 = 1, \; \omega \neq 1,$$

therefore

$$\frac{1}{x}\#\big\{n \le x \;\mid\; G(n+1) = G(n)\big\} \to 0 \quad (x \to \infty).$$

Let $\mathcal{R} = \{n \in \mathbb{N} \mid G(n+1) = G(n)\}$.

Let us consider the sequence:

$$G(12m-4),\ G(12m-3),\ G(12m-2),\ G(12m-1),\ G(12m).$$

We shall prove that at least one of $12m-\ell \in \mathcal{R}$ ($\ell = 1, 2, 3, 4$). In the opposite case:

$$G(12m) \neq G(12m-2),\ G(12m-4),\ G(12m-3)$$

and

$$G(12m-4),\ G(12m-3),\ G(12m-2),\ G(12m) \quad \text{are distinct values.}$$

It is impossible. Consequently

$$\liminf_{x\to\infty} \frac{1}{x} \#\{n \leq x \mid n \in \mathcal{R}\} \geq \frac{1}{12},$$

which shows that such a case cannot occur.

Assume that $1 \in \mathbb{A}$. Then $\mathbb{A} = \{1, \beta\}$, $\beta \neq 1$. Let $I_k = [2^k, 2^{k+1}-1]$. It is clear that

$$M_\beta(k) = \#\{n \in I_k \mid B(n) = \beta\} = \rho(\beta)2^k + o(2^k),$$

$$M_1(k) = \#\{n \in I_k \mid B(n) = 1\} = \rho(1)2^k + o(2^k).$$

Consider the set of those n for which $B(n) = \beta$. Then

$$\big(B(2n), B(2n+1)\big) \neq (\beta,\beta),\ (1,1) \quad \text{for almost all } n.$$

If $1 = B(n)$. Then $1 = B(2n)B(2n+1)$, and so $(B(2n), B(2n+1)) = (\beta,\beta)$ or $(B(2n), B(2n+1)) = (1,1)$. The first case implies that $\beta^2 = 1$, consequently $\beta = -1$. Assume that $(B(2n), B(2n+1)) = (1,1)$. We obtain that

$$M_\beta(k) = M_\beta(k-1) + o(2^k). \tag{2.5}$$

Since $M_\beta(k) = \rho(\beta)2^k + o(2^k)$, thus $\rho(\beta)2^k = \rho(\beta)2^{k-1} + o(2^k)$. This is impossible if $\rho(\beta) \neq 0$. Thus $\beta = -1$ and we are done.

Case $k = 3$. Assume that $\mathbb{A} = \{\alpha,\beta,\gamma\}$. First we consider the case $\alpha = 1$. If $\beta = \gamma^2$ and $\gamma = \beta^2$, then $\beta = \beta^4$, $\beta^3 = 1$, $\gamma^3 = 1$ and we are done. Assume that $\beta \neq \gamma^2$. If $B(n) = \beta$, then for almost all such n

$$\big(B(2n),\ B(2n+1)\big) \in \big\{(1,\beta),\ (\beta,1)\big\}.$$

Since $\beta \neq \overline{\gamma}$, we obtain that

$$M_1(k) \geq 2M_1(k-1) + M_\beta(k-1),$$

which contradicts the assumption that $\rho(\beta) \neq 0$. We also obtain a contradiction if $\gamma \neq \beta^2$. So we have

$$\mathbb{A} = \{\omega \mid \omega^3 = 1\}.$$

Now we consider the case $1 \notin \mathbb{A}$.

a) Let $\alpha = \beta^2$, $\beta = \gamma^2$, $\gamma = \alpha^2$. Then $\mathbb{A}^7 = \{1\}$, and we are done.

b) Let $\alpha = \beta^2$, $\beta = \gamma^2$, $\gamma = \alpha\beta$. Then $\alpha^5 = \beta^5 = \gamma^5 = 1$, consequently $\mathbb{A}^5 = \{1\}$, and we are done.

c) Let $\alpha = \beta^2$, $\beta = \alpha\gamma$, $\gamma = \alpha\beta$. In this case, we have

$$\beta\gamma = \alpha\gamma\alpha\beta, \quad \alpha = -1, \quad \beta = -\gamma \quad \text{and} \quad \beta^4 = \gamma^4 = 1.$$

Therefore, $\mathbb{A}^4 = \{1\}$.

d) Let $\alpha = \beta\gamma$, $\beta = \alpha\gamma$, $\gamma = \alpha\beta$. Then

$$\alpha\beta\gamma = 1,\ \alpha = \beta\alpha\gamma^2,\ \gamma = -1,\ \alpha = -\beta,\ \alpha\beta = -1 \text{ and } \alpha^2 = \beta^2 = 1,$$

which cannot occur.

All the possible cases are covered above by a permutation of $\mathbb{A}$. ■

3. Further remarks

Let $0 \leq u_1 < u_2 < \cdots < u_k < 1$; $p_1, \ldots, p_k > 0$ with $\sum_{i=1}^{k} p_i = 1$.

Let $\mathcal{H}$ be the class of the distribution functions of the random variables ξ satisfying the conditions $P(\xi = u_i) = p_i$.

Conjecture 5. (I. Kátai and B. M. Phong) *Let $h(n)$ be an additive function, $\delta(n) = h(n+1) - h(n) \pmod 1$. Assume that it has a limit distribution, and that its distribution $F \in \mathcal{H}$,*

$$F(y) = P(\xi \leq y), \quad P(\xi = u_i) = p_i \quad (i = 1, \ldots, k).$$

Then

$$h(n) \equiv \tau \log n + E(n)$$

and

$$\frac{1}{x}\#\big\{n \leq x \;\big|\; E(n+1) - E(n) \pmod 1 = u_i\big\} = p_i \quad (i = 1, \ldots, k),$$

furthermore $\ell E(n) \equiv 0 \pmod 1$ for a suitable $\ell \in \mathbb{N}$.

Conjecture 5 is a variant of Conjecture 4.

Conjecture 6. (I. Kátai and B. M. Phong) *Let* $h(n)$, $\delta(n)$ *be as above. Let* $I=[\alpha,\beta)\subseteq[0,1)$ *and assume that*

$$\lim_{x\to\infty}\frac{1}{x}\#\big\{n\le x \;\big|\; \delta(n)\in I\big\}=0.$$

Then

$$h(n)\equiv\tau\log n+E(n)\pmod 1$$

and $\ell E(n)\equiv 0 \pmod 1$ *for a suitable* $\ell\in\mathbb{N}$.

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