

ON AN OPEN QUESTION REGARDING
GENERALIZED NUMBER SYSTEMS
IN EUCLIDEAN SPACES

Jean-Marie De Koninck¹ (Québec, Canada)

Imre Kátai (Budapest, Hungary)

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Abstract. We raise an open question regarding generalized number systems in Euclidean spaces and formulate a partial answer.

1. Introduction

Given an integer M , let $t = |M|$ and assume that $t \neq 0, 1$. Let

$$\mathcal{A} := \{a_0 = 0, a_1, \dots, a_{t-1}\}$$

be a complete residue system modulo M . Further set

$$\mathcal{B} := \mathcal{A} - \mathcal{A} = \{a_i - a_j : a_i, a_j \in \mathcal{A}\}.$$

Michalek [4], [5] proved the following.

Theorem A. *Let \mathcal{A} and \mathcal{B} be as above. Any integer n can be written in the form*

$$(1.1) \quad n = c_0 + c_1M + \dots + c_hM^h \text{ with each } c_i \in \mathcal{B}$$

for some positive integer h if and only if $\text{GCD}(a_1, a_2, \dots, a_{t-1}) = 1$.

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Why is Theorem A interesting? Consider the sets

$$\begin{aligned} H &:= \left\{ \sum_{\nu=1}^{\infty} \frac{c_{\nu}}{M^{\nu}} : c_{\nu} \in \mathcal{A} \right\}, \\ \Gamma_{\ell} &:= \left\{ \sum_{\nu=0}^{\ell} c_{\nu} M^{\nu} : c_{\nu} \in \mathcal{A} \right\}, \\ \Gamma &:= \bigcup_{\ell=0}^{\infty} \Gamma_{\ell}. \end{aligned}$$

The following statements were proved in [1], [2] and [3]:

- (1) H is a compact set and $\lambda(H) > 0$ (here, λ is the Lebesgue measure in \mathbb{R}).
- (2) $\lambda(H + \gamma_1 \cap H + \gamma_2) = 0$ if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$.

In light of the above setup, we say that (\mathcal{A}, M) is *generalized number system* if $\Gamma = \mathbb{Z}$, or in other words, if every $n \in \mathbb{Z}$ can be written as $n = \sum_{\nu=0}^h c_{\nu} M^{\nu}$ for some positive integer h . In this case,

$$(1.2) \quad \lambda(H + n_1 \cap H + n_2) = 0 \text{ for all } n_1, n_2 \in \mathbb{Z}, n_1 \neq n_2.$$

It may occur that (1.2) holds despite the fact that $\Gamma \neq \mathbb{Z}$. Observe that it was proved earlier that (1.2) holds if and only if

$$(1.3) \quad \Gamma - \Gamma = \mathbb{Z},$$

that is if every $n \in \mathbb{Z}$ can be written in the form (1.1). In this case, we say that (\mathcal{A}, M) is a *just touching covering system*.

2. Generalized number systems in Euclidean spaces

Given a positive integer k , let \mathbb{R}_k and \mathbb{Z}_k stand respectively for the k -dimensional real Euclidean space and the ring of k -dimensional vectors with integer entries. Fix k and let M be a $k \times k$ matrix with integer elements. Assume that M has k distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ such that $|\lambda_1| > |\lambda_2| > \dots > |\lambda_k| > 1$. Let $\mathcal{L} := M\mathbb{Z}_k$. Then, \mathcal{L} is a subgroup of \mathbb{Z}_k . Let t stand for the order of \mathbb{Z}_k/\mathcal{L} , so that $t = |\det M|$. Further let A_0, A_1, \dots, A_{t-1} stand for the residue classes mod \mathcal{L} and let $A_0 = \mathcal{L}$. For each $j \in \{0, 1, \dots, t-1\}$, choose an arbitrary element $\underline{a}_j \in A_j$ such that the vector \underline{a}_0 is the zero vector $\underline{0} = (0, 0, \dots, 0)$, and then write

$$\mathcal{A} := \{\underline{a}_0, \underline{a}_1, \dots, \underline{a}_{t-1}\},$$

so that \mathcal{A} is a k -dimensional complete residue system modulo M .

We now introduce the sets

$$\begin{aligned} H &:= \left\{ \sum_{\nu=1}^{\infty} M^{-\nu} \underline{c}_{\nu} : \underline{c}_{\nu} \in \mathcal{A} \right\}, \\ \Gamma_{\ell} &:= \left\{ \sum_{\nu=0}^{\ell} M^{\nu} \underline{d}_{\nu} : \underline{d}_{\nu} \in \mathcal{A} \right\}, \\ \Gamma &:= \bigcup_{\ell=0}^{\infty} \Gamma_{\ell}. \end{aligned}$$

We will say that (\mathcal{A}, M) is *just touching covering system* (JTCS) if

$$\lambda(H + \underline{n}_1 \cap H + \underline{n}_2) = 0 \text{ for all } \underline{n}_1, \underline{n}_2 \in \mathbb{Z}_k, \underline{n}_1 \neq \underline{n}_2.$$

In [3], it was proved that (\mathcal{A}, M) is a JTCS if and only if $\Gamma - \Gamma = \mathbb{Z}_k$, that is if every $\underline{n} \in \mathbb{Z}_k$ can be written as

$$(2.1) \quad \underline{n} = \sum_{j=0}^h M^j \underline{d}_j,$$

where the \underline{d}_j 's belong to the set $\mathcal{B} = \mathcal{A} - \mathcal{A}$.

We now state the following.

Open question. *Under what condition is it true that (\mathcal{A}, M) is a JTCS ?*

Let us consider the special case where $M = \text{diag}(m_1, \dots, m_k)$, that is the “diagonal matrix”, whose elements are all 0 except the elements on the diagonal whose values are m_1, \dots, m_k . Let $t = |m_1 \cdots m_k|$ and assume that $|m_j| > 1$ for $j = 1, \dots, k$. Let $\mathcal{A} = \{\underline{a}_0 = 0, \underline{a}_1, \dots, \underline{a}_{t-1}\}$ be the complete residue system modulo t . If (\mathcal{A}, M) is a JTCS, let $\mathcal{A}^{(j)}$ be the set of the j -th coordinates of \mathcal{A} , then $(\mathcal{A}^{(j)}, m_j)$ should be a JTCS in \mathbb{R} , in which case every $n \in \mathbb{Z}$ can be written as

$$n = \sum_{\nu=0}^h b_{\nu} m_{\nu}^{\nu} \quad \text{with } b_{\nu} \in \mathcal{A}^{(j)} - \mathcal{A}^{(j)}.$$

This means that if (\mathcal{A}, M) is a JTCS, then

$$\text{GCD}(a_1^{(j)}, \dots, a_{t-1}^{(j)}) = 1 \quad \text{for each } j = 1, \dots, k.$$

The question is: Is this condition sufficient ?

References

- [1] **Kátai, I.**, *Generalized Number Systems and Fractal Geometry*, Lecture Notes Janus Pannonius Universitas, Pécs, 1985, 1–40.
- [2] **Kátai, I.**, Generalized number systems in Euclidean spaces, *Mathematical and Computer Modelling*, **38** (2003), 883–892.
- [3] **Indlekofer, K.-H., I. Kátai and P. Racsó,** Some remarks on generalized number systems, *Acta Sci. Math. (Szeged)*, **57(1–4)** (1993), 543–553.
- [4] **Michalek, G.E.**, Base three just touching covering systems, *Publ. Math. Debrecen*, **51(3–4)** (1997), 241–263.
- [5] **Michalek, G.E.**, Base N just touching covering systems, *Publ. Math. Debrecen*, **58** (2001), 549–557.

J.-M. De Koninck

Dép. de mathématiques et de statistique
Université Laval
Québec
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

I. Kátai

Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest
Pázmány Péter sétány 1/C
Hungary
katali@inf.elte.hu