

# ON AN OPEN QUESTION REGARDING GENERALIZED NUMBER SYSTEMS IN EUCLIDEAN SPACES

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**Abstract.** We raise an open question regarding generalized number systems in Euclidean spaces and formulate a partial answer.

## 1. Introduction

Given an integer  $M$ , let  $t = |M|$  and assume that  $t \neq 0, 1$ . Let

$$\mathcal{A} := \{a_0 = 0, a_1, \dots, a_{t-1}\}$$

be a complete residue system modulo  $M$ . Further set

$$\mathcal{B} := \mathcal{A} - \mathcal{A} = \{a_i - a_j : a_i, a_j \in \mathcal{A}\}.$$

Michalek [4], [5] proved the following.

**Theorem A.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be as above. Any integer  $n$  can be written in the form*

$$(1.1) \quad n = c_0 + c_1M + \dots + c_hM^h \text{ with each } c_i \in \mathcal{B}$$

*for some positive integer  $h$  if and only if  $\text{GCD}(a_1, a_2, \dots, a_{t-1}) = 1$ .*

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Why is Theorem A interesting? Consider the sets

$$\begin{aligned} H &:= \left\{ \sum_{\nu=1}^{\infty} \frac{c_{\nu}}{M^{\nu}} : c_{\nu} \in \mathcal{A} \right\}, \\ \Gamma_{\ell} &:= \left\{ \sum_{\nu=0}^{\ell} c_{\nu} M^{\nu} : c_{\nu} \in \mathcal{A} \right\}, \\ \Gamma &:= \bigcup_{\ell=0}^{\infty} \Gamma_{\ell}. \end{aligned}$$

The following statements were proved in [1], [2] and [3]:

- (1)  $H$  is a compact set and  $\lambda(H) > 0$  (here,  $\lambda$  is the Lebesgue measure in  $\mathbb{R}$ ).
- (2)  $\lambda(H + \gamma_1 \cap H + \gamma_2) = 0$  if  $\gamma_1, \gamma_2 \in \Gamma$ ,  $\gamma_1 \neq \gamma_2$ .

In light of the above setup, we say that  $(\mathcal{A}, M)$  is *generalized number system* if  $\Gamma = \mathbb{Z}$ , or in other words, if every  $n \in \mathbb{Z}$  can be written as  $n = \sum_{\nu=0}^h c_{\nu} M^{\nu}$  for some positive integer  $h$ . In this case,

$$(1.2) \quad \lambda(H + n_1 \cap H + n_2) = 0 \text{ for all } n_1, n_2 \in \mathbb{Z}, n_1 \neq n_2.$$

It may occur that (1.2) holds despite the fact that  $\Gamma \neq \mathbb{Z}$ . Observe that it was proved earlier that (1.2) holds if and only if

$$(1.3) \quad \Gamma - \Gamma = \mathbb{Z},$$

that is if every  $n \in \mathbb{Z}$  can be written in the form (1.1). In this case, we say that  $(\mathcal{A}, M)$  is a *just touching covering system*.

## 2. Generalized number systems in Euclidean spaces

Given a positive integer  $k$ , let  $\mathbb{R}_k$  and  $\mathbb{Z}_k$  stand respectively for the  $k$ -dimensional real Euclidean space and the ring of  $k$ -dimensional vectors with integer entries. Fix  $k$  and let  $M$  be a  $k \times k$  matrix with integer elements. Assume that  $M$  has  $k$  distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$  such that  $|\lambda_1| > |\lambda_2| > \dots > |\lambda_k| > 1$ . Let  $\mathcal{L} := M\mathbb{Z}_k$ . Then,  $\mathcal{L}$  is a subgroup of  $\mathbb{Z}_k$ . Let  $t$  stand for the order of  $\mathbb{Z}_k/\mathcal{L}$ , so that  $t = |\det M|$ . Further let  $A_0, A_1, \dots, A_{t-1}$  stand for the residue classes mod  $\mathcal{L}$  and let  $A_0 = \mathcal{L}$ . For each  $j \in \{0, 1, \dots, t-1\}$ , choose an arbitrary element  $\underline{a}_j \in A_j$  such that the vector  $\underline{a}_0$  is the zero vector  $\underline{0} = (0, 0, \dots, 0)$ , and then write

$$\mathcal{A} := \{\underline{a}_0, \underline{a}_1, \dots, \underline{a}_{t-1}\},$$

so that  $\mathcal{A}$  is a  $k$ -dimensional complete residue system modulo  $M$ .

We now introduce the sets

$$\begin{aligned} H &:= \left\{ \sum_{\nu=1}^{\infty} M^{-\nu} \underline{c}_{\nu} : \underline{c}_{\nu} \in \mathcal{A} \right\}, \\ \Gamma_{\ell} &:= \left\{ \sum_{\nu=0}^{\ell} M^{\nu} \underline{d}_{\nu} : \underline{d}_{\nu} \in \mathcal{A} \right\}, \\ \Gamma &:= \bigcup_{\ell=0}^{\infty} \Gamma_{\ell}. \end{aligned}$$

We will say that  $(\mathcal{A}, M)$  is *just touching covering system* (JTCS) if

$$\lambda(H + \underline{n}_1 \cap H + \underline{n}_2) = 0 \text{ for all } \underline{n}_1, \underline{n}_2 \in \mathbb{Z}_k, \underline{n}_1 \neq \underline{n}_2.$$

In [3], it was proved that  $(\mathcal{A}, M)$  is a JTCS if and only if  $\Gamma - \Gamma = \mathbb{Z}_k$ , that is if every  $\underline{n} \in \mathbb{Z}_k$  can be written as

$$(2.1) \quad \underline{n} = \sum_{j=0}^h M^j \underline{d}_j,$$

where the  $\underline{d}_j$ 's belong to the set  $\mathcal{B} = \mathcal{A} - \mathcal{A}$ .

We now state the following.

**Open question.** *Under what condition is it true that  $(\mathcal{A}, M)$  is a JTCS ?*

Let us consider the special case where  $M = \text{diag}(m_1, \dots, m_k)$ , that is the “diagonal matrix”, whose elements are all 0 except the elements on the diagonal whose values are  $m_1, \dots, m_k$ . Let  $t = |m_1 \cdots m_k|$  and assume that  $|m_j| > 1$  for  $j = 1, \dots, k$ . Let  $\mathcal{A} = \{\underline{a}_0 = 0, \underline{a}_1, \dots, \underline{a}_{t-1}\}$  be the complete residue system modulo  $t$ . If  $(\mathcal{A}, M)$  is a JTCS, let  $\mathcal{A}^{(j)}$  be the set of the  $j$ -th coordinates of  $\mathcal{A}$ , then  $(\mathcal{A}^{(j)}, m_j)$  should be a JTCS in  $\mathbb{R}$ , in which case every  $n \in \mathbb{Z}$  can be written as

$$n = \sum_{\nu=0}^h b_{\nu} m_j^{\nu} \quad \text{with } b_{\nu} \in \mathcal{A}^{(j)} - \mathcal{A}^{(j)}.$$

This means that if  $(\mathcal{A}, M)$  is a JTCS, then

$$\text{GCD}(a_1^{(j)}, \dots, a_{t-1}^{(j)}) = 1 \quad \text{for each } j = 1, \dots, k.$$

The question is: Is this condition sufficient ?

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