Annales Univ. Sci. Budapest., Sect. Comp. 52 (2021) 13–28

UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH DEFICIENT VALUES AND UNIQUE RANGE SETS OF SMALL CARDINALITIES

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Communicated by Bui Minh Phong

(Received March 14, 2021; accepted May 13, 2021)

Abstract. We give sufficient conditions for a subset S to be a unique range set for meromorphic functions, in term of relations between the degree and the derivative index of the associated polynomial, and the deficient values of functions at ∞ and 0. As consequences, we obtained again some classes of URS for meromorphic functions with 11 elements. Moreover, for classes of meromorphic functions, satisfying some hypotheses on the deficient values at ∞ and 0, we present unique range sets with 6, 7, 8, 9, 10 elements. Similar results for derivatives of meromorphic functions are also obtained.

1. Introduction. Main results

First of all, let us recall some basic notions and describe the main results of the paper.

We assume that the reader is familiar with the notations of the Nevanlinna theory (see, for example [9]). Let f be a meromorphic function in $\mathbb{C}$ $a \in \mathbb{C} \cup \{\infty\}$. Denote by $E_f(a)$ the set of $z \in \mathbb{C} \cup \infty$ such that $f(z) = a$, where every such z is counted with its multiplicity.

Key words and phrases: Meromorphic function, uniqueness, deficient value.
2010 Mathematics Subject Classification: 30D35.
This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2018.301.

For a nonempty subset $S \subset \mathbb{C} \cup \infty$, define

$$E_f(S) = \bigcup_{a \in S} E_f(a).$$

Let $\mathcal{M}(\mathbb{C})$ be the set of meromorphic functions in $\mathbb{C}$, and $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{C})$. Two functions $f, g \in \mathcal{F}$ are said to *share S, counting multiplicities* (share S CM) if $E_f(S) = E_g(S)$.

In 1976 Gross [6] proved that there exist three finite sets S_j ($j = 1, 2, 3$) such that any two entire functions f and g, sharing S_j CM ($j = 1, 2, 3$), must be identical. In the same paper Gross posed the following question:

Question A. ([6]) *Can one find two (or possible even one) finite sets S_j, $j = 1, 2$, such that any two entire functions f and g, satisfying $E_f(S_j) = E_g(S_j)$ ($j = 1, 2$), must be identical.*

If any two functions $f, g \in \mathcal{F}$, sharing S CM, must be identical, then S is called a *unique range set for functions in $\mathcal{F}$*.

In [12]–[14] H. X. Yi first gave an affirmative answer to Question A. Since then, many results have been obtained for this and related topics.

Concerning Question A, a natural question is the following.

Question B. *What is the smallest cardinality for such a finite set S, such that any two meromorphic functions f and g, satisfying $E_f(S) = E_g(S)$, must be identical?*

So far, the best answer to Question B was obtained by Frank and Reinders [3]. They proved the following result.

Theorem C. ([3]) *The set*

$$S = \Big\{ z \in \mathbb{C} : \frac{(n-1)(n-2)}{2} z^n + n(n-2) z^{n-1} + \frac{(n-1)n}{2} z^{n-2} + b = 0 \Big\},$$

where $n \geq 11$ and $b \neq 0, 1$, is a unique range set for meromorphic functions (S is a URSM).

In [10], P. Li and C. C. Yang introduced the following notation:

$$\lambda_M = \inf \big\{ \#(S) \mid S \text{ is a } URSM \big\},$$

where $\#$ denotes the cardinal number of the set S.

They proved that $\lambda_M \geq 6$. In [7] Ha Huy Khoai proposed that this minimum should be 7. Since a natural question is the following.

Question D. *Can one find a finite set S with $6 \leq \#(S) \leq 11$ and a class $\mathcal{F} \subset \mathcal{M}(\mathbb{C})$ such that any two meromorphic functions $f, g \in \mathcal{F}$, satisfying $E_f(S) = E_g(S)$, must be identical.*

Concerning this question, in [8] Ha Huy Khoai, Vu Hoai An and Nguyen Xuan Lai proved the following theorem.

Theorem E. ([8]) *Let $\mathcal{F}_2$ be the class of meromorphic functions, whose zeros and poles are of multiplicities at least 2. There exist unique range sets for functions in the class $\mathcal{F}_2$ with 7 elements.*

In this paper, we apply the arguments used in [8] to more general settings, and establish unique range sets for meromorphic functions in term of deficient values. As consequences, we obtained again some classes of URS for meromorphic functions with 11 elements. Moreover, for classes of meromorphic functions, satisfying some hypotheses on the deficient values at ∞ and 0, we present unique range sets with 5, 6, 7, 8, 9, 10 elements. Similar results for derivatives of meromorphic functions are also obtained.

Now let us describe the main results of the paper.

A polynomial $P(z)$ is called a *strong uniqueness polynomial for meromorphic functions* if for arbitrary two non-constant meromorphic functions f and g, and a nonzero constant c, the condition $P(f) = cP(g)$ implies $f = g$ (see [1], [4], [5], [8]). In this case we call $P(z)$ *a SUPM* .

For a finite subset $S = \{a_1, a_2, ..., a_q\} \subset \mathbb{C}$, we consider its *associated polynomial*

$$P(z) = (z - a_1)(z - a_2) \dots (z - a_q).$$

Assume that the derivative of $P(z)$ has mutually distinct k zeros $d_1, d_2, \dots, d_k$ with multiplicities $q_1, q_2, \dots, q_k$, respectively. We often consider polynomials, satisfying the following condition, introduced by Fujimoto [4]:

$$P(d_i) \neq P(d_j), \quad 1 \leq i < j \leq k.$$

Then we say P is *critically injective*. The number k is called the *derivative index* of P.

Let $f_1, \dots, f_N$ be nonconstant meromorphic functions. We define the *deficient value of* $(f_1, \dots, f_N)$ *at* ∞ by

$$\Theta\big(\infty, (f_1, \dots, f_N)\big) = 1 - \lim_{r \to +\infty} \sup \frac{\overline{N}(r, f_1) + \cdots + \overline{N}(r, f_N)}{T(r, f_1) + \cdots + T(r, f_N)},$$

and the *deficient value of* $(f_1, \dots, f_N)$ *at* $a \in \mathbb{C}$ by

$$\Theta\big(a, (f_1, \dots, f_N)\big) = 1 - \lim_{r \to +\infty} \sup \frac{\overline{N}(r, \frac{1}{f_1 - a}) + \cdots + \overline{N}(r, \frac{1}{f_N - a})}{T(r, f_1) + \cdots + T(r, f_N)}.$$

Note that for $a \in \mathbb{C} \cup \{\infty\}$,

$$\Theta\left(a, (f_1, \ldots, f_N)\right) \leq \sum_{j=1}^{N} \Theta(a, f_j).$$

The following theorem gives a sufficient condition for a subset S to be a unique range set for meromorphic functions in term of relations between the degree and the derivative index of the associated polynomial, and the deficient values of functions at ∞ and 0.

Theorem 1. *Let $S = \{a_1, a_2, \ldots, a_q\} \subset \mathbb{C}$, let $P(z)$ be the associated polynomial to S, $P^{'}(z) = qz^{q_1}(z - d_2)^{q_2} \ldots (z - d_k)^{q_k}$. Assume that P is critically injective, and is a strong uniqueness polynomial for meromorphic functions. If $k \geq 3$, or $k = 2$ and $\min\{q_1, q_2\} \geq 2$, then for two non-constant meromorphic functions f and g, the condition $E_f(S) = E_g(S)$, $q > 2k + 6 - 4(\Theta(\infty, (f, g)) + +\Theta(0, (f, g)))$ implies $f = g$.*

For the case of derivatives, in 1997 Yang and Hua [11] studied the uniqueness problem for meromorphic functions and differential monomials of the form $f^n f^{'}$, sharing only one value. S. S. Bhoosnurmath, R. S. Dyavanal [2] extended Yang-Hua's result to the case of $(f^n)^{(k)}$. H.-X. Yi and W.-C. Lin [14] studied the uniqueness problem for derivatives of meromorphic functions sharing a finite set. Next we give some applications of Theorem 1 to the uniqueness problem for meromorphic functions and their derivatives, sharing a subset.

Theorem 2. *Let $S = \{a_1, a_2, \ldots, a_q\} \subset \mathbb{C}$, $P(z)$ be the associated polynomial to S, $P^{'}(z) = qz^{q_1}(z-d_2)^{q_2} \ldots (z-d_k)^{q_k}$. Assume that P is critically injective, and is a strong uniqueness polynomial for meromorphic functions. If $k \geq 3$, or $k = 2$ and $\min\{q_1, q_2\} \geq 2$, then we have:*

1. *If $E_{(f^n)^{(m)}}(S) = E_{(g^n)^{(m)}}(S)$, $n \geq m + 2$ and $q > 2k + 6 - \frac{8(n-1)}{n+m}$, then $f = cg$ with $c^n = 1$.*
2. *If $E_{f^{(m)}}(S) = E_{g^{(m)}}(S)$ and $q > 2k + 6 - \frac{4m}{m+1}$, then $f^{(m)} = g^{(m)}$.*

Theorem 1, Theorem 2 and their corollaries give an affirmative answers to Question D.

2. Some lemmas

If C is some condition, we denote by $N(r, \frac{1}{f-a}; C)$ the counting function of a-points of f, counted with multiplicities, at which f satisfies the condition C. The counting functions $\overline{N}(r, \frac{1}{f-a}; C)$, $\overline{N}(r, f; C)$ are similarly defined.

Lemma 2.1. *Let f, g be two non-constant meromorphic functions. Set*

$$F = \frac{1}{f}, \quad G = \frac{1}{g}, \quad H = \frac{F^{''}}{F'} - \frac{G^{''}}{G'}.$$

If $E_f(0) = E_g(0)$, and $H \not\equiv 0$, then

$$N(r, H) \leq \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + \overline{N}(r, \frac{1}{f'}; f \neq 0) + \overline{N}(r, \frac{1}{g'}; g \neq 0),$$

where $\overline{N}_{(2}(r, f)$ denote the counting functions of multiple poles of f, each pole is counted once.

Moreover, if a is a common simple zero of f and g, then $H(a) = 0$.

Lemma 2.1 is proved by the same arguments in the proof of Lemma 2.3 in [8], taking into account $E_f(0) = E_g(0)$.

Lemma 2.2. *Let f be a non-constant meromorphic function, n, m be positive integers, $n > m$, and let α be a pole of f of order p. Then*

$$(f^n)^{(m)} = \frac{\varphi_m}{(z-\alpha)^{np+m}}, \text{ where } \varphi_m(\alpha) \neq 0.$$

Lemma 2.3. *Let f be a non-constant meromorphic function, n, m be positive integers, $n > m$, and let α be a pole of order p, β be a zero of multiplicity l of f. Then*

1. $\dfrac{(f^n)^{(m)}}{f^n} = \dfrac{h_m}{(z-\alpha)^m}$, *where* $h_m(\alpha) \neq 0$;
2. $\dfrac{(f^n)^{(m)}}{f^n} = \dfrac{S_m}{(z-\beta)^m}$, *where* $S_m(\beta) \neq 0$.

Lemmas 2.2 and 2.3 can be proved by strength calculations, using the Laurent expansion.

Lemma 2.4. ([4]) *Let $P(z)$ be a polynomial of the above form. Assume $P(z)$ is critically injective, $q \geq 5$, and for two non-constant meromorphic functions f and g, two some constants $c \neq 0$ and c_1,*

$$\frac{1}{P(f)} = \frac{c}{P(g)} + c_1.$$

If $k \geq 3$ or if $k = 2, \min\{q_1, q_2\} \geq 2$, then $c_1 = 0$.

Fujimoto in [5] gave the following class of strong uniqueness polynomials.

Lemma 2.5. ([5]) *Let P_F be a polynomial of the above form, having derivative index $k = 3$. Assume that P_F is critically injective, and* $\max\{m_1, m_2, m_3\} \geq 2$, $\frac{P_F(d_l)}{P_F(d_m)} \neq -1$, $\frac{P_F(d_l)}{P_F(d_m)} \neq 1$, $1 \leq l < m \leq 3$; *and* $\frac{P_F(d_l)}{P_F(d_m)} \neq \frac{P_F(d_m)}{P_F(d_i)}$ *for any permutation* (l, m, i) *of* $(1, 2, 3)$. *Then P_F is a strong uniqueness polynomial for meromorphic functions.*

In [8] we gave another class of strong uniqueness polynomials.

Lemma 2.6. ([8]) *Set*

$$P_B(z) = (n+p+1)\left(\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{n+p+1-i} z^{n+p+1-i} a^i\right) + 1.$$

Suppose that

$$(n+p+1)\left(\sum_{i=0}^{p} \binom{p}{i} \frac{(-1)^i}{n+p+1-i}\right) a^{n+p+1} \neq -2, \quad np > n+p.$$

Then P_B is a strong uniqueness polynomial for meromorphic functions.

Note that polynomials of the above form were investigated first by A. Banerjee [1].

3. The proofs

3.1. The proof of Theorem 1. Recall that

$$P(z) = (z-a_1)...(z-a_q), \quad q = q_1 + + \cdots + q_k + 1,$$

$$P^{'}(z) = qz^{q_1}(z-d_2)^{q_2} \ldots (z-d_k)^{q_k}, \quad q = q_1 + q_2 + \cdots + q_k + 1.$$

We first prove the existence of a constant $C \neq 0$ such that $P(f) = CP(g)$. Set

$$F = \frac{1}{P(f)}, \quad G = \frac{1}{P(g)}, \quad H = \frac{F^{''}}{F^{'}} - \frac{G^{''}}{G^{'}},$$

$$T(r) = T(r,f) + T(r,g), \quad S(r) = S(r,f) + S(r,g).$$

Then

$$T(r, P(f)) = qT(r,f) + S(r,f), \quad T(r, P(g)) = qT(r,g) + S(r,f),$$

and hence $S(r, P(f)) = S(r,f)$ and $S(r, P(g)) = S(r,g)$, since $P(f)$ and f, and $P(g)$ and g have the same growth estimates, respectively.

Suppose that $H \not\equiv 0$. We have

$[P(f)]' = qf^{q_1}(f-d_2)^{q_2}\dots(f-d_k)^{q_k}f',\ [P(g)]' = qg^{q_1}(g-d_2)^{q_2}\dots(g-d_k)^{q_k}g'.$

Note that for a function f, $N_{1)}(r, \frac{1}{f})$ is the counting function of the simple zeros of f. Since $E_f(S) = E_g(S)$, we have

$$N_{1)}\left(r, \frac{1}{F}\right) = N_{1)}\left(r, \frac{1}{G}\right),$$

and we denote this common value by $N_1(r)$. By Lemma 2.1, each common simple zero of f and g is a zero of H. Therefore,

$$N_1(r) \le \overline{N}\left(r, \frac{1}{H}\right) \le T(r, H) \le N(r, H) + S(r),$$

because by Lemma on logarithmic derivatives, $m(r, H) = o(T(r, H))$.

On the other hand, H has only simple zeros, then we have:

$$N_1(r) \le \overline{N}(r, H).$$

Claim 1. *We have*

$$qT(r) \le \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \\ + \overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right) - N_o\left(r, \frac{1}{f'}\right) - N_o\left(r, \frac{1}{g'}\right) + S(r),$$

where $N_o(r, \frac{1}{f'})$ (resp., $N_o(r, \frac{1}{g'})$) is the counting function of those zeros of f' (resp., g'), which are not zeros of any function $(f-a_j)$, $(f-d_l)$ (resp., $(g-a_j)$, $(g-d_l)$) for $j \in \{1, 2, \dots, q\}$; $l \in \{2, \dots, k\}$.

Proof of Claim 1. Applying the Second Main Theorem to f and the values $a_1, a_2, \dots, a_q, 0, d_2, \dots, d_k, \infty$ we obtain

$$(q+k-1)T(r, f) \le \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \sum_{i=1}^{q} \overline{N}\left(r, \frac{1}{f-a_i}\right) + \\ + \sum_{i=2}^{k} \overline{N}\left(r, \frac{1}{f-d_i}\right) - N_o\left(r, \frac{1}{f'}\right) + S(r, f).$$

On the other hand

$$\sum_{i=2}^{k} \overline{N}\left(r, \frac{1}{f-d_i}\right) \le (k-1)T(r, f) + S(r, f),$$

and

$$\sum_{i=1}^{q}\overline{N}\left(r,\frac{1}{f-a_i}\right)=\overline{N}\left(r,\frac{1}{P(f)}\right).$$

Then we have

$$qT(r,f)\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{P(f)}\right)-N_o\left(r,\frac{1}{f'}\right)+S(r,f).$$

Combining this inequality with the similar inequality for g, we obtain

$$qT(r)\le\overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{P(f)}\right)+$$
$$+\overline{N}\left(r,\frac{1}{P(g)}\right)-N_o\left(r,\frac{1}{f'}\right)-N_o\left(r,\frac{1}{g'}\right)+S(r).$$

Claim 2. *We have*

$$\overline{N}\left(r,\frac{1}{P(f)}\right)+\overline{N}\left(r,\frac{1}{P(g)}\right)\le\frac{q}{2}T(r)+N_1(r)+S(r).$$

Proof of Claim 2. It is easy to see that

$$\overline{N}\left(r,\frac{1}{P(f)}\right)\le\frac{1}{2}N\left(r,\frac{1}{P(f)}\right)+\frac{1}{2}N_{1)}\left(r,\frac{1}{P(f)}\right)\le$$
$$\le\frac{q}{2}T(r,f)+\frac{1}{2}N_{1)}\left(r,\frac{1}{P(f)}\right)+S(r,f).$$

Claim 2 is proved by adding this inequality with the similar inequality for g.

Claim 1 and Claim 2 give us

$$\frac{q}{2}T(r)\le\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+\overline{N}(r,g)+$$
$$+N_1(r)-N_o\left(r,\frac{1}{f'}\right)-N_o\left(r,\frac{1}{g'}\right)+S(r).$$

Claim 3. *We have*

$$N_1(r)\le(k-1)T(r)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+\overline{N}(r,g)+$$
$$+N_o\left(r,\frac{1}{f'}\right)+N_o\left(r,\frac{1}{g'}\right)+S(r).$$

Proof of Claim 3. Since $N_1(r) \le N(r,H)$, by Lemma 2.1 we have:

$$N_1(r) \le \overline{N}_{(2}(r,P(f)) + \overline{N}_{(2}(r,P(g)) + \\ + \overline{N}\left(r, \frac{1}{[P(f)]'}; P(f) \ne 0\right) + \overline{N}\left(r, \frac{1}{[P(g)]'}; P(g) \ne 0\right) = \\ = \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{[P(f)]'}; P(f) \ne 0\right) + \\ + \overline{N}\left(r, \frac{1}{[P(g)]'}; P(g) \ne 0\right).$$

On the other hand,

$$\overline{N}\left(r, \frac{1}{[P(f)]'}; P(f) \ne 0\right) = \\ = \overline{N}\left(r, \frac{1}{f^{q_1}(f-d_2)^{q_2}\cdots(f-d_k)^{q_k}f'}; (f-a_1)\cdots(f-a_q) \ne 0\right) \le \\ \le \overline{N}\left(r, \frac{1}{f}\right) + \sum_{i=2}^{k} \overline{N}\left(r, \frac{1}{f-d_i}\right) = \\ = \overline{N}\left(r, \frac{1}{f}\right) + \sum_{i=2}^{k} \overline{N}\left(r, \frac{1}{f-d_i}\right) + \overline{N}_o\left(r, \frac{1}{f'}\right) \le \\ \le (k-1)T(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}_o\left(r, \frac{1}{f'}\right) + S(r,f).$$

Similarly,

$$\overline{N}\left(r, \frac{1}{[P(g)]'}; P(g) \ne 0\right) \le \\ \le (k-1)T(r,g) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}_o\left(r, \frac{1}{g'}\right) + S(r,g).$$

Claim 3 is proved.

From Claims 1, 2, 3 we get:

$$\frac{q}{2}T(r) \le (k-1)T(r) + 2\big[\overline{N}(r,f) + \overline{N}(r,g)\big] + \\ +2\left[\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right)\right] + S(r).$$

Therefore,

$$\frac{q}{2} \le (k-1) + 2\big[1 - \Theta\big(\infty,(f,g)\big)\big] + 2\big[1 - \Theta\big(0,(f,g)\big)\big].$$

We have a contradiction to the hypothesis $q > 2k + 6 - 4[\Theta(\infty, (f, g)) + +\Theta(0, (f, g))]$.

So $H \equiv 0$. Therefore, $\frac{1}{P(f)} = \frac{c}{P(g)} + c_1$ for some constants $c \neq 0$ and c_1 By Lemma 2.4 we obtain $c_1 = 0$ and $P(g) = cP(f)$.

Since $P(z)$ is a strong uniqueness polynomial of meromorphic functions, we obtain $f = g$

Theorem 1 is proved. ∎

3.2. The proof of Theorem 2. We first prove the following:

Let f, g be non-constant meromorphic functions and n, m be positive integers, $n \geq m + 2$. Then

$$\Theta\Big(\infty,\ \big((f^n)^{(m)},\ (g^n)^{(m)}\big)\Big) + \Theta\Big(0,\ \big((f^n)^{(m)},\ (g^n)^{(m)}\big)\Big) \geq \frac{2(n-1)}{n+m}. \tag{3.1}$$

$$\Theta\Big(\infty,\ \big((f^n)^{(m)},\ (g^n)^{(m)}\big)\Big) \geq \frac{m}{m+1}. \tag{3.2}$$

Proof of (3.1). Because $n \geq m+2$, $(f^n)^{(m)}$ is not a constant. Write $(f^n)^{(m)} = = f^{n-m}F$. Then

$$\frac{F}{f^m} = \frac{(f^n)^{(m)}}{f^n}.$$

We see that if z_0 is a pole of $\frac{(f^n)^{(m)}}{f^n}$, then z_0 is either a pole of or a zero of f. By Lemma 2.3, if α, β are a pole and a zero of f, g, respectively, then

$$\frac{(f^n)^{(m)}}{f^n} = \frac{B}{(z-\alpha)^m}, \quad \frac{(f^n)^{(m)}}{f^n} = \frac{C}{(z-\beta)^m},$$

where $B(\alpha), C(\beta) \neq 0$.

From this it follows that

$$\begin{aligned} N\left(r, \frac{1}{F}; f \neq 0\right) &= N\left(r, \frac{1}{F/f^m}\right) \leq T\left(r, \frac{F}{f^m}\right) + S(r,f) \leq \\ &\leq N\left(r, \frac{F}{f^m}\right) + m\left(r, \frac{F}{f^m}\right) + S(r,f) \leq \\ &\leq m\overline{N}(r,f) + m\overline{N}\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned}$$

Therefore,

$$N\left(r, \frac{1}{F}; f \neq 0\right) \leq m\overline{N}(r,f) + m\overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

So we have

$$\overline{N}\left(r,\frac{1}{(f^n)^{(m)}}\right)==\overline{N}\left(r,\ \frac{1}{f^{n-m}F}\right)\le\overline{N}\left(r,\ \frac{1}{f}\right)+N\left(r,\ \frac{1}{F};\ f\neq 0\right)$$

$$\le\overline{N}\left(r,\frac{1}{f}\right)+m\overline{N}(r,f)+m\overline{N}\left(r,\ \frac{1}{f}\right)+S(r,f) \tag{3.3}$$

$$\le (m+1)\overline{N}\left(r,\ \frac{1}{f}\right)+m\overline{N}(r,f)+S(r,f).$$

By Lemmas 2.2 and 2.3, if α is a pole of order p, and β is a zero of multiplicity l, then $(f^n)^{(m)}=\frac{B_m}{(z-\alpha)^{np+m}}$, $B_m(\alpha)\neq 0$, and $(f^n)^{(m)}=C_m(z-\beta)^{nl-m}$, $C_m(\beta)\neq 0$.

Thus, we see that

$$N\left(r,\ \frac{1}{(f^n)^{(m)}}\right)-\overline{N}\left(r,\ \frac{1}{(f^n)^{(m)}}\right)\ge(n-1)(m+1)\overline{N}_{(m+1}\left(r,\ \frac{1}{f}\right)+ \tag{3.4}$$

$$+(n-m-1)\overline{N}_{m)}\left(r,\ \frac{1}{f}\right).$$

On the other hand,

$$\overline{N}\left(r,\ \frac{1}{f}\right)=\overline{N}_{m)}\left(r,\ \frac{1}{f}\right)+\overline{N}_{(m+1}\left(r,\ \frac{1}{f}\right).$$

From this and (3.3), (3.4) we obtain

$$\overline{N}_{m)}\left(r,\ \frac{1}{f}\right)\le\frac{1}{n-m-1}\left[N\left(r,\frac{1}{(f^n)^{(m)}}\right)-\overline{N}\left(r,\ \frac{1}{(f^n)^{(m)}}\right)-\right.$$

$$\left.-(n-1)(m+1)\overline{N}_{(m+1}\left(r,\ \frac{1}{f}\right)\right],$$

and

$$\overline{N}\left(r,\frac{1}{(f^n)^{(m)}}\right)\le(m+1)\overline{N}\left(r,\frac{1}{f}\right)+m\overline{N}(r,f)+S(r,f)\le$$

$$\le(m+1)\overline{N}_{m)}\left(r,\frac{1}{f}\right)+(m+1)\overline{N}_{(m+1}\left(r,\frac{1}{f}\right)+m\overline{N}(r,f)+S(r,f)\le$$

$$\le(m+1)\overline{N}_{(m+1}\left(r,\frac{1}{f}\right)+m\overline{N}(r,f)+\frac{m+1}{n-m-1}\left[N\left(r,\frac{1}{(f^n)^{(m)}}\right)-\right.$$

$$\left.-\overline{N}\left(r,\frac{1}{(f^n)^{(m)}}\right)-(n-1)(m+1)\overline{N}_{(m+1}\left(r,\frac{1}{f}\right)\right]+S(r,f).$$

Thus,

$$\frac{n}{n-m-1}\overline{N}\Big(r, \frac{1}{(f^n)^{(m)}}\Big) \le \frac{m+1}{n-m-1}N\Big(r, \frac{1}{(f^n)^{(m)}}\Big) + m\overline{N}(r,f) +$$
$$+\Big(m+1-\frac{(m+1)^2(n-1)}{n-m-1}\Big)\overline{N}\Big(r, \frac{1}{f}\Big) + S(r,f).$$

Noting that

$$m+1-\frac{(m+1)^2(n-1)}{n-m-1} < 0,$$

we have

$$\overline{N}\Big(r, \frac{1}{(f^n)^{(m)}}\Big) \le \frac{m+1}{n}N\Big(r, \frac{1}{(f^n)^{(m)}}\Big) + \frac{m(n-m-1)}{n}\overline{N}(r,f) + S(r,f).$$

Moreover, each pole of f of order p is a pole of $(f^n)^{(m)}$ of order $np+m \ge n+m$. Thus,

$$\frac{1}{n+m}N\big(r,(f^n)^{(m)}\big) \ge \overline{N}(r,f), \quad \overline{N}\big(r,(f^n)^{(m)}\big) = \overline{N}(r,f).$$

Therefore,

$$\overline{N}\big(r,(f^n)^{(m)}\big) \le \frac{1}{n+m}T\big(r,(f^n)^{(m)}\big) + S(r,f), \tag{3.5}$$

$$\begin{aligned}\overline{N}\Big(r, \frac{1}{(f^n)^{(m)}}\Big) &\le \frac{m+1}{n}T\big(r,(f^n)^{(m)}\big) + \\ &+\frac{m(n-m-1)}{n(n+m)}T\big(r,(f^n)^{(m)}\big) + S(r,f).\end{aligned} \tag{3.6}$$

Similarly,

$$\overline{N}\big(r,(g^n)^{(m)}\big) \le \frac{1}{n+m}T\big(r,(g^n)^{(m)}\big) + S(r,g), \tag{3.7}$$

$$\begin{aligned}\overline{N}\Big(r, \frac{1}{(g^n)^{(m)}}\Big) &\le \frac{m+1}{n}T\big(r,(g^n)^{(m)}\big) + \\ &+\frac{m(n-m-1)}{n(n+m)}T\big(r,(g^n)^{(m)}\big) + S(r,g).\end{aligned} \tag{3.8}$$

From (3.5)–(3.8) we get

$$\overline{N}\Big(r, \frac{1}{(f^n)^{(m)}}\Big) + \overline{N}\Big(r, \frac{1}{(g^n)^{(m)}}\Big) + \overline{N}\big(r,(f^n)^{(m)}\big) + \overline{N}(r,(g^n)^{(m)}) \le$$
$$\le \frac{2(m+1)}{n+m}\big(T(r,(f^n)^{(m)}) + T(r,(g^n)^{(m)})\big) + S(r,f) + S(r,g).$$

Therefore,

$$\big[1-\Theta\big(0,\ ((f^n)^{(m)},(g^n)^{(m)})\big)\big]+\big[1-\Theta\big(\infty,\ ((f^n)^{(m)},(g^n)^{(m)})\big)\big]\le\frac{2(m+1)}{n+m}.$$

It implies

$$\Theta\big(\infty,\ \big((f^n)^{(m)},(g^n)^{(m)}\big)\big)+\Theta\big(0,\ \big((f^n)^{(m)},(g^n)^{(m)}\big)\big)\ge\frac{2(n-1)}{n+m}.$$

Proof of (3.2). From (3.5) we get

$$\overline{N}\big(r,(f)^{(m)}\big)\le\frac{1}{m+1}T\big(r,(f)^{(m)}\big)+S(r,f),$$
$$\overline{N}\big(r,(g)^{(m)}\big)\le\frac{1}{m+1}T\big(r,(g)^{(m)}\big)+S(r,g).$$

From this we obtain

$$\overline{N}\big(r,(f)^{(m)}\big)+\overline{N}\big(r,(g)^{(m)}\big)\le$$
$$\le\frac{1}{m+1}\Big(T\big(r,(f)^{(m)}\big)+T\big(r,(g)^{(m)}\big)\Big)+S(r,f)+S(r,g).$$

Therefore,

$$\Theta\big(\infty,\ \big((f^n)^{(m)},(g^n)^{(m)}\big)\big)\ge\frac{m}{m+1}.$$

Now we return to the proof of Theorem 2.

Proof of Part 1 of Theorem 2. Suppose $n\ge m+2, q>2k+6-\dfrac{8(n-1)}{n+m}$ and for two meromorphic functions f,g we have

$$E_{(f^n)^{(m)}}(S)=E_{(g^n)^{(m)}}(S).$$

By (3.1),

$$\Theta\big(\infty,\ \big((f^n)^{(m)},(g^n)^{(m)}\big)\big)+\Theta\big(0,\ \big((f^n)^{(m)},(g^n)^{(m)}\big)\big)\ge\frac{2(n-1)}{n+m}.$$

Therefore,

$$q>2k+6-4(\Theta(\infty,((f^n)^{(m)},(g^n)^{(m)}))+\Theta(0,((f^n)^{(m)},(g^n)^{(m)}))). \tag{3.9}$$

Part 1 of Theorem 2 is proved by applying Theorem 1 and (3.9).

Proof of Part 2 of Theorem 2. It suffices to apply Theorem 1 and (3.2). ∎

4. Corollaries

Now we give some corollaries of Theorem 1 and Theorem 2.

In Theorem 1, if we take $k = 2$, then the hypothesis is satisfied for $q \geq 11$, and we obtain again URS's with 11 elements. As it is mentioned, so far the smallest URS for meromorphic functions has 11 elements.

Note that, for the polynomial P_B, defined in Lemma 2.6 we have:

$$P'(z) = (n + p + 1)z^n(z - a)^p,$$

and in this case $k = 2$.

Corollary 4.1. *Let P_F, P_B be polynomials defined in Lemmas 2.5 and 2.6, and let S_F, S_B, respectively, be their zero sets. If $k = 2, q \geq 11$, then S_F, S_B, are URS for meromorphic functions.*

Moreover, for the classes of meromorphic functions, satisfying the hypothesis of Theorem 1, we present URS with 5, 6, 7, 8, 9, 10 elements. Similar results for derivatives of meromorphic functons are also obtained.

Corollary 4.2. *Let f and g be two non-constant meromorphic functions, P_F, P_B be polynomials defined in Lemma 2.5 and Lemma 2.6, and let S_F, S_B be their zero sets, respectively. Assume one of the following conditions is satisfied:*

1. $E_f(S_F) = E_g(S_F)$, $q \geq 5, k = 2$ *and* $\Theta\big(\infty, (f, g)\big) + \Theta\big(0, (f, g)\big) > \frac{3}{2}$.
2. $E_f(S_B) = E_g(S_B)$ *and*
   i) $q \geq 6$, $\Theta\big(\infty, (f, g)\big) + \Theta\big(0, (f, g)\big) > 1$, *or*
   ii) $q \geq 7$, $\Theta\big(\infty, (f, g)\big) + \Theta\big(0, (f, g)\big) > \frac{3}{4}$, *or*
   iii) $q \geq 8$, $\Theta\big(\infty, (f, g)\big) + \Theta\big(0, (f, g)\big) > \frac{1}{2}$, *or*
   iv) $q \geq 9$, $\Theta\big(\infty, (f, g)\big) + \Theta\big(0, (f, g)\big) > \frac{1}{4}$, *or*
   v) $q \geq 10$, $\Theta\big(\infty, (f, g)\big) + \Theta\big(0, (f, g)\big) > 0$.

Then $f = g$.

For the proof, it suffices to apply Theorem 1.

Let f and g be two non-constant meromorphic functions. Note that, if all zeros and poles of f and g have multiplicities at least s, l, respectively, then $\frac{1}{s} + \frac{1}{l} \geq 2 - \big(\Theta(\infty, (f, g)) + \Theta(0, (f, g)\big)$. From this and Corollary 4.2 we obtain the following corollary.

Corollary 4.3. *Let f and g be two non-constant meromorphic functions and P_F, P_B be polynomials defined in Lemma 2.5, Lemma 2.6, and let S_F, S_B be* *their zero sets, respectively. Assume that* $E_f(S_B) = E_g(S_B)$ *and one of the following conditions holds:*

i) $q \geq 6$, $\frac{1}{s} + \frac{1}{l} < 1$, *or*
ii) $q \geq 7$, $\frac{1}{s} + \frac{1}{l} < \frac{5}{4}$, *or*
iii) $q \geq 8$, $\frac{1}{s} + \frac{1}{l} < \frac{3}{2}$, *or*
iv) $q \geq 9$, $\frac{1}{s} + \frac{1}{l} < \frac{7}{4}$, *or*
v) $q = 10$, $\frac{1}{s} + \frac{1}{l} < 2$.

Then $f = g$.

From Theorem 2 we obtain the following corollaries for derivatives of meromorphic functions.

Corollary 4.4. *Let* f *and* g *be two non-constant meromorphic functions and* P_F, P_B *be polynomials defined in Lemma 2.5, Lemma 2.6, and let* S_F, S_B *be their zero sets, respectively. Assume that* $E_{(f^n)^{(m)}}(S_B) = E_{(g^n)^{(m)}}(S_B)$ *and one of the following conditions holds:*

i) $q \geq 6$, $n \geq m + 3$, *or*
ii) $q \geq 7$, $n \geq m + 2$.

Then $f = cg$ *with* $c^n = 1$.

Corollary 4.5. *Let* f *and* g *be two non-constant meromorphic functions,* n, m *be positive integers,* P_B *be the polynomial defined in Lemma 2.6,* S_B *be the zero set of* P_B. *Assume* $E_{(f)^{(m)}}(S_B) = E_{(g)^{(m)}}(S_B)$ *and one of the following conditions holds:*

1. $q \geq 7$ *and* $m > 3$.
2. $q \geq 8$ *and* $m > 1$.
3. $q \geq 9$.

Then $f^{(m)} = g^{(m)}$.

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