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ON SINHA'S NOTE ON PERFECT NUMBERS

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Abstract. We shall show that there is no odd perfect number of the form $2^n + 1$ or $n^n + 1$.

1. Introduction

A positive integer N is called perfect if $\sigma(N) = 2N$, where $\sigma(N)$ denotes the sum of divisors of N. As is well known, an even integer N is perfect if and only if $N = 2^{k-1}(2^k - 1)$ with $2^k - 1$ prime. In contrast, one of the oldest unsolved problems is whether there exists an odd perfect number or not. Moreover, it is also unknown whether there exists an odd m-perfect number for an integer $m \geq 2$, i.e., an integer N with $\sigma(N) = mN$ or not.

Sinha [5] showed that 28 is the only even perfect number of the form $x^n + y^n$ with $\gcd(x, y) = 1$ and $n \geq 2$ and also the only even perfect number of the form $a^n + 1$ with $n \geq 2$. On the other hand, it is not even proved or disproved that there exists no odd perfect number of the form $x^2 + 1$ with x an integer. Klurman [1] proved that if $P(x)$ is a polynomial of degree ≥ 3 without repeated factors, then there exist only finitely many odd perfect numbers of the form $P(x)$ with x an integer. Luca [4] (cited in Theorem 9.8 of [2]) showed that no Fermat number can be perfect.

In this article, we would like to prove that there exists no odd perfect number of the form $2^n + 1$ or $n^n + 1$.

Indeed, we prove a more general result.

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Theorem 1.1. *Let m and U be nonnegative integers. We put* $s_0 = \lfloor 2^U \log a/(U+1)\log 2 \rfloor$ *and* $t_0 = 2s_0 + 1$ *if* $U = 0$ *and* $a+1$ *is square and* $t_0 = 2s_0$ *otherwise. Let* $c = 1.093\cdots = (\log 2)/2 + (\log 3)/3 - (\log^2 3)/2$ *and* $C = C(U)$ *be the constant defined by*

$$C = \sum_{2^{U+1}(2m+1)<16,} \frac{1 - \log\log(2^{U+1}m)}{2^{U+1}m}.$$

If $a^n + 1$ is an odd $(4m+2)$-perfect number and $n = 2^U$, then

$$\log a > \frac{((4m+2)/e^C)^{2^{U+1}}}{2^U}. \tag{1.1}$$

If $a^n + 1$ is an odd $(4m+2)$-perfect number and $n = 2^U v$ with $v > 1$ odd, then

$$\log(4m+2) - C < \tag{1.2}$$

$$< \frac{\exp\left(\frac{1+\log t_0}{2^{U+1}}\right)}{2^{U+1}} \left(\log(2^U \log a) + (U+1)(1+\log t_0)\log 2 + \frac{\log^2 t_0}{2} + c\right).$$

Moreover, no integer of the form $2^n + 1$ can be $(4m+2)$-perfect.

For example, if $a^{128s} + 1$ is odd $(4m+2)$-perfect, then $a \geq 10$ and, if $a^{256s} + 1$ is odd $(4m+2)$-perfect, then $a \geq 18$. Furthermore, if $a^{16} + 1$ is odd $(4m+2)$-perfect, then $a > \exp\exp 19.4$ and, if $a^{32} + 1$ is odd $(4m+2)$-perfect, then $a > \exp\exp 40.8$. We note that $C(0) = 0.9807\cdots$, $C(1) = 0.1758\cdots$, $C(2) = 0.03348\cdots$ and $C(U) = 0$ for $U \geq 3$.

We shall prove that an odd perfect number of the form $n^n + 1$ must be of the form $2^m + 1$ and deduce the following result from the above result.

Theorem 1.2. 28 *is the only $(4m+2)$-perfect number of the form $n^n + 1$ with $m, n \geq 0$ an integer.*

Thus, we conclude that 28 is the only perfect number of the form $n^n + 1$.

2. Proof of Theorem 1.1

Assume that $a^n + 1$ is an odd $(4m+2)$-perfect number. By Euler's result, we must have $a^n + 1 = px^2$ for a prime p and an integer x.

Write $n = 2^U p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$ with $p_1 > p_2 > \cdots > p_r$ odd primes and let $P_i = p_i^{e_i}$ for $i = 1, 2, \ldots, r$ and $s = \omega(a^{2^U} + 1)$. We put $o_p(x)$ to be the multiplicative order of x modulo p.

We can factor $a^n + 1 = M_0 M_1 \cdots M_r$, where $M_0 = a^{2^U} + 1$ and

$$M_i = \frac{a^{2^U P_1 P_2 \cdots P_i} + 1}{a^{2^U P_1 P_2 \cdots P_{i-1}} + 1}$$

for $i = 1, 2, \ldots, r$. Moreover, let

$$L_i = M_0 M_1 \ldots M_i = a^{2^U P_1 P_2 \cdots P_i} + 1$$

and $M_i = E_i Y_i^2, L_i = D_i X_i^2$ with D_i and E_i squarefree. Clearly, we have $a^n + 1 = L_r = px^2$ and therefore $D_r = p$.

We begin by showing that $p_i \equiv 1 \pmod{2^{U+1}}$ for every i. If $\gcd((a^n + +1)/(a^{n/P_i} + 1), a^{n/P_i} + 1) = 1$, then

$$a^{n/P_i} + 1 = X^2, \frac{a^n + 1}{a^{n/P_i} + 1} = pY^2 \tag{2.1}$$

or

$$a^{n/P_i} + 1 = pX^2, \frac{a^n + 1}{a^{n/P_i} + 1} = Y^2 \tag{2.2}$$

for some integers X and Y. If $U = 0$, then we clearly have $p_i \equiv 1 \pmod{2^{U+1}}$. If $U > 0$, then $n/p_i^{e_i}$ is even and (2.1) is clearly impossible. The impossibility of (2.2) follows from Ljunggren's result [3] that $(a^f + 1)/(a + 1)$ with $a \geq 2, f \geq 3$ cannot be square.

Hence, we must have $\gcd((a^n + 1)/(a^{n/P_i} + 1), a^{n/P_i} + 1) > 1$. Observing that

$$\frac{a^n + 1}{a^{n/P_i} + 1} = \sum_{j=0}^{P_i - 1} (-1)^j a^{j(n/P_i)} \equiv P_i \pmod{a^{n/P_i} + 1},$$

p_i must divide $a^{n/P_i} + 1$. Thus, proceeding as in the proof of Theorem 4.12 of [2], we see that 2^{U+1} divides $o_{p_i}(a)$ and $o_{p_i}(a)$ divides $2n/P_i$. In particular, $p_i \equiv 1 \pmod{2^{U+1}}$ for every i.

Nextly, we show that for each $i = 1, 2, \ldots, r$, we have either

(i) $\gcd(L_{i-1}, M_i) = 1$ and $\omega(D_{i-1}) < \omega(D_i)$ or

(ii) p_i is the only prime dividing $\gcd(L_{i-1}, M_i)$ and p_i divides $a^{2^U} + 1$.

If $\gcd(L_{i-1}, M_i) = 1$, then we must have $D_i = D_{i-1} E_{i-1}$ and $X_i = = X_{i-1} Y_{i-1}$. It follows from Ljunggren's result mentioned above that $E_{i-1} \neq 1$. Since D_i is squarefree, we have $\omega(D_{i-1}) < \omega(D_i)$.

Assume that $\gcd(L_{i-1}, M_i) > 1$. Since

$$M_i = \sum_{j=0}^{P_i - 1} (-1)^j 2^{2^U P_1 P_2 \ldots P_{i-1} j} \equiv P_i \pmod{L_{i-1}},$$

we see that p_i is the only prime dividing both L_{i-1} and M_i.

Now p_i must divide L_{i-1} and therefore, proceeding as above, we see that 2^{U+1} divides $o_{p_i}(a)$ and $o_{p_i}(a)$ divides $2^{U+1}P_1P_2\cdots P_{i-1}$. Hence, $o_{p_i}(a) = 2^{U+1}d$ and therefore $p_i \equiv 1 \pmod{2^{U+1}d}$ for some d dividing $P_1P_2\cdots P_{i-1}$. But, since $p_1 > \cdots > p_{i-1} > p_i$, we must have $o_{p_i}(a) = 2^{U+1}$ and therefore p_i must divide $a^{2^U}+1$.

It is clear that (ii) occurs at most s times. Moreover, we observe that in the case (ii), p_i is the only possible prime which divides D_{i-1} but not D_i. Hence, we must have $\omega(D_{i-1}) \le \omega(D_i)+1$ for each i. Now we see that (i) also occurs at most s times.

We can easily see that $\omega(D_0) = 0$ if and only if $U = 0$ and $a+1$ is a square. Thus we conclude that $r \le 2s+1$ if $D_0 = a+1$ with $U = 0$ is square and $r \le 2s$ otherwise.

If a prime p divides $a^{2^U d}+1$ but $a^{2^U e}+1$ for any $e < d$, then the multiplicative order of $2 \pmod p$ is equal to $2^{U+1}d$ and therefore $p = 2^{U+1}kd+1$ for some integer k. Moreover, the number of such primes is at most $k_0(d) = \lfloor 2^U d\log a/\log(2^{U+1}d)\rfloor$ and therefore $s \le s_0$.

Hence, for each d,

$$\prod_{o_p(a)=2^{U+1}d} \frac{p}{p-1} < \exp \sum_{o_p(a)=2^{U+1}d} \frac{1}{p-1} \le \sum_{k=1}^{k_0(d)} \frac{1}{2^{U+1}kd} \le \le \exp \frac{1+\log(2^U d\log a/\log(2^{U+1}d))}{2^{U+1}d}, \tag{2.3}$$

so that

$$\frac{\sigma(a^n+1)}{a^n+1} = \prod_{\substack{o_p(a)=2^{U+1}d,\\ d\mid P_1P_2\ldots P_r}} \frac{p}{p-1} < \exp\left(C + \sum_{d\mid P_1P_2\ldots P_r} \frac{\log(2^U d\log a)}{2^{U+1}d}\right). \tag{2.4}$$

If $r = 0$, then we immediately see that

$$\sum_{d\mid P_1P_2\ldots P_r} \frac{\log(2^U d\log a)}{2^{U+1}d} = \frac{U\log 2+\log\log a}{2^{U+1}}. \tag{2.5}$$

If $r > 0$, then, observing that

$$\sum_{i=0}^{\infty} \frac{i}{q^i} = \sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} \frac{1}{q^i} = \sum_{j=0}^{\infty} \frac{1}{q^j(q-1)} = \frac{q}{(q-1)^2}, \tag{2.6}$$

we have

$$
\begin{aligned}
(2.7)\qquad & \sum_{d \mid P_1 P_2 \ldots P_r} \frac{\log(2^U d \log a)}{2^{U+1} d} < \\
& < \sum_{f_1, f_2, \ldots, f_r \geq 0} \frac{\log(2^U \log a) + f_1 \log p_1 + f_2 \log p_2 + \cdots + f_r \log p_r}{2^{U+1} p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r}} = \\
& = \prod_{i=1}^{t} \frac{p_i}{p_i - 1} \left(\frac{\log(2^U \log a)}{2^{U+1}} + \sum_{k=1}^{t} \frac{\log p_k}{2^{U+1}(p_k - 1)} \right) = \\
& = \left(\frac{1}{2^{U+1}} \prod_{i=1}^{r} \frac{p_i}{p_i - 1} \right) \left(\frac{\log(2^U \log a)}{2^{U+1}} + \sum_{k=1}^{r} \frac{\log p_k}{p_k - 1} \right).
\end{aligned}
$$

Since each $p_i \equiv 1 \pmod{2^{U+1}}$, we have

$$
(2.8)\qquad \prod_{i=1}^{r} \frac{p_i}{p_i - 1} < \prod_{k=1}^{r} \frac{2^{U+1} k + 1}{2^{U+1} k} < \exp \frac{1 + \log r}{2^{U+1}}
$$

and observing that $\sum_{k=1}^{t} \log k / k \leq (\log t)^2/2 + c$ for $t \geq 1$,

$$
\begin{aligned}
(2.9)\qquad & \sum_{k=1}^{r} \frac{\log p_k}{p_k - 1} < \sum_{k=1}^{r} \frac{\log k + (U+1) \log 2}{2^{U+1} k} < \\
& < \frac{1}{2^{U+1}} \left((U+1)(1 + \log r) \log 2 + \frac{\log^2 r}{2} + c \right).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
(2.10)\qquad & \sum_{d \mid P_1 P_2 \ldots P_r} \frac{\log(2^U d \log a)}{2^{U+1} d} < \\
& < \frac{\exp\left(\frac{1 + \log r}{2^{U+1}}\right)}{2^{U+1}} \left(\log(2^U \log a) + (U+1)(1 + \log r) \log 2 + \frac{\log^2 r}{2} + c \right).
\end{aligned}
$$

We see that $r \leq t_0$, where we recall that $s \leq s_0 = \left\lfloor 2^U \log a / (U+1) \log 2 \right\rfloor$. Hence, we conclude that

$$
(2.11)\qquad \log(4m + 2) = \log \frac{\sigma(a^n + 1)}{a^n + 1} < C + \frac{U \log 2 + \log \log a}{2^{U+1}}
$$

if $r = 0$ and

$$
\begin{aligned}
(2.12)\qquad & \log(4m + 2) - C < \\
& < \frac{\exp\left(\frac{1 + \log t_0}{2^{U+1}}\right)}{2^{U+1}} \left(\log(2^U \log a) + (U+1)(1 + \log t_0) \log 2 + \frac{\log^2 t_0}{2} + c \right)
\end{aligned}
$$

otherwise. Thus (1.1) and (1.2) follows.

Now we consider the case $a = 2$. If $U \geq 4$, then the right-hand side of (1.1) and (1.2) is $< 0.53 < \log 2$ and therefore $a^n + 1$ cannot be $(4m+2)$-perfect.

If $U \leq 3$, then $2^{2^U} + 1$ is prime and therefore $s = 1$. Clearly, for $n = 2^U$ with $U \leq 3$, $2^n + 1 = 2^{2^U} + 1$ is not $(4m+2)$-perfect. Hence, we must have $r \leq 2$ and $n = 2^U p_1^{e_1}$ or $2^U p_1^{e_1} p_2^{e_2}$.

If $n = 2^U p_1^{e_1}$, then, iterating the argument given before, we must have $p_1 = 2^{2^U} + 1$. Thus, $n = 3^{e_1}$, 2×5^{e_1}, $2^2 \times 17^{e_1}$ or $2^3 \times 257^{e_1}$.

However, for $n = 3^{e_1}$ with $e_1 \geq 3$, we see that both primes 19 and 87211 divide $2^n + 1$ exactly once since 19 and 87211 divide $2^{27} + 1$ exactly once and the only prime dividing both $(2^n + 1)/(2^{27} + 1)$ and $2^{27} + 1$ is 3. This implies that $2^n + 1$ cannot be of the form px^2 and therefore $2^n + 1$ cannot be $(4m+2)$-perfect if $n = 3^{e_1}$ with $e_1 \geq 3$. Similarly, 41 and 101 divide $2^n + 1$ exactly once if $n = 2 \times 5^{e_1}$ and $e_1 \geq 2$. Clearly, none of $2^3 + 1, 2^9 + 1, 2^{10} + 1$ is $(4m+2)$-perfect. Thus $2^n + 1$ cannot be $(4m+2)$-perfect if $n = 3^{e_1}$ or 2×5^{e_1}. Similarly, $2^n + 1$ cannot be $(4m+2)$-perfect if $n = 2^2 \times 17^{e_1}$ or $2^3 \times 257^{e_1}$.

If $n = 2^U p_1^{e_1} p_2^{e_2}$, then, iterating the argument given before, $p_1 > p_2 = = 2^{2^U} + 1$.

If $U = 1$ and $n = 10 p_1^{e_1}$, then we must have

$$2^{10} + 1 = 5^2 \times 41, \frac{2^n + 1}{2^{10} + 1} = 41 p y^2$$

since $(2^n + 1)/(2^{10} + 1)$ cannot be square by Ljunggren's result. Thus, we must have $p_1 = 41$. However, this implies that $2^n + 1$ must be divisible by 821 and 10169 exactly once, which contradicts to the fact that $2^n + 1 = px^2$. If $U = 1$ and $n = 2 \times 5^{e_2} p_1^{e_1}$ with $e_2 \geq 2$, then, since three primes $41, 101, 8101$ divide $2^{50} + 1$ exactly once, at least two of these primes divide $2^n + 1$. Thus $2^n + 1$ cannot be $(4m+2)$-perfect if $n = 2 p_1^{e_1} p_2^{e_2}$. Similarly, $2^n + 1$ cannot be $(4m+2)$-perfect for $n = 2^U p_1^{e_1} p_2^{e_2}$ with $U = 2, 3$.

Now we assume that $n = 3^{e_2} p_1^{e_1}$.

If $n = 3^{e_2} p_1^{e_1}$ with $e_2 \geq 4$, then, at least two of three primes $19, 163, 87211$ divide $2^n + 1$ exactly once and therefore $2^n + 1$ cannot be $(4m+2)$-perfect for such n. If $n = 27 p_1^{e_1}$, then we must have $p_1 = 19$ or 87211. We cannot have $p_1 = 19$ since 571 and 87211 divide $2^n + 1$ exactly once for $n = 27 \times 19^{e_1}$. Assume that $p_1 = 87211$. We observe that, for $d = 3^{f_2} 87211^{f_1}$ with $f_1 > 0$, we have

$$\prod_{o_p(a)=2d} \frac{p}{p-1} < \exp \frac{1 + \log(d \log 2 / \log(2d))}{2d} < \exp \frac{\log d}{2d} \tag{2.13}$$

and, proceeding as in (2.7),

$$\sum_{\substack{d=3^{f_2}87211^{f_1},\\ f_1>0, f_2\geq 0}} \frac{\log d}{2d} < \frac{87211}{116280}\left(\frac{\log 3}{174422} + \frac{\log 87211}{87210}\right) < \frac{1}{9000}. \tag{2.14}$$

Thus, $\sigma(2^n+1)/(2^n+1) < e^{1/9000}\sigma(2^{27}+1)/(2^{27}+1) < 2$ and therefore 2^n+1 cannot be $(4m+2)$-perfect.

If $n = 9p_1^{e_1}$, then we must have $p_1 = 19$ and therefore two primes 571 and 174763 divide 2^n+1 exactly once, which is a contradiction.

Finally, assume that $n = 3p_1^{e_1}$. If $p_1 \geq 11$, then, like (2.14),

$$\sum_{\substack{d=3^{f_2}p_1^{f_1},\\ f_1>0, f_2\geq 0}} \frac{\log d}{2d} < \frac{3p_1}{2(p_1-1)}\left(\frac{\log 3}{2p_1} + \frac{\log p_1}{p_1-1}\right) < 0.24 \tag{2.15}$$

and $\sigma(2^n+1)/(2^n+1) < (13/9)e^{0.24} < 2$, which is a contradiction.

The only remaining case is $n = 3p_1^{e_1}$ with $p_1 = 5$ or 7. We observe that $2^{15}+1 = 3^2 \times 11 \times 331$ and $2^{21}+1 = 3^2 \times 43 \times 5419$. Thus 2^n+1 must be divisible by at least two distinct primes exactly once, which is a contradiction again. Now we conclude that 2^n+1 can never be $(4m+2)$-perfect. ∎

3. Proof of Theorem 1.2

Sinha's result clearly implies that 28 is the only even perfect number of the form n^n+1. Thus, we may assume that n^n+1 is an odd $(4m+2)$-perfect number. Clearly n must be even and we can write $n = 2^u s$ with $u > 0$ and s odd.

As before, we must have $n^n+1 = px^2$ for some prime p and integer x.

Assume that $s > 1$. Then we must have

$$n^n + 1 = (n^{2^u}+1) \times \frac{n^{2^u s}+1}{n^{2^u}+1} = N_1 N_2, \tag{3.1}$$

say.

If N_1 and N_2 have a common prime factor p, then p divides d_2 and therefore p divides $2^u s = n$. This is impossible since $\gcd(n^n+1, n) = 1$. Thus, we see that $\gcd(N_1, N_2) = 1$ and therefore $N_1 = X^2, N_2 = pY^2$ or $N_1 = pX^2, N_2 = Y^2$.

We can easily see that $n^{2^u}+1$ cannot be square since $u > 0$ and therefore

$$\frac{n^{2^u s}+1}{n^{2^u}+1} = Z^2. \tag{3.2}$$

However, this is also impossible from Ljunggren's result.

Now we must have $s = 1$ and $n^n + 1 = 2^{u2^u} + 1$, which we have just proved not to be $(4m + 2)$-perfect in Theorem 1.1. This proves Theorem 1.2. ■

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