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ABSTRACT FRACTIONAL LANDAU INEQUALITIES

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Abstract. We present uniform and L_p mixed Caputo–Bochner abstract fractional Landau inequalities over $\mathbb{R}$ of fractional orders $2 < \nu \leq 3$. These estimate the size of first and second derivatives of a Banach space valued function over $\mathbb{R}$. We give applications when $\nu = 2.5$.

1. Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1], [5].

The research on these inequalities started by E. Landau [11] in 1913. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2}, \tag{2}$$

are the best constants in (1).

In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1. \tag{3}$$

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In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [8] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \quad \text{for } p \in [1, \infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

In fact, in [6] and [9] was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

In this work we are greatly inspired by interesting articles [10], [14].

We need the following concepts from abstract fractional calculus.

Our integrals next are of Bochner type [12].

We need

Definition 1. ([4], p. 150) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo–Bochner left fractional derivative of order α:

$$(D_{*a}^{\alpha} f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad \forall\, x \in [a,b], \tag{5}$$

where Γ is the gamma function.

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [13], p. 83), and also set $D_{*a}^{0} f := f$.

By ([4], p. 2), $(D_{*a}^{\alpha} f)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in$ $\in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, then by ([4], p. 3), $D_{*a}^{\alpha} f \in C([a,b], X)$, hence $\| D_{*a}^{\alpha} f \| \in C([a,b])$.

We also need

Definition 2. ([4], p. 150) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo–Bochner right fractional derivative of order α:

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b]. \tag{6}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) =$ $= f(x)$.

By ([4], p. 34), $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in$ $\in L_1([a,b], X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by ([4], p. 37), $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\|D_{b-}^{\alpha} f\right\| \in C([a,b])$.

We mention the important:

Corollary 3. ([4], p. 157) *Let $f \in C^m([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in$ $\in [a,b]$. Then $D_{*x_0}^{a} f(x)$, $D_{x_0-}^{a} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into X, X is a Banach space.*

By convention we suppose that

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0,$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0,$$

for all $x, x_0 \in [a,b]$.

The author has already done an extensive body of work on fractional Landau inequalities, see [3], and on abstract fractional Landau inequalities, see [4]. However there the proving methods came out of applications of fractional Ostrowski inequalities ([2], [4]) and the derived inequalities were for small fractional orders, i.e. $\alpha \in (0,1)$. Usually there the domains where $[A, +\infty)$ or $(-\infty, B]$, with $A, B \in \mathbb{R}$ and in one mixed case the domain was all of $\mathbb{R}$.

In this work with less assumptions we establish uniform and L_p type mixed Caputo–Bochner abstract fractional Landau inequalities over $\mathbb{R}$ for fractional orders $2 < \nu \leq 3$. The method of proving is based on left and right Caputo–Bochner fractional Taylor's formulae with integral remainders, see [4], pp. 151 – 152.

We give also applications when $\nu = 2.5$. Regardless to say that we are also inspired by [3], [4].

2. Main results

We present the following abstract mixed fractional Landau inequalities over $\mathbb{R}$.

Theorem 4. *Let $2 < \nu \leq 3$ and $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. We assume that $\|\|f\|\|_{\infty,\mathbb{R}} < \infty$, and that*

$$K := \max\left\{\left\|\left\|D_{*a}^{\nu} f(z)\right\|\right\|_{\infty,\mathbb{R}^2}, \left\|\left\|D_{a-}^{\nu} f(z)\right\|\right\|_{\infty,\mathbb{R}^2}\right\} < \infty, \tag{7}$$

where $(a,z) \in \mathbb{R}^2$.

Then

$$\left\Vert \left\Vert f^{\prime }\right\Vert \right\Vert _{\infty ,\mathbb{R}}\leq \nu \left(\frac{K}{\Gamma \left(\nu +1\right) }\right) ^{\frac{1}{\nu }}\left(\frac{\left\Vert \left\Vert f\right\Vert \right\Vert _{\infty ,\mathbb{R}}}{\nu -1}\right) ^{\frac{\nu -1}{\nu }}, \tag{8}$$

and

$$\left\Vert \left\Vert f^{\prime \prime }\right\Vert \right\Vert _{\infty ,\mathbb{R}}\leq \nu \left(\frac{K}{\Gamma \left(\nu +1\right) }\right) ^{\frac{2}{\nu }}\left(\frac{4\left\Vert \left\Vert f\right\Vert \right\Vert _{\infty ,\mathbb{R}}}{\nu -2}\right) ^{\frac{\nu -2}{\nu }}. \tag{9}$$

That is $\left\Vert \left\Vert f^{\prime }\right\Vert \right\Vert _{\infty ,\mathbb{R}},\left\Vert \left\Vert f^{\prime \prime }\right\Vert \right\Vert _{\infty ,\mathbb{R}}<\infty$.

Proof. Here $2<\nu \leq 3$, i.e. $\lceil \nu \rceil =3$. Let $f\in C^{3}\left(\mathbb{R},X\right)$, where $\left(X,\left\Vert \cdot \right\Vert \right)$ is a Banach space, $a\in \mathbb{R}$.

We need the following abstract fractional Taylor formulae.

By Theorem 5.8, p. 151 ([4]) we get,

$$\begin{gathered} f\left(x\right) -f\left(a\right) -\left(x-a\right) f^{\prime }\left(a\right) -\frac{\left(x-a\right) ^{2}}{2}f^{\prime \prime }\left(a\right) = \\ =\frac{1}{\Gamma \left(\nu \right) }\int_{a}^{x}\left(x-z\right) ^{\nu -1}\left(D_{\ast a}^{\nu }f\right) \left(z\right) dz,\quad \forall \ x\geq a. \end{gathered} \tag{10}$$

And by Theorem 5.9, p. 152 ([4]), see also Definition 5.10, p. 152 ([4]):

$$\begin{gathered} f\left(x\right) -f\left(a\right) -\left(x-a\right) f^{\prime }\left(a\right) -\frac{\left(x-a\right) ^{2}}{2}f^{\prime \prime }\left(a\right) = \\ =\frac{1}{\Gamma \left(\nu \right) }\int_{x}^{a}\left(z-x\right) ^{\nu -1}\left(D_{a-}^{\nu }f\right) \left(z\right) dz,\quad \forall \ x\leq a. \end{gathered} \tag{11}$$

The remainders are continuous functions.

Let $x_{1}>a$, then

$$\begin{gathered} f\left(x_{1}\right) -f\left(a\right) -\left(x_{1}-a\right) f^{\prime }\left(a\right) -\frac{\left(x_{1}-a\right) ^{2}}{2}f^{\prime \prime }\left(a\right) = \\ =\frac{1}{\Gamma \left(\nu \right) }\int_{a}^{x_{1}}\left(x_{1}-z\right) ^{\nu -1}\left(D_{\ast a}^{\nu }f\right) \left(z\right) dz, \end{gathered} \tag{12}$$

and for $x_{2}<a$, we get

$$\begin{gathered} f\left(x_{2}\right) -f\left(a\right) -\left(x_{2}-a\right) f^{\prime }\left(a\right) -\frac{\left(x_{2}-a\right) ^{2}}{2}f^{\prime \prime }\left(a\right) = \\ =\frac{1}{\Gamma \left(\nu \right) }\int_{x_{2}}^{a}\left(z-x_{2}\right) ^{\nu -1}\left(D_{a-}^{\nu }f\right) \left(z\right) dz. \end{gathered} \tag{13}$$

That is, we have the system of two equations with unknowns $f'(a), f''(a)$:

$$(x_1 - a) f'(a) + \frac{(x_1 - a)^2}{2} f''(a) = \tag{14}$$
$$= f(x_1) - f(a) - \frac{1}{\Gamma(\nu)} \int_a^{x_1} (x_1 - z)^{\nu - 1} (D_{*a}^{\nu} f)(z) \, dz =: A,$$

and

$$(x_2 - a) f'(a) + \frac{(x_2 - a)^2}{2} f''(a) = \tag{15}$$
$$= f(x_2) - f(a) - \frac{1}{\Gamma(\nu)} \int_{x_2}^{a} (z - x_2)^{\nu - 1} \left(D_{a-}^{\nu} f\right)(z) \, dz =: B.$$

The coefficient determinant is $D = \frac{1}{2}(x_1 - a)(x_2 - a)(x_2 - x_1)$. It is $D \neq 0$.

We have the system:

$$\begin{aligned} (x_1 - a) f'(a) + \tfrac{(x_1 - a)^2}{2} f''(a) &= A, \\ (x_2 - a) f'(a) + \tfrac{(x_2 - a)^2}{2} f''(a) &= B. \end{aligned} \tag{16}$$

Solving the system (16), it has a unique non-trivial solution, we get:

$$f'(a) = \frac{\begin{vmatrix} A & \frac{(x_1 - a)^2}{2} \\ B & \frac{(x_2 - a)^2}{2} \end{vmatrix}}{\frac{1}{2}(x_1 - a)(x_2 - a)(x_2 - x_1)} = \frac{A (x_2 - a)^2 - B (x_1 - a)^2}{(x_1 - a)(x_2 - a)(x_2 - x_1)}, \tag{17}$$

and

$$f''(a) = \frac{\begin{vmatrix} x_1 - a & A \\ x_2 - a & B \end{vmatrix}}{\frac{1}{2}(x_1 - a)(x_2 - a)(x_2 - x_1)} = \frac{2 \left[B (x_1 - a) - A (x_2 - a)\right]}{(x_1 - a)(x_2 - a)(x_2 - x_1)}. \tag{18}$$

We derive that

$$f'(a) = \left[\left[f(x_1) - f(a) - \frac{1}{\Gamma(\nu)} \int_a^{x_1} (x_1 - z)^{\nu - 1} (D_{*a}^{\nu} f)(z) \, dz\right] (x_2 - a)^2 - \right.$$
$$\left. - \left[f(x_2) - f(a) - \frac{1}{\Gamma(\nu)} \int_{x_2}^{a} (z - x_2)^{\nu - 1} \left(D_{a-}^{\nu} f\right)(z) \, dz\right] (x_1 - a)^2\right] \cdot \tag{19}$$
$$\cdot \frac{1}{(x_1 - a)(x_2 - a)(x_2 - x_1)},$$

and

$$f''(a)=2\left[\left[f(x_2)-f(a)-\frac{1}{\Gamma(\nu)}\int_{x_2}^{a}(z-x_2)^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right](x_1-a)-\right.$$

$$(20)\qquad \left.-\left[f(x_1)-f(a)-\frac{1}{\Gamma(\nu)}\int_{a}^{x_1}(x_1-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz\right](x_2-a)\right]\cdot \cdot\frac{1}{(x_1-a)(x_2-a)(x_2-x_1)}.$$

Simplifying things we have

$$(21)\qquad f'(a)=\frac{\left[f(x_1)-f(a)-\frac{1}{\Gamma(\nu)}\int_{a}^{x_1}(x_1-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz\right](x_2-a)}{(x_1-a)(x_2-x_1)}- -\frac{\left[f(x_2)-f(a)-\frac{1}{\Gamma(\nu)}\int_{x_2}^{a}(z-x_2)^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right](x_1-a)}{(x_2-a)(x_2-x_1)},$$

and

$$(22)\qquad f''(a)=\frac{2\left[f(x_2)-f(a)-\frac{1}{\Gamma(\nu)}\int_{x_2}^{a}(z-x_2)^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right]}{(x_2-a)(x_2-x_1)}- -\frac{2\left[f(x_1)-f(a)-\frac{1}{\Gamma(\nu)}\int_{a}^{x_1}(x_1-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz\right]}{(x_1-a)(x_2-x_1)}.$$

Rewriting above solutions we have

$$(23)\qquad f'(a)=\frac{(f(x_1)-f(a))(x_2-a)}{(x_1-a)(x_2-x_1)}-\frac{(f(x_2)-f(a))(x_1-a)}{(x_2-a)(x_2-x_1)}- -\frac{\left(\int_{a}^{x_1}(x_1-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz\right)(x_2-a)}{\Gamma(\nu)(x_1-a)(x_2-x_1)}+ +\frac{\left(\int_{x_2}^{a}(z-x_2)^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right)(x_1-a)}{\Gamma(\nu)(x_2-a)(x_2-x_1)},$$

and

$$f''(a)=\frac{2(f(x_2)-f(a))}{(x_2-a)(x_2-x_1)}-\frac{2(f(x_1)-f(a))}{(x_1-a)(x_2-x_1)}- \tag{24}$$

$$-\frac{2\left(\int\limits_{x_2}^{a}(z-x_2)^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right)}{\Gamma(\nu)(x_2-a)(x_2-x_1)}+\frac{2\left(\int\limits_{a}^{x_1}(x_1-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz\right)}{\Gamma(\nu)(x_1-a)(x_2-x_1)}.$$

Let now $h>0$, and choose $x_1=a+h$ and $x_2=a-h$. That is $x_1-a=h$, $x_2-a=-h$, and $x_2-x_1=-2h$.

We derive the following:

$$f'(a)=\frac{(f(a+h)-f(a))}{2h}-\frac{(f(a-h)-f(a))}{2h}- \tag{25}$$

$$-\frac{\left(\int\limits_{a}^{a+h}(a+h-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz\right)}{2h\Gamma(\nu)}+\frac{\left(\int\limits_{a-h}^{a}(z-a+h)^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right)}{2h\Gamma(\nu)},$$

and similarly,

$$f''(a)=\frac{(f(a-h)-f(a))}{h^2}-\frac{(f(a+h)-f(a))}{h^2}- \tag{26}$$

$$-\frac{\left(\int\limits_{a-h}^{a}(z-a+h)^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right)}{h^2\Gamma(\nu)}-\frac{\left(\int\limits_{a}^{a+h}(a+h-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz\right)}{h^2\Gamma(\nu)}.$$

Putting things together we derive

$$f'(a)=\frac{(f(a+h)-f(a-h))}{2h}-\frac{1}{2h\Gamma(\nu)}\cdot \tag{27}$$

$$\cdot\left[\int\limits_{a}^{a+h}(a+h-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz-\int\limits_{a-h}^{a}(z-(a-h))^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right],$$

and

$$f''(a)=\frac{(f(a+h)+f(a-h)-2f(a))}{h^2}-\frac{1}{h^2\Gamma(\nu)}\cdot \tag{28}$$

$$\cdot\left[\int\limits_{a}^{a+h}(a+h-z)^{\nu-1}\left(D_{*a}^{\nu}f\right)(z)\,dz+\int\limits_{a-h}^{a}(z-(a-h))^{\nu-1}\left(D_{a-}^{\nu}f\right)(z)\,dz\right].$$

By Corollary 3 we obtain that $(D_{*a}^{\nu}f)(z)$, $\left(D_{a-}^{\nu}f\right)(z)$ are jointly continuous functions in (z,a) from $\mathbb{R}^2$ into X, where X is a Banach space. We assumed here that $\left\|\left\|D_{*a}^{\nu}f(z)\right\|\right\|_{\infty,\mathbb{R}^2}$, $\left\|\left\|D_{a-}^{\nu}f(z)\right\|\right\|_{\infty,\mathbb{R}^2}<\infty$.

By (27) we get that

$$\|f'(a)\|\leq\frac{\|f(a+h)-f(a-h)\|}{2h}+$$
$$+\frac{1}{2h\Gamma(\nu)}\left[\int_a^{a+h}(a+h-z)^{\nu-1}\|(D_{*a}^{\nu}f)(z)\|\,dz+\right.$$
$$\left.+\int_{a-h}^{a}(z-(a-h))^{\nu-1}\left\|\left(D_{a-}^{\nu}f\right)(z)\right\|dz\right]\leq\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+ \tag{29}$$
$$+\frac{K}{2h\Gamma(\nu)}\left[\int_a^{a+h}(a+h-z)^{\nu-1}dz+\int_{a-h}^{a}(z-(a-h))^{\nu-1}dz\right]=$$
$$=\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\frac{K}{2h\Gamma(\nu)}\left[\frac{2h^{\nu}}{\nu}\right]=\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\frac{Kh^{\nu-1}}{\Gamma(\nu+1)}.$$

That is

$$\|f'(a)\|\leq\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\frac{Kh^{\nu-1}}{\Gamma(\nu+1)}, \tag{30}$$

$\forall\ a\in\mathbb{R}$, $\forall\ h>0$.

I.e. it holds

$$\|\|f'\|\|_{\infty,\mathbb{R}}\leq\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\frac{Kh^{\nu-1}}{\Gamma(\nu+1)}, \tag{31}$$

$\forall\ h>0$, $2<\nu\leq3$.

Call

$$\mu:=\|\|f\|\|_{\infty,\mathbb{R}}\,,\qquad \theta=\tfrac{K}{\Gamma(\nu+1)}, \tag{32}$$

both are greater than zero.

Set also $\rho:=\nu-1>1$.

We consider the function

$$y(h):=\frac{\mu}{h}+\theta h^{\rho},\quad\forall\ h>0. \tag{33}$$

We have

$$y'(h) = -\frac{\mu}{h^2} + \rho\theta h^{\rho-1} = 0, \tag{34}$$

then

$$\rho\theta h^{\rho-1} = \frac{\mu}{h^2},$$

and

$$\rho\theta h^{\rho+1} = \mu,$$

with a unique solution

$$h_0 := h_{crit.no} = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}. \tag{35}$$

We have that

$$y'(h) = -\mu h^{-2} + \rho\theta h^{\rho-1},$$

and

$$y''(h) = 2\mu h^{-3} + \rho(\rho-1)\theta h^{\rho-2}. \tag{36}$$

We see that

$$\begin{aligned} y''(h_0) &= 2\mu\left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}} + \rho(\rho-1)\theta\left(\frac{\mu}{\rho\theta}\right)^{\frac{((\rho+1)-3)}{\rho+1}} = \\ &= \left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}}\left[2\mu + \rho(\rho-1)\theta\left(\frac{\mu}{\rho\theta}\right)\right] = \\ &= \left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}}\left[2\mu + \mu(\rho-1)\right] = \mu\left(\frac{\mu}{\rho\theta}\right)^{-\frac{3}{\rho+1}}(\rho+1) > 0. \end{aligned} \tag{37}$$

Therefore y has a global minimum at $h_0 = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}$, which is

$$\begin{aligned} y(h_0) &= \frac{\mu}{\left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}} + \theta\left(\frac{\mu}{\rho\theta}\right)^{\frac{\rho}{\rho+1}} = \\ &= (\rho\theta)^{\frac{1}{\rho+1}}\frac{\mu}{\mu^{\frac{1}{\rho+1}}} + \frac{\theta\mu^{\frac{\rho}{\rho+1}}}{\rho^{\frac{\rho}{\rho+1}}\theta^{\frac{\rho}{\rho+1}}} = (\rho\theta)^{\frac{1}{\rho+1}}\mu^{1-\frac{1}{\rho+1}} + \frac{\theta^{1-\frac{\rho}{\rho+1}}\mu^{\frac{\rho}{\rho+1}}}{\rho^{\frac{\rho}{\rho+1}}} = \\ &= \rho^{\frac{1}{\rho+1}}\theta^{\frac{1}{\rho+1}}\mu^{\frac{\rho}{\rho+1}} + \theta^{\frac{1}{\rho+1}}\mu^{\frac{\rho}{\rho+1}}\rho^{-\frac{\rho}{\rho+1}} = \theta^{\frac{1}{\rho+1}}\mu^{\frac{\rho}{\rho+1}}\left[\rho^{\frac{1}{\rho+1}} + \rho^{-\frac{\rho}{\rho+1}}\right] = \\ &= (\theta\mu^{\rho})^{\frac{1}{\rho+1}}\left(\rho^{\frac{1}{\rho+1}} + \frac{1}{\rho^{\frac{\rho}{\rho+1}}}\right) = (\theta\mu^{\rho})^{\frac{1}{\rho+1}}\left(\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\right) = \\ &= (\theta\mu^{\rho})^{\frac{1}{\rho+1}}(\rho+1)\rho^{-\frac{\rho}{\rho+1}}. \end{aligned} \tag{38}$$

That is

$$y\left(h_{0}\right)=\left(\theta\mu^{\rho}\right)^{\frac{1}{\rho+1}}\left(\rho+1\right)\rho^{-\frac{\rho}{\rho+1}}. \tag{39}$$

Consequently

$$y\left(h_{0}\right)=\left(\frac{K}{\Gamma\left(\nu+1\right)}\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}^{\nu-1}\right)^{\frac{1}{\nu}}\nu\left(\nu-1\right)^{-\left(\frac{\nu-1}{\nu}\right)}. \tag{40}$$

We have proved that

$$\left\|\left\|f'\right\|\right\|_{\infty,\mathbb{R}}\leq\left(\frac{K}{\Gamma\left(\nu+1\right)}\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}^{\nu-1}\right)^{\frac{1}{\nu}}\nu\left(\nu-1\right)^{-\left(\frac{\nu-1}{\nu}\right)}. \tag{41}$$

From (28) we derive

$$\begin{gathered}\left\|f''\left(a\right)\right\|\leq\frac{4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h^{2}}+\\ +\frac{K}{h^{2}\Gamma\left(\nu\right)}\left[\int_{a}^{a+h}\left(a+h-z\right)^{\nu-1}dz+\int_{a-h}^{a}\left(z-\left(a-h\right)\right)^{\nu-1}dz\right]=\\ =\frac{4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h^{2}}+\frac{K}{h^{2}\Gamma\left(\nu\right)}\left[\frac{2h^{\nu}}{\nu}\right]=\frac{4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h^{2}}+\frac{2Kh^{\nu-2}}{\Gamma\left(\nu+1\right)}.\end{gathered} \tag{42}$$

That is

$$\left\|f''\left(a\right)\right\|\leq\frac{4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h^{2}}+\frac{2Kh^{\nu-2}}{\Gamma\left(\nu+1\right)}, \tag{43}$$

$\forall\ a\in\mathbb{R}$, $\forall\ h>0$.

Hence it holds

$$\left\|\left\|f''\right\|\right\|_{\infty,\mathbb{R}}\leq\frac{4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}}}{h^{2}}+\frac{2Kh^{\nu-2}}{\Gamma\left(\nu+1\right)}, \tag{44}$$

$\forall\ h>0$, $2<\nu\leq3$.

Call

$$\begin{gathered}\xi:=4\left\|\left\|f\right\|\right\|_{\infty,\mathbb{R}},\\ \psi=\tfrac{2K}{\Gamma(\nu+1)},\end{gathered} \tag{45}$$

both are greater than zero.

Set also $\varphi := \nu - 2 > 0$.

We consider the function

$$\gamma(h) := \frac{\xi}{h^2} + \psi h^{\varphi} = \xi h^{-2} + \psi h^{\varphi}, \quad \forall\ h > 0. \tag{46}$$

We have

$$\gamma'(h) = -2\xi h^{-3} + \varphi\psi h^{\varphi-1} = 0, \tag{47}$$

then

$$\varphi\psi h^{\varphi-1} = 2\xi h^{-3},$$

and

$$\varphi\psi h^{\varphi+2} = 2\xi,$$

with a unique solution

$$h_0 := h_{crit.no} = \left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}. \tag{48}$$

We have that

$$\gamma''(h) = 6\xi h^{-4} + \varphi(\varphi-1)\psi h^{\varphi-2}. \tag{49}$$

We see that

$$\begin{aligned} \gamma''(h_0) &= 6\xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}} + \varphi(\varphi-1)\psi\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{(\varphi+2)-4}{\varphi+2}} = \\ &= \left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}\left[6\xi + \varphi(\varphi-1)\psi\frac{2\xi}{\varphi\psi}\right] = \\ &= \left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}\left[6\xi + (\varphi-1)2\xi\right] = 2\xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{4}{\varphi+2}}(\varphi+2) > 0. \end{aligned} \tag{50}$$

Therefore γ has a global minimum at $h_0 = \left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}$, which is

$$\begin{aligned} \gamma(h_0) &= \xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{2}{\varphi+2}} + \psi\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{\varphi+2-2}{\varphi+2}} = \\ &= \left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{2}{\varphi+2}}\left[\xi + \psi\frac{2\xi}{\varphi\psi}\right] = \xi\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{2}{\varphi+2}}\left[1 + \frac{2}{\varphi}\right] = \\ &= \frac{\xi}{\varphi}\left(\frac{2\xi}{\varphi\psi}\right)^{-\frac{2}{\varphi+2}}(\varphi+2) = \frac{\xi}{\varphi}\left(\frac{\xi}{\varphi}\right)^{-\frac{2}{\varphi+2}}\left(\frac{2}{\psi}\right)^{-\frac{2}{\varphi+2}}(\varphi+2) = \\ &= \left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}}\left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}}(\varphi+2). \end{aligned} \tag{51}$$

That is

$$\gamma(h_0) = \left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}} \left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}} (\varphi + 2). \tag{52}$$

Consequently

$$\gamma(h_0) = \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\nu - 2}\right)^{\frac{\nu-2}{\nu}} \left(\frac{K}{\Gamma(\nu+1)}\right)^{\frac{2}{\nu}} \nu. \tag{53}$$

We have proved that

$$\|\|f''\|\|_{\infty,\mathbb{R}} \leq \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\nu - 2}\right)^{\frac{\nu-2}{\nu}} \left(\frac{K}{\Gamma(\nu+1)}\right)^{\frac{2}{\nu}} \nu. \tag{54}$$

The theorem is proved. ■

We also give an L_p analog of a fractional Landau inequality

Theorem 5. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $2 < \nu \leq 3$ and $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. We assume that $\|\|f\|\|_{\infty,\mathbb{R}} < \infty$, and that*

$$M := \max\left\{\sup_{a\in\mathbb{R}} \|\|D_{*a}^{\nu} f\|\|_{p,\mathbb{R}},\ \sup_{a\in\mathbb{R}} \left\|\left\|D_{a-}^{\nu} f\right\|\right\|_{p,\mathbb{R}}\right\} < \infty. \tag{55}$$

Then

1)

$$\|\|f'\|\|_{\infty,\mathbb{R}} \leq$$

$$\leq \left(\nu - \frac{1}{p}\right) \left(\frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}\right)^{\frac{1}{\left(\nu-\frac{1}{p}\right)}} \left(\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{\nu - 1 - \frac{1}{p}}\right)^{\left(\frac{\nu-1-\frac{1}{p}}{\nu-\frac{1}{p}}\right)}, \tag{56}$$

and

2) *under the additional assumption $2 + \frac{1}{p} < \nu \leq 3$, we have*

$$\|\|f''\|\|_{\infty,\mathbb{R}} \leq$$

$$\leq \left(\nu - \frac{1}{p}\right) \left(\frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\nu-\frac{1}{p}}\right)} \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\nu - 2 - \frac{1}{p}}\right)^{\left(\frac{\nu-2-\frac{1}{p}}{\nu-\frac{1}{p}}\right)}. \tag{57}$$

That is $\|\|f'\|\|_{\infty,\mathbb{R}}, \|\|f''\|\|_{\infty,\mathbb{R}} < \infty$.

Proof. From (27) we get

$$
\begin{aligned}
\|f'(a)\| &\leq \frac{\|f(a+h)-f(a-h)\|}{2h}+\\
&+\frac{1}{2h\Gamma(\nu)}\left[\int_a^{a+h}(a+h-z)^{\nu-1}\left\|\left(D_{*a}^{\nu}f\right)(z)\right\|dz+\right.\\
&\left.+\int_{a-h}^{a}(z-(a-h))^{\nu-1}\left\|\left(D_{a-}^{\nu}f\right)(z)\right\|dz\right]\leq\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\\
&+\frac{1}{2h\Gamma(\nu)}\left[\left(\int_a^{a+h}(a+h-z)^{q(\nu-1)}dz\right)^{\frac{1}{q}}\left(\int_a^{a+h}\left\|\left(D_{*a}^{\nu}f\right)(z)\right\|^p dz\right)^{\frac{1}{p}}+\right.\\
&\left.+\left(\int_{a-h}^{a}(z-(a-h))^{q(\nu-1)}dz\right)^{\frac{1}{q}}\left(\int_{a-h}^{a}\left\|\left(D_{a-}^{\nu}f\right)(z)\right\|^p dz\right)^{\frac{1}{p}}\right]\leq\\
&\leq\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\frac{1}{2h\Gamma(\nu)}\frac{h^{\nu-\frac{1}{p}}}{(q(\nu-1)+1)^{\frac{1}{q}}}\cdot\\
&\cdot\left(\sup_{a\in\mathbb{R}}\|\|D_{*a}^{\nu}f\|\|_{p,\mathbb{R}}+\sup_{a\in\mathbb{R}}\left\|\left\|D_{a-}^{\nu}f\right\|\right\|_{p,\mathbb{R}}\right)\leq\\
&\leq\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\frac{Mh^{\left(\nu-1-\frac{1}{p}\right)}}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}.
\end{aligned}
\tag{58}
$$

That is

$$
\|f'(a)\|\leq\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}h^{\left(\nu-1-\frac{1}{p}\right)},
\tag{59}
$$

$\forall\ a\in\mathbb{R}$, $\forall\ h>0$.

I.e. it holds

$$
\|\|f'\|\|_{\infty,\mathbb{R}}\leq\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{h}+\frac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}h^{\left(\nu-1-\frac{1}{p}\right)},
\tag{60}
$$

$\forall\ h>0$, $2<\nu\leq 3$.

Call

$$
\begin{aligned}
\mu &:= \|\|f\|\|_{\infty,\mathbb{R}},\\
\theta &= \tfrac{M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}},
\end{aligned}
\tag{61}
$$

both are greater than zero.

Set also $\rho := \nu - 1 - \frac{1}{p} > \frac{1}{q} > 0$.

We consider the function

$$y(h) := \frac{\mu}{h} + \theta h^{\rho}, \quad \forall\, h > 0. \tag{62}$$

As in the proof of Theorem 4 it has only one critical number

$$h_0 := h_{crit.no} = \left(\frac{\mu}{\rho\theta}\right)^{\frac{1}{\rho+1}}, \tag{63}$$

and a global minimum

$$y(h_0) = \theta^{\frac{1}{\rho+1}} \mu^{\frac{\rho}{\rho+1}} (\rho+1)\, \rho^{-\frac{\rho}{\rho+1}}. \tag{64}$$

Consequently

$$\begin{aligned} y(h_0) = &\left(\frac{M}{\Gamma(\nu)\left(q(\nu-1)+1\right)^{\frac{1}{q}}}\right)^{\frac{1}{\left(\nu-\frac{1}{p}\right)}} \left(\|\|f\|\|_{\infty,\mathbb{R}}\right)^{\left(\frac{\nu-1-\frac{1}{p}}{\nu-\frac{1}{p}}\right)} \cdot \\ &\cdot \left(\nu - \frac{1}{p}\right)\left(\nu - 1 - \frac{1}{p}\right)^{-\left(\frac{\nu-1-\frac{1}{p}}{\nu-\frac{1}{p}}\right)}. \end{aligned} \tag{65}$$

We have proved that

$$\begin{aligned} &\|\|f'\|\|_{\infty,\mathbb{R}} \leq \\ &\left(\frac{M}{\Gamma(\nu)\left(q(\nu-1)+1\right)^{\frac{1}{q}}}\right)^{\frac{1}{\left(\nu-\frac{1}{p}\right)}} \left(\frac{\|\|f\|\|_{\infty,\mathbb{R}}}{\nu-1-\frac{1}{p}}\right)^{\left(\frac{\nu-1-\frac{1}{p}}{\nu-\frac{1}{p}}\right)} \left(\nu - \frac{1}{p}\right). \end{aligned} \tag{66}$$

From (28) we derive

$$\begin{aligned} &\|f''(a)\| \leq \frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{h^2} + \\ &+ \frac{1}{h^2\Gamma(\nu)}\left[\int_a^{a+h} (a+h-z)^{\nu-1} \left\|\left(D_{*a}^{\nu} f\right)(z)\right\| dz + \right. \\ &\left. + \int_{a-h}^{a} (z-(a-h))^{\nu-1} \left\|\left(D_{a-}^{\nu} f\right)(z)\right\| dz\right] \leq \frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{h^2} + \\ &+ \frac{1}{h^2\Gamma(\nu)}\left[\left(\int_a^{a+h} (a+h-z)^{q(\nu-1)} dz\right)^{\frac{1}{q}} \sup_{a\in\mathbb{R}} \|\|D_{*a}^{\nu} f\|\|_{p,\mathbb{R}} + \right. \end{aligned} \tag{67}$$

$$+\left(\int_{a-h}^{a}(z-(a-h))^{q(\nu-1)}\,dz\right)^{\frac{1}{q}}\sup_{a\in\mathbb{R}}\left\|\left\|D_{a-}^{\nu}f\right\|\right\|_{p,\mathbb{R}}\Bigg]\leq$$

$$\leq\frac{4\left\|\|f\|\right\|_{\infty,\mathbb{R}}}{h^2}+\frac{M}{h^2\Gamma(\nu)}\left(\frac{2h^{\nu-\frac{1}{p}}}{(q(\nu-1)+1)^{\frac{1}{q}}}\right)= \tag{67}$$

$$=\frac{4\left\|\|f\|\right\|_{\infty,\mathbb{R}}}{h^2}+\frac{2Mh^{\nu-2-\frac{1}{p}}}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}.$$

That is

$$\|f''(a)\|\leq\frac{4\left\|\|f\|\right\|_{\infty,\mathbb{R}}}{h^2}+\left(\frac{2M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}\right)h^{\nu-2-\frac{1}{p}}, \tag{68}$$

$\forall\ a\in\mathbb{R}$, $\forall\ h>0$.

I.e. it holds

$$\left\|\|f''\|\right\|_{\infty,\mathbb{R}}\leq\frac{4\left\|\|f\|\right\|_{\infty,\mathbb{R}}}{h^2}+\left(\frac{2M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}\right)h^{\nu-2-\frac{1}{p}}, \tag{69}$$

$\forall\ h>0$, under $2+\frac{1}{p}<\nu\leq 3$.

Call

$$\xi:=4\left\|\|f\|\right\|_{\infty,\mathbb{R}},$$
$$\psi:=\frac{2M}{\Gamma(\nu)(q(\nu-1)+1)^{\frac{1}{q}}}, \tag{70}$$

both are greater than zero.

Set also $\varphi:=\nu-2-\frac{1}{p}>0$.

We consider the function

$$\gamma(h):=\frac{\xi}{h^2}+\psi h^{\varphi},\quad\forall\ h>0. \tag{71}$$

We have as in the proof of Theorem 4 that γ has a global minimum at

$$h_0=\left(\frac{2\xi}{\varphi\psi}\right)^{\frac{1}{\varphi+2}}, \tag{72}$$

which is

$$\gamma(h_0)=\left(\frac{\xi}{\varphi}\right)^{\frac{\varphi}{\varphi+2}}\left(\frac{\psi}{2}\right)^{\frac{2}{\varphi+2}}(\varphi+2). \tag{73}$$

Consequently

$$\gamma(h_0) =$$

(74)
$$= \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\nu-2-\frac{1}{p}}\right)^{\left(\frac{\nu-2-\frac{1}{p}}{\nu-\frac{1}{p}}\right)} \left(\frac{M}{\Gamma(\nu)\,(q(\nu-1)+1)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\nu-\frac{1}{p}}\right)} \left(\nu-\frac{1}{p}\right).$$

We have proved that

$$\|\|f''\|\|_{\infty,\mathbb{R}} \leq$$

(75)
$$\leq \left(\frac{4\,\|\|f\|\|_{\infty,\mathbb{R}}}{\nu-2-\frac{1}{p}}\right)^{\left(\frac{\nu-2-\frac{1}{p}}{\nu-\frac{1}{p}}\right)} \left(\frac{M}{\Gamma(\nu)\,(q(\nu-1)+1)^{\frac{1}{q}}}\right)^{\left(\frac{2}{\nu-\frac{1}{p}}\right)} \left(\nu-\frac{1}{p}\right).$$

The theorem is proved. ■

We give

Corollary 6. *(case of $\nu = 2.5$) Let $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. We assume that $\|\|f\|\|_{\infty,\mathbb{R}} < \infty$, and that*

(76)
$$K_{2.5} := \max\left\{\left\|\left\|D_{*a}^{2.5} f(z)\right\|\right\|_{\infty,\mathbb{R}^2}, \left\|\left\|D_{a-}^{2.5} f(z)\right\|\right\|_{\infty,\mathbb{R}^2}\right\} < \infty,$$

where $(a, z) \in \mathbb{R}^2$.

Then

(77)
$$\|\|f'\|\|_{\infty,\mathbb{R}} \leq 1.21136\,(K_{2.5})^{0.4}\left(\|\|f\|\|_{\infty,\mathbb{R}}\right)^{0.6},$$

and

(78)
$$\|\|f''\|\|_{\infty,\mathbb{R}} \leq 1.44713\,(K_{2.5})^{0.8}\left(\|\|f\|\|_{\infty,\mathbb{R}}\right)^{0.2}.$$

That is $\|\|f'\|\|_{\infty,\mathbb{R}}$, $\|\|f''\|\|_{\infty,\mathbb{R}} < \infty$.

Proof. By Theorem 4. ■

We finish with

Corollary 7. *(case of $\nu = 2.5$, $p = q = 2$) Let $f \in C^3(\mathbb{R}, X)$, where $(X, \|\cdot\|)$ is a Banach space. We assume that $\|\|f\|\|_{\infty,\mathbb{R}} < \infty$, and that*

(79)
$$M_{2.5} := \max\left\{\sup_{a\in\mathbb{R}}\left\|\left\|D_{*a}^{2.5} f\right\|\right\|_{2,\mathbb{R}}, \sup_{a\in\mathbb{R}}\left\|\left\|D_{a-}^{2.5} f\right\|\right\|_{2,\mathbb{R}}\right\} < \infty.$$

Then

$$\| \| f' \| \|_{\infty,\mathbb{R}} \leq 1.226583057 \sqrt{M_{2.5} \| \| f \| \|_{\infty,\mathbb{R}}}. \tag{80}$$

That is $\| \| f' \| \|_{\infty,\mathbb{R}} < \infty$.

Proof. By Theorem 5. ■

References

[1] **Aglic Aljinovic, A.A, Lj. Marangunic and J. Pecaric,** On Landau type inequalities via Ostrowski inequalities, *Nonlinear Funct. Anal. Appl.*, **10(4)** (2005), 565–579.

[2] **Anastassiou, G.,** *Fractional Differentiation inequalities*, Research monograph, Springer, New York, 2009.

[3] **Anastassiou, G.A.,** *Advances on Fractional Inequalities*, Springer, New York, 2011.

[4] **Anastassiou, G.A.,** *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[5] **Barnett, N.S. and S.S. Dragomir,** Some Landau type inequalities for functions whose derivatives are of locally bounded variation, *Tamkang Journal of Mathematics*, **37(4)** (2006), 301–308.

[6] **Ditzian, Z.,** Remarks, questions and conjectures on Landau–Kolmogorov-type inequalities, *Math. Inequal. Appl.*, **3** (2000), 15–24.

[7] **Hardy, G.H. and J.E. Littlewood,** Some integral inequalities connected with the calculus of variations, *Quart. J. Math. Oxford Ser.*, **3** (1932), 241–252.

[8] **Hardy, G.H., E. Landau and J.E. Littlewood,** Some inequalities satisfied by the integrals or derivatives of real or analytic functions, *Math. Z.*, **39** (1935), 677–695.

[9] **Kallman, R.R. and G.C. Rota,** On the inequality $\|f'\|^2 \leq 4 \|f\| \cdot \|f''\|$, in *Inequalities*, Vol. II, (O. Shisha, Ed.), 187–192, Academic Press, New York, 1970.

[10] **Kraljevic, H. and J. Pecaric,** Some Landau's type inequalities for infinitesimal generators, *Aequationes Mathematicae*, **40** (1990), 147–153.

[11] **Landau, E.,** Einige Ungleichungen für zweimal differentzierban funktionen, *Proc. London Math. Soc.*, **13** (1913), 43–49.

[12] **Mikusinski, J.,** *The Bochner integral*, Academic Press, New York, 1978.

[13] **Shilov, G.E.,** *Elementary Functional Analysis*, Dover Publications Inc., New York, 1996.

[14] **Xiao, Yingxiong,** Landau type inequalities for Banach space valued functions, *Journal of Mathematical Inequalities*, **7(1)** (2013), 103–114.

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