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LIMIT THEOREMS FOR CONTAMINATED RUNS OF HEADS

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Abstract. The classical coin tossing experiment is studied. Limit theorems are obtained concerning the head-runs containing certain number of tails. It is proved that the limit of the number of those runs of length n which contain at most T tails is compound Poisson. Accompanying distributions are obtained for the length of the longest head-runs containing at most T tails. To this end a two parameter family of accompanying distributions is offered.

1. Introduction

In this paper, we consider the length of consecutive heads interrupted by several tails in the usual coin tossing experiment. So let $p \in (0,1)$ be the probability of heads and $q = 1 - p$ the probability of tails. During the paper p is fixed. We toss the coin N times independently. We write 1, when the result is head and 0, when the result is tail. So the experiment can be described using independent identically distributed random variables $X_1, X_2, \ldots, X_N$ with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = q$, $i = 1, 2, \ldots, N$. Let $T \geq 0$ be a fixed integer. We shall study the precisely and the at most T-contaminated (in other words T-interrupted) runs of heads having length n.

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Before going into the details, we list some results of the huge literature of the topic. One of the most famous classical results is the following theorem of Erdős and Rényi [3]. Consider the case of a fair coin, that is let $p = 1/2$. Let $\mu(N)$ denote the length of the longest pure head run during N tosses. Let $0 < c_1 < 1 < c_2 < \infty$. Then for almost all elementary event ω there exists a finite $N_0 = N_0(\omega, c_1, c_2)$ such that

$$[c_1 \log N] \leq \mu(N) \leq [c_2 \log N]$$

if $N \geq N_0$. Here log denotes logarithm base 2 and $[\,.\,]$ is the integer part. More precise result was obtained by Erdős and Révész in [4]. Furthermore, in [4], almost sure limit results were proved for the length of the longest runs containing at most T tails when $p = 1/2$.

Földes, in [7], studied also the case of the fair coin. She proved asymptotic results for the distributions of the number of pure head runs, the first hitting time of a run having a fixed length, and the length of the longest pure head-run. Then she announced the extensions of the above mentioned results for T contaminated runs of heads. Földes presented the proofs for her results on T contaminated runs of heads in [6]. The aim of our paper is to extend the results of Földes [7] to the case of possibly biased coins and to give detailed proofs. In our proofs we use the ideas given in [6] and [7].

Gordon, Schilling, and Waterman in [8] used extreme value theory to find the asymptotic behaviour of the expectation and the variance of the length of the longest precisely T-contaminated head run. Also in [8], accompanying distributions were obtained for the length of the longest precisely T-contaminated head run. We shall show, that the accompanying distributions of [8] are simple consequences of our approach.

Móri in [11] also studied the longest precisely T-contaminated head run. He presented the limiting distribution for the first hitting time (without proof) and a so called almost sure limit theorem (see Corollary 5.1 in [11]). We emphasize that, contrary to [11] and [8], our main results concern the case of at most T-contaminated head runs and not the precisely T-contaminated head runs.

There are several extensions and applications of the results on the longest head-run, see e.g. [1] and [2]. The Markovian case is also studied. For example, in [12], the accuracy of the approximation to the distribution of the length of the longest head run in a Markov chain is considered. See also the references in [12].

In [9], [5] and [13] recursive formulae were used to describe the properties of the longest head run. It is known that the properties of longest runs are used to test random number generators, see e.g. [10].

2. Notation and limit theorems for the number of runs

Consider the classical coin tossing experiment. Let $p \in (0,1)$ be the probability of heads and $q = 1-p$ the probability of tails. Here p is fixed. We toss a coin N times independently. We shall write 1, when the result is head and 0, when the result is tail. Therefore consider independent identically distributed random variables $X_1, X_2, \dots, X_N$ with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = q$, $i = 1, 2, \dots, N$. Let $T \geq 0$ be a fixed integer.

Let $\widetilde{\xi} = \widetilde{\xi}^T(n, N)$ denote the number of those precisely T-contaminated n-length runs of heads for which the preceding element is a tail. More precisely let

$$\widetilde{\eta}_i = \widetilde{\eta}_i^T(n) = \begin{cases} 1, & \text{if there are precisely } T \ \ 0 \text{ values among} \\ & \qquad X_i, \dots, X_{i+n-1} \text{ and } X_{i-1} = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Here X_0 is defined as $X_0 \equiv 0$. Now let

$$\widetilde{\xi} = \widetilde{\xi}^T(n, N) = \sum_{i=1}^{N-n+1} \widetilde{\eta}_i^T(n). \tag{2.2}$$

So $\widetilde{\xi}$ can be considered as the number of those precisely T-contaminated head-runs having length n for which the preceding value is tail.

Our main condition is the following. Let $p \in (0,1)$ be fixed. Let T be a fixed non-negative integer. Let $N \to \infty$ and $n \to \infty$ so that

$$\frac{N q^{T+1} p^{n-T} n^T}{T!} \to \lambda > 0, \tag{2.3}$$

where λ is fixed. We remark that condition (2.3) implies that $N/n \to \infty$.

Now we show that the distribution of $\widetilde{\xi}$ converges to the λ parameter Poisson distribution.

Theorem 2.1. *Let T be fixed. Let $N \to \infty$ and $n \to \infty$ so that condition (2.3) is satisfied. Then*

$$\lim_{N\to\infty} \mathbb{P}\left(\widetilde{\xi}^T(n, N) = k\right) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \dots.$$

The above theorem is proved in [6] for $p = 1/2$ (see Theorem 3.1 in [6], see also Theorem 1.A in [7]).

To obtain the proofs of our theorems we need the following known result.

Proposition 2.2. (See [14].) *Let* $Y_i^{(m)}$, $i=1,2,\dots,l_m$, $m=1,2,\dots,$ *be a triangular array of Bernoulli random variables, i.e. the values of* $Y_i^{(m)}$ *are* 0 *or* 1*. Let*

$$Z_m = Y_1^{(m)} + Y_2^{(m)} + \dots + Y_{l_m}^{(m)}, \quad m = 1,2,\dots$$

be the row sums. Let

$$b_{i_1,\dots,i_r}^{(m)} = \mathbb{P}(Y_{i_1}^{(m)} = Y_{i_2}^{(m)} = \dots = Y_{i_r}^{(m)} = 1),$$

where $(i_1,\dots,i_r)$ *denotes an* r *dimensional vector such that the integers* $i_1,\dots,i_r$ *are pairwise different with* $1 \le i_t \le l_m$, $t=1,\dots,r$, $r=1,2,\dots$. *Assume that for each* $r=2,3,\dots,$ $m=1,2,\dots$ *there exists an exceptional set* $I_r(m)$ *consisting of certain vectors* $\alpha_r = (i_1,\dots,i_r)$ *such that the numbers* $i_1,\dots,i_r$ *are pairwise different with* $1 \le i_t \le l_m$, $t=1,\dots,r$. *Assume that*

$$\lim_{m\to\infty} \max_{1\le i\le l_m} b_i^{(m)} = 0, \tag{2.4}$$

$$\lim_{m\to\infty} \sum_{i=1}^{l_m} b_i^{(m)} = \lambda > 0, \tag{2.5}$$

$$\lim_{m\to\infty} \sum_{\alpha_r\in I_r(m)} b_{i_1,\dots,i_r}^{(m)} = 0, \tag{2.6}$$

$$\lim_{m\to\infty} \sum_{\alpha_r\in I_r(m)} b_{i_1}^{(m)} \cdots b_{i_r}^{(m)} = 0, \tag{2.7}$$

and uniformly for all $\alpha_r \notin I_r(m)$

$$\lim_{m\to\infty} \frac{b_{i_1,\dots,i_r}^{(m)}}{b_{i_1}^{(m)} \cdots b_{i_r}^{(m)}} = 1. \tag{2.8}$$

Then

$$\lim_{m\to\infty} \mathbb{P}(Z_m = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k=0,1,2,\dots. \tag{2.9}$$

Proof of Theorem 2.1. We apply Proposition 2.2 for $l_m = N-n+1$ and $Y_i = \tilde{\eta}_i$, $i=1,2,\dots,l_m$. So we check the conditions of Proposition 2.2. Condition (2.4) is satisfied because

$$\begin{aligned} \max_{1\le i\le N-n+1} b_i &= \max_{1\le i\le N-n+1} \mathbb{P}(\widetilde{\eta}_i = 1) = \\ &= \max\{1,q\}\binom{n}{T} q^T p^{n-T} \le c n^T p^n \to 0 \end{aligned} \tag{2.10}$$

as $N, n \to \infty$, because $0 < p < 1$. Condition (2.5) is satisfied because, by condition (2.3),

$$
\begin{aligned}
\sum_{i=1}^{N-n+1} b_i &= \sum_{i=1}^{N-n+1} \mathbb{P}(\widetilde{\eta}_i = 1) = \\
&= (1 + (N-n)q)\binom{n}{T} q^T p^{n-T} \approx N \frac{n^T}{T!} q^{T+1} p^{n-T} \to \lambda \qquad (2.11)
\end{aligned}
$$

as $n, N \to \infty$. Here we applied that, by (2.3), $n/N \to 0$.

Now let $\alpha_r = (i_1, \ldots, i_r)$ denote an r dimensional vector such that the numbers $i_1, \ldots, i_r$ are pairwise different with $1 \le i_t \le N - n + 1$, $t = 1, \ldots, r$. Define the set $I_r(n, N)$ of exceptional indices as the set of those indices $\alpha_r = = (i_1, \ldots, i_r)$ such that there are $i, j \in \{1, \ldots, r\}$, $j \ne l$ with $|i_j - i_l| < n + 1$.

The random vectors $X_{i-1}, X_i, \ldots, X_{i+n-1}$ and $X_{j-1}, X_j, \ldots, X_{j+n-1}$ are independent if $n < j - i$. Therefore $\widetilde{\eta}_{i_1}, \ldots, \widetilde{\eta}_{i_r}$ are independent if $\alpha_r = = (i_1, \ldots, i_r) \notin I_r(n, N)$. So (2.8) is satisfied.

Now turn to (2.7). By the definition of $I_r(n, N)$, we should choose r elements out of N elements so that there should be a pair among them with distance being not greater than n. Therefore

$$
\sum_{\alpha_r \in I_r(n,N)} b_{i_1} \cdots b_{i_r} = \sum_{\alpha_r \in I_r(n,N)} \mathbb{P}(\widetilde{\eta}_{i_1} = 1) \cdots \mathbb{P}(\widetilde{\eta}_{i_r} = 1) \le \qquad (2.12)
$$

$$
\le \binom{N}{r-1}(r-1)2n \left[\binom{n}{T} q^T p^{n-T}\right]^r \le c\frac{n}{N}\left(N n^T p^n\right)^r \to 0
$$

as $n, N \to \infty$, because of condition (2.3).

Now consider condition (2.6). For $r = 1$ conditions (2.6) and (2.7) are equivalent. For $r \ge 2$ and $T = 0$ we have

$$
\sum_{\alpha_r \in I_r(n,N)} b_{i_1,\ldots,i_r} = \sum_{\alpha_r \in I_r(n,N)} \mathbb{P}(\widetilde{\eta}_{i_1} = 1, \ldots, \widetilde{\eta}_{i_r} = 1) = 0, \qquad (2.13)
$$

because in the definition of $\widetilde{\eta}_i$ we claim $X_{i-1} = 0$.

Now we shall prove condition (2.6) for $T \ne 0$. For any $\alpha_r = (i_1, \ldots, i_r)$ the indices of X_i variables involved belong to the intervals

$$
[i_1 - 1, i_1 + n - 1],\ [i_2 - 1, i_2 + n - 1], \ldots, [i_r - 1, i_r + n - 1]. \qquad (2.14)
$$

If $\alpha_r \in I_r(n, N)$, then at least two of the above intervals have a common point. So we can divide the family of intervals (2.14) into disjoint components so that inside each component the intervals are connected. The random variables having indices in disjoint components are independent. Therefore the term $\sum_{\alpha_r \in I_r(n,N)} \mathbb{P}(\widetilde{\eta}_{i_1} = 1, \dots, \widetilde{\eta}_{i_r} = 1)$ is a sum of the products of terms corresponding to connected components. Using this fact and (2.11), we can see that it is enough to prove (2.6) for a connected set of intervals (2.14). So let $I_r^* = I_r^*(n,N)$ be the set of indices $\alpha \in I_r(n,N)$ with a connected family (2.14) of intervals. Denote by s the overall length of these intervals. Then $n + r \le s \le rn + 1$. Divide I_r^* into two parts; $\alpha_r \in I_r^*(1) = I_r^*(1,n,N)$ if and only if $s \le 2n+1$ while $\alpha_r \in I_r^*(2) = I_r^*(2,n,N)$ if and only if $s > 2n+1$. So in the case of $I_r^*(1)$ there is a common point of the intervals (2.14) but in the case of $I_r^*(2)$ the first and the last intervals are disjoint. In the case of $I_r^*(2)$ a rough upper bound will do.

$$\begin{aligned}
&\sum_{\alpha_r \in I_r^*(2)} \mathbb{P}(\widetilde{\eta}_{i_1} = 1, \dots, \widetilde{\eta}_{i_r} = 1) \le N \sum_{s=2n+2}^{rn+1} \sum_{j=2T}^{rT} \binom{s}{j} q^j p^{s-j} \le \\
&\le cNp^{2n-rT}(rn+1)^{rT}(rn+1)(rT) \le cNp^{2n}n^{rT+1} = \\
&= c(Np^n n^T)(p^n n^{(r-1)T+1}) \to 0
\end{aligned}$$

as $n, N \to \infty$, because of condition (2.3). Now consider the case of $I_r^*(1)$ that is when $s \le 2n+1$. Then the intersection of all the intervals in (2.14) is not empty. The case $r > T+1$ is impossible as then at least $r-1$ tails $(r-1 > T)$ would be in the first interval. So let $r \le T+1$. Concerning the location of the intervals (2.14) let $l_j = i_{j+1} - i_j$, $j = 1, \dots, r-1$. If $l_1, \dots, l_{r-1}$ are fixed, then the locations of $r-1$ tails are given. So in the first interval we can choose the locations of $T - r + 1$ tails. The starting point of the first interval can be chosen less than N different ways. Moreover, the probability that at most T tails occur from l tosses is not greater than $q_0^l (T+1) l^T$, where $q_0 = \max\{q, p\} < 1$. Therefore

$$\sum_{\alpha_r \in I_r^*(1)} \mathbb{P}(\widetilde{\eta}_{i_1} = 1, \dots, \widetilde{\eta}_{i_r} = 1) \le$$

$$\le N \sum_{1 \le l_1, \dots, l_{r-1} \le n} \left[\binom{n}{T-r+1} q^T p^{n-T} \right] q_0^{l_1} (T+1) l_1^T \dots q_0^{l_{r-1}} (T+1) l_{r-1}^T \le$$

$$\le cNn^{T-r+1} q^T p^{n-T} \left(\sum_{l=1}^{n} l^T q_0^l \right)^{r-1} (T+1)^{r-1} \le cNp^n n^T n^{1-r} \to 0$$

as $N \to \infty$, because $r = 2, 3, \dots$. Above we applied that $\sum_{l=1}^n l^T q_0^l \le c \int_0^\infty x^T q_0^x dx < \infty$. So we obtained

$$\sum_{\alpha_r \in I_r(n,N)} b_{i_1,\dots,i_r} = \sum_{\alpha_r \in I_r(n,N)} \mathbb{P}(\widetilde{\eta}_{i_1} = 1, \dots, \widetilde{\eta}_{i_r} = 1) \to 0 \tag{2.15}$$

as $N \to \infty$. ∎

Now we define the number of at most T-contaminated runs of heads having length n as follows. Let

$$(2.16) \qquad \eta_i = \eta_i^T(n) = \begin{cases} 1, & \text{if there are at most } T \ \ 0 \text{ values among } X_i, \dots, X_{i+n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Now let

$$(2.17) \qquad \xi = \xi^T(n, N) = \sum_{i=1}^{N-n+1} \eta_i^T(n).$$

ξ can be considered as the number of head-runs being at most T-contaminated and having length n.

Now we prove that the distribution of ξ converges to a compound Poisson distribution.

Theorem 2.3. *Let T be fixed. Let $N \to \infty$ and $n \to \infty$ so that condition (2.3) is satisfied. Then for the generator functions we have*

$$\lim_{N\to\infty} \mathbb{E}\left(z^{\xi^T(n,N)}\right) = \exp\left[\lambda\left(\frac{qz}{1-pz} - 1\right)\right].$$

The above theorem is proved in [6] for $p = 1/2$ (see Theorem 3.2 in [6], see also Theorem 2.A in [7]).

Proof. First recall the notion of the compound Poisson distribution. Here we need its particular version, that is the so called geometric Poisson distribution. Let γ have Poisson distribution $\mathbb{P}(\gamma = k) = \lambda^k e^{-\lambda}/k!$, $k = 0, 1, 2, \dots$. Let $\varrho_1, \varrho_2, \dots,$ be independent random variables each having q parameter geometric distribution: $\mathbb{P}(\varrho_i = l) = p^{l-1}q$, $l = 1, 2, \dots,$ $q \in (0, 1)$, $p = 1 - q$. Let the distribution of ϱ be the same as $\varrho_1 + \dots + \varrho_k$ when $\gamma = k$. (Here an empty sum is defined as 0, i.e. $\varrho = 0$ when $\gamma = 0$.) Then ϱ has generator function $\mathbb{E}\left(z^{\varrho}\right) = \exp\left[\lambda\left(\frac{qz}{1-pz} - 1\right)\right]$ for $|zp| < 1$.

It is easy to see that for any run of length n containing at most T zeros either there exists a preceding run of length n containing precisely T zeros or all preceding runs contain zeros less than T. To give a formal explanation of this fact let

$$(2.18) \qquad \eta'_i = {\eta'}_i^T(n) = \begin{cases} \widetilde{\eta}_i^T(n) \cdot X_{i+n-1}, & \text{if } i > 1 \\ \eta_i^T(n), & \text{if } i = 1. \end{cases}$$

Therefore $\eta'_i = 1$ either if $i = 1$ and among the first n tosses there are at most T tails, or $i > 1$ and $X_{i-1} = 0$, $X_{i+n-1} = 1$ and between these locations there are precisely T zeros. We see that in the second case shifting left by 1 the interval $i, i+1, \dots, i+n-1$ we obtain an interval containing $T+1$ zeros. Now starting at an arbitrary interval $i, i+1, \dots, i+n-1$ containing at most T tails, shift left this interval step by step until there are more than T tails in it. If $k+1$ denotes the number of steps until that situation is reached, then $\eta'_{i-k} = 1$. (Moreover, $\eta'_{i-k} = 1$ means always the end of the above shifting procedure.) Therefore we can find the longest sequences of overlapping intervals of type $i, i+1, \dots, i+n-1$ containing at most T tails. We shall refer to these sequences of intervals as chains of intervals. So to count the number of head-runs being at most T-contaminated and having length n we should find the number of these chains of intervals and their length. To be more precise, we should consider the following representation of $\xi = \xi^T(n, N)$.

$$\xi = \xi^T(n, N) = \sum_{i=1}^{N-n+1} \gamma_i^T(n) = \sum_{i=1}^{N-n+1} \gamma_i, \tag{2.19}$$

where

$$\gamma_i = \gamma_i^T(n) = \eta'_i \left[\min \left\{ k > 0 \ : \quad \begin{array}{l} \text{either } \eta_{i+k} = 0 \\ \text{or } i+k+n-1 > N \end{array} \right\} \right]. \tag{2.20}$$

Let $\gamma = \gamma^T(n)$ denote the number of non-zero γ_i's (i.e. the number of non-zero η'_i elements). We know that $\widetilde{\xi}$ is the number of precisely T-contaminated head-runs of length n so that the preceding element is 0. So $\gamma \neq \widetilde{\xi}$ only in the following two cases. The first case is when $\eta_1 = 1$ and $\widetilde{\eta}_1 = 0$. The second case is $\eta'_i \neq \widetilde{\eta}_i$ for some $i > 1$. The probability of these events is not greater than

$$\mathbb{P}(\eta_1 = 1, \widetilde{\eta}_1 = 0) + \sum_{i=2}^{N-n+1} \mathbb{P}(\eta'_i \neq \widetilde{\eta}_i) \leq$$

$$\begin{aligned}
\leq \mathbb{P}(&\text{at the beginning of the tosses there is a run containing tails less than } T) \\
&+ N\mathbb{P}(\text{there are precisely } T \text{ tails among } n \text{ tosses,} \\
&\qquad \text{the last and the previous are tails}) \leq \\
&\leq \sum_{i=0}^{T-1} \binom{n}{i} p^{n-i} q^i + Nq \binom{n-1}{T-1} p^{n-T} q^{T-1} q \leq \\
&\leq cTn^T p^{n-T+1} + cNn^{T-1} p^{n-T} q^{T+1} \to 0
\end{aligned}$$

as $N, n \to \infty$, because of condition (2.3).

Therefore $\mathbb{P}(\gamma = \widetilde{\xi}) \to 1$ if $N, n \to \infty$. By Theorem 2.1, the limit distribution of $\widetilde{\xi}$ is λ parameter Poisson. So, by Slutsky's lemma, the limit distribution of γ is also λ parameter Poisson.

We shall show that

(2.21) $$\lim_{N,n\to\infty} \mathbb{P}(\gamma_i > k|\gamma_i > 0) = p^k, \quad k = 0, 1, 2, \dots,$$

i.e. the (conditional) limiting distribution of γ_i is the q parameter geometric distribution. To this end we shall use the following elementary fact. When $\mathbb{P}(B_nC_n) \neq 0$, then $\lim_{n\to\infty} \mathbb{P}(B_nC_n)/\mathbb{P}(B_n) = 1$ implies that

(2.22) $$\lim_{n\to\infty} \mathbb{P}(A_n|B_n) = \lim_{n\to\infty} \mathbb{P}(A_n|B_nC_n)$$

in the sense that if one side of the above equation exists, then the other side also exists and the two sides are equal.

For any fixed $i \in \{1, 2, \dots\}$ and $k_0 \geq 1$ let $A_n = \{\gamma_i^T(n) > k_0\}$, $B_n = = \{\gamma_i^T(n) > 0\}$, $C_n = \{\prod_{j=i}^{i+k_0-1} X_j = 1\}$. First let $i > 1$. Then

$$B_n = \{X_{i-1} = 0, X_{i+n-1} = 1, \text{ there are precisely } T \text{ zeros among } X_i, \dots, X_{i+n-2}\}.$$

Then

$$\frac{\mathbb{P}(B_nC_n)}{\mathbb{P}(B_n)} = \frac{qp^{k_0}\binom{n-k_0-1}{T}q^T p^{n-k_0-1-T}p}{q\binom{n-1}{T}q^T p^{n-1-T}p} \to 1.$$

So, by (2.22),

$$\lim_{n\to\infty} \mathbb{P}(\gamma_i > k_0|\gamma_i > 0) = \lim_{n\to\infty} \mathbb{P}(A_n|B_n) =$$
$$= \lim_{n\to\infty} \mathbb{P}(A_n|B_nC_n) = \lim_{n\to\infty} \mathbb{P}(X_{i+n} = \dots = X_{i+n+k_0-1} = 1) = p^{k_0}.$$

Now let $i = 1$. Then

$$B_n = \{\gamma_1^T(n) > 0\} = \{\text{ there are at most } T \text{ zeros among } X_1, \dots, X_n\}.$$

Let

$$B'_n = \{\text{ there are precisely } T \text{ zeros among } X_1, \dots, X_n\}.$$

Then $\mathbb{P}(B'_n) = \binom{n}{T}q^T p^{n-T}$, $\mathbb{P}(B_n) = \sum_{j=0}^{T}\binom{n}{j}q^j p^{n-j}$. We see that $B'_n \subseteq B_n$ and $\frac{\mathbb{P}(B_nB'_n)}{\mathbb{P}(B_n)} = \frac{\mathbb{P}(B'_n)}{\mathbb{P}(B_n)} \to 1$ as $n \to \infty$. So, using (2.22) with $C_n = B'_n$, we see that $\lim_{n\to\infty} \mathbb{P}(A_n|B_n) = \lim_{n\to\infty} \mathbb{P}(A_n|B'_n)$. Then with $C_n = \{\prod_{j=1}^{k_0} X_j = 1\}$ we have

$$\frac{\mathbb{P}(B'_nC_n)}{\mathbb{P}(B'_n)} = \frac{p^{k_0}\binom{n-k_0}{T}q^T p^{n-k_0-T}}{\binom{n}{T}q^T p^{n-T}} \to 1.$$

So we can use (2.22) with B'_n instead of B_n. Then we obtain

$$\lim_{n\to\infty} \mathbb{P}(A_n|B_n) = \lim_{n\to\infty} \mathbb{P}(A_n|B'_n) = \lim_{n\to\infty} \mathbb{P}(A_n|B'_nC_n) =$$
$$= \lim_{n\to\infty} \mathbb{P}(X_{n+1} = \dots = X_{n+k_0} = 1) = p^{k_0}.$$

So we obtained (2.21).

Now let $\alpha_r = (i_1, \dots, i_r)$ be a vector of indices with $i_j \neq i_l$ for $j \neq l$. Introduce notation

$$C(\alpha_r) = \{\gamma = r, \eta'_{i_1} = \eta'_{i_2} = \cdots = \eta'_{i_r} = 1\}.$$

The meaning of $C(\alpha_r)$ is that there are r above mentioned chains of intervals and they start at positions $i_1, i_2, \dots, i_r$. Obviously $C(\alpha_r) \cap C(\alpha'_r) = \emptyset$ for $\alpha_r \neq \alpha'_r$, moreover $\{\gamma = r\} = \cup_{\alpha_r} C(\alpha_r)$. Therefore

$$\mathbb{P}(\{\gamma = r\}) = \sum_{\alpha_r} \mathbb{P}(C(\alpha_r)).$$

Now let $K = (k_1, \dots, k_r)$, $|K| = k_1 + \cdots + k_r$, and let

$$C_1(\alpha_r, K) = C(\alpha_r) \cap \{\gamma_{i_1} > k_1, \dots, \gamma_{i_r} > k_r\}.$$

The meaning of $C_1(\alpha_r, K)$ is that lengths of the above mentioned chains of intervals are greater than $k_1, \dots, k_r$. Using event $C_1(\alpha_r, K)$, we can describe the asymptotic joint distribution of the positive γ_i's.

To finish the proof we have to prove the following. Given that there are r positive γ_i variables, then the asymptotic joint distribution of the positive γ_i variables is equal to the joint distribution of r independent geometrically distributed random variables. That is we have to prove

$$\frac{\sum_{\alpha_r} \mathbb{P}(C_1(\alpha_r, K))}{\mathbb{P}(\gamma = r)} \to p^{|K|}. \tag{2.23}$$

Let $k = \max_{1 \le i \le r} k_i$ and let the exceptional set $I_r(n+k, N)$ be defined as in the proof of Theorem 2.1. That is $\alpha_r \in I_r(n+k, N)$ if and only if there exists $i_j, i_l \in \alpha_r$ such that $|i_j - i_l| \le n + k$. Now we show that

$$\sum_{\alpha_r \in I_r(n+k,N)} \mathbb{P}(C(\alpha_r)) \to 0. \tag{2.24}$$

We have

$$\mathbb{P}(C(\alpha_r)) \le \mathbb{P}(\{\eta'_{i_1} = \eta'_{i_2} = \cdots = \eta'_{i_r} = 1\}) \le \mathbb{P}(\{\widetilde{\eta}_{i_1} = \widetilde{\eta}_{i_2} = \cdots = \widetilde{\eta}_{i_r} = 1\})$$

because of the inclusion relations among the above events. Using this relation and the fact that $\widetilde{\eta}_{i_1}, \widetilde{\eta}_{i_2}, \dots, \widetilde{\eta}_{i_r}$ are independent for $\alpha_r \notin I_r(n, N)$, we obtain

$$\begin{aligned} \sum_{\alpha_r \in I_r(n+k,N)} \mathbb{P}(C(\alpha_r)) \quad &\le \quad \sum_{\alpha_r \in I_r(n,N)} \mathbb{P}(\{\widetilde{\eta}_{i_1} = \widetilde{\eta}_{i_2} = \cdots = \widetilde{\eta}_{i_r} = 1\}) \\ &\quad + |I_r(n+k, N)| \left[q \binom{n}{T} q^T p^{n-T} \right]^r . \end{aligned}$$

During the proof of Theorem 2.1 we obtained that the limit of the first term is 0 (see (2.15)). On the other hand, for the second term we have

$$\lim_{N\to\infty} |I_r(n+k,N)| \left[q\binom{n}{T}q^T p^{n-T}\right]^r \leq$$
$$\leq \lim_{N\to\infty} c\left(\binom{N}{r-1}(r-1)2(n+k)\right)\left[n^T q^{(T+1)} p^{(n-T)}\right]^r \leq$$
$$\leq \lim_{N\to\infty} c\frac{n}{N}| \left[Nn^T p^n\right]^r = 0$$

by condition (2.3). So we obtained (2.24). Now we have

$$\sum_{\alpha_r} \mathbb{P}(C_1(\alpha_r, K)) =$$
$$= \sum_{\alpha_r\in I_r(n+k,N)} \mathbb{P}(C_1(\alpha_r, K)) + \sum_{\alpha_r\notin I_r(n+k,N)} \mathbb{P}(C_1(\alpha_r, K)). \tag{2.25}$$

By (2.24), the first term in (2.25) converges to 0. Now we consider the second term. By the definition of $C_1(\alpha_r, K)$, we have

$$\sum_{\alpha_r\notin I_r(n+k,N)} \mathbb{P}(C_1(\alpha_r, K)) =$$
$$= \sum_{\alpha_r\notin I_r(n+k,N)} \mathbb{P}\left(\gamma_{i_1} > k_1, \dots, \gamma_{i_r} > k_r \,|\, \gamma = r, \eta'_{i_1} = \eta'_{i_2} = \cdots = \eta'_{i_r} = 1\right) \times$$
$$\times \quad \mathbb{P}\left(\gamma = r, \eta'_{i_1} = \eta'_{i_2} = \cdots = \eta'_{i_r} = 1\right).$$

By independence and (2.21),

$$\sum_{\alpha_r\notin I_r(n+k,N)} \mathbb{P}(C_1(\alpha_r, K)) \approx$$
$$\approx \sum_{\alpha_r\notin I_r(n+k,N)} \left(\prod_{i=1}^r p^{k_i}\right) \mathbb{P}\left(\gamma = r, \eta'_{i_1} = \eta'_{i_2} = \cdots = \eta'_{i_r} = 1\right) =$$
$$= p^{|K|}\left(\sum_{\alpha_r} \mathbb{P}(C(\alpha_r)) - \sum_{\alpha_r\in I_r(n+k,N)} \mathbb{P}(C(\alpha_r))\right) \approx p^{|K|}\mathbb{P}(\gamma = r). \tag{2.26}$$

In the last step we applied (2.24). As the limit distribution of γ is Poisson, that is the limit of $\mathbb{P}(\gamma = r)$ is non-zero, we obtain from (2.25), (2.24) and (2.26) that (2.23) is satisfied.

So we obtained for relation (2.19) that is for $\xi = \sum_{i=1}^{N-n+1} \gamma_i$ the following facts. The number γ of the non-zero terms γ_i is asymptotically Poisson with parameter λ. Moreover, the positive ones out of the variables $\gamma_1, \gamma_2, \dots$ are asymptotically geometric and they are asymptotically independent. Therefore ξ is asymptotically compound Poisson. ■

3. First arrival time and longest run

Let

$$\tau = \tau^T(n) = \min\{N \ : \ \xi^T(n, N) > 0\}. \tag{3.1}$$

τ is the first hitting time of the run having length n and containing at most T tails. We show that the appropriate normalised version of τ has exponential limiting distribution.

Theorem 3.1. *Let T be fixed. Then for any $0 < x < \infty$*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\tau^T(n)n^T}{T!} q^{T+1}p^{n-T} \leq x\right) = 1 - e^{-x}.$$

The above theorem is proved in [6] for $p = 1/2$ (see Theorem 3.3 in [6], see also Theorem 3.A in [7]).

Proof.

$$\mathbb{P}\left(\frac{\tau^T(n)n^T}{T!} q^{T+1}p^{n-T} > x\right) =$$
$$= \mathbb{P}\left(\tau^T(n) > \frac{xT!}{n^T q^{T+1}p^{n-T}}\right) = \mathbb{P}\left(\xi^T(n, N(x)) = 0\right),$$

where $N(x) = \left[\frac{xT!}{n^T q^{T+1}p^{n-T}}\right]$. By Theorem 2.3, the asymptotic distribution of ξ is compound Poisson, it is obtained from a λ parameter Poisson and q parameter geometric distributions. Therefore (using notation from the beginning of the proof of Theorem 2.3) the limiting distribution is

$$\mathbb{P}(\varrho = 0) = \sum_{k=0}^{\infty} \mathbb{P}(\varrho = 0 \,|\, \gamma = k)\mathbb{P}(\gamma = k) =$$
$$= \mathbb{P}(0 = 0)\mathbb{P}(\gamma = 0) + \sum_{k=1}^{\infty} \mathbb{P}(\varrho_1 + \cdots + \varrho_k = 0)\mathbb{P}(\gamma = k) = 1e^{-\lambda} + 0.$$

We have to check condition (2.3) and to find the value of λ.

$$\frac{N(x)n^T q^{T+1}p^{n-T}}{T!} = \left[\frac{xT!}{n^T q^{T+1}p^{n-T}}\right] \frac{n^T q^{T+1}p^{n-T}}{T!} \to x$$

as $n \to \infty$. Therefore the λ parameter is equal to x. ∎

Let

$$\mu = \mu^T(N) = \max\{n \ : \ \xi^T(n, N) > 0\}. \tag{3.2}$$

Considering the result of tossing a coin N times, μ is the length of the longest run of heads containing at most T tails. The following theorem describes the accompanying distribution of $\mu^T(N)$. We offer a two parameter family of distributions to approximate the distribution of μ. We shall use the following notation. Let B be a fixed positive number. For any positive x we have that

$$x = kB + r,$$

where k is integer and r is the residual for which $0 \le r < B$. Here k and r are uniquely determined. We define $[x]_B$ and $\{x\}_B$ as $[x]_B = kB$ and $\{x\}_B = r$.

Theorem 3.2. *Let T be fixed. Let B be a fixed positive number and let S be a fixed number. Then for any integer k we have*

$$\mathbb{P}\left(\mu^T(N) - [\log N + T\log(\log N + S\log\log N)]_B < k\right) =$$

$$= \exp\left(-q^{T+1}p^{(k-T-\{\log N+T\log(\log N+S\log\log N)\}_B}/T!\right) + \mathrm{o}(1). \tag{3.3}$$

Here log *denotes logarithm of base* $1/p$.

For $B = S = 1$ and $p = 1/2$, the above theorem is proved in [6] (see Theorem 3.4 in [6], see also Theorem 4.A in [7]).

Proof. For any integer k let

$$f(N) = \mathbb{P}\left(\mu^T(N) - [\log N + T\log(\log N + S\log\log N)]_B < k\right).$$

Then

$$f(N) = \mathbb{P}\left(\mu^T(N) < n(k)\right) = \mathbb{P}\left(\xi^T(n(k), N) = 0\right),$$

where $n(k) = k + [\log N + T\log(\log N + S\log\log N)]_B$. For any fixed k, the sequence $n(k)$ converges to infinity as $N \to \infty$. Let $\lambda_0 \in [0, B]$ be fixed and choose a subsequence

$$N_j \uparrow \infty \quad \text{such that} \quad \{\log N_j + T\log(\log N_j + S\log\log N_j)\}_B \to \lambda_0. \tag{3.4}$$

For this subsequence N_j and for

$$n_j(k) = k + [\log N_j + T\log(\log N_j + S\log\log N_j)]_B,$$

condition (2.3) is satisfied in the following form:

$$\frac{N_j q^{T+1} p^{n_j(k)-T}(n_j(k))^T}{T!} \to \frac{q^{T+1}p^{k-T}p^{-\lambda_0}}{T!}. \tag{3.5}$$

Therefore, by Theorem 2.3, and using the argument of the proof of Theorem 3.1, we obtain

$$\lim_{j\to\infty} f(N_j) = \lim_{j\to\infty} \mathbb{P}\left(\mu^T(N_j) - \log N_j + T\log(\log N_j + S\log\log N_j)]_B < k\right) =$$

$$= \lim_{j\to\infty} \mathbb{P}\left(\xi^T(n_j(k), N_j) = 0\right) = \exp(-q^{T+1}p^{k-T}p^{-\lambda_0}/T!). \tag{3.6}$$

If (3.4) is satisfied, then for

$$g(N) = \exp\left(-q^{T+1}p^{(k-T-\{\log N + T\log(\log N + S\log\log N)\}_B)/T!}\right)$$

$$\lim_{j\to\infty} g(N_j) = \exp(-q^{T+1}p^{k-T}p^{-\lambda_0}/T!) \tag{3.7}$$

is true. To obtain (3.3), we have to show that $f(N) - g(N) \to 0$. Suppose that it is not satisfied, i.e. there exists $\varepsilon > 0$ such that for certain subsequence N'_j we have $|f(N'_j) - g(N'_j)| > \varepsilon$ for any j. As the sequence $\left\{\log N'_j + T\log(\log N'_j + S\log\log N'_j)\right\}_B$ has an accumulation point λ_0 in $[0, B]$, so there exists a further subsequence N_j of N'_j so that

$$\{\log N_j + T\log(\log N_j + S\log\log N_j)\}_B \to \lambda_0.$$

Now, by (3.6) and (3.7), $f(N_j) - g(N_j) \to 0$. It is a contradiction, so (3.3) is satisfied. ∎

Now we give a new proof for Theorem 1 of [8]. In Theorem 1 of [8] the longest head run containing (precisely) T tails was studied. However, we have the following

Remark. The limiting distribution of the length of the longest head run containing T tails is the same as the limiting distribution of the length of the longest head run containing at most T tails. To prove it, let A be the event that the length of the longest head run containing at most T tails is greater than n. Then $A = B \cup C$, where B is the event that the length of the longest head run containing precisely T tails is greater than n and C is the event that the length of a head run containing less than T tails is greater than n and it is not possible to add some tails to it. But

$$\mathbb{P}(C) \le \sum_{i=0}^{T-1} \binom{N}{i} p^{N-i}q^i \le cp^N N^{T-1} \to 0$$

as $N \to \infty$.

In [8], the original proof was based on extreme value theory, but here we give a new proof using the method of our Theorem 3.2. Let $[x]$ denote the usual integer part of x and $\{x\}$ its fractional part.

Proposition 3.3. (Theorem 1 of [8].)

$$\mathbb{P}\left(\mu^T(N) - \mu_T(qN) \le t\right) = \mathbb{P}\left(\left[\frac{W}{\ln(\frac{1}{p})} + \{\mu_T(qN)\}\right] - \{\mu_T(qN)\} \le t\right) + \mathrm{o}(1)$$

for all t*, where*

$$\mu_T(qN) = \log(qN) + T\log\log(qN) + T\log(q/p) - \log(T!)$$

and the distribution of W *is* $\mathbb{P}(W \le t) = \exp(-e^{-t})$.

Proof. Some algebraic calculation shows, that we have to prove that

$$\mathbb{P}\left(\mu^T(N) - [\mu_T(qN)] < k\right) = \mathbb{P}\left(\left[\frac{W}{\ln(\frac{1}{p})} + \{\mu_T(qN)\}\right] < k\right) + \mathrm{o}(1)$$

for all integers k. Using the definition of the distribution of W, this relation is equivalent to

$$\mathbb{P}\left(\mu^T(N) - [\mu_T(qN)] < k\right) = \exp\left(-p^{k-\{\mu_T(qN)\}}\right) + \mathrm{o}(1).$$

The remaining part of the proof is the same as that of Theorem 3.2. ■

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