

# EXTENDING THE APPLICABILITY OF ITERATIVE METHODS FREE OF DERIVATIVES BY MEANS OF MEMORY

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**Abstract.** Iterative methods free of derivatives are useful for generating a sequence approximating a solution of nonlinear equations involving Banach space valued operators. The high convergence order of these methods is important too. The aim of this article is to extend the convergence of these methods in cases not covered before. Numerical experiments test our theoretical results.

## 1. Introduction

A plethora of applications from various disciplines results to finding a solution  $x_*$  of the equation

$$(1.1) \quad F(x) = 0,$$

with  $F : D \subset B_1 \rightarrow B_2$  is being differentiable as by Frechét, where  $B_1, B_2$  stand for Banach spaces and  $D$  being an open and convex set. The solution  $x_*$  is elusive in many cases as a closed form. Therefore, researchers and practitioners develop iterative methods producing sequences converging to the solution under some sufficient semilocal criteria on the initial data. Some popular and well studied methods are:

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Newton's method:

$$(1.2) \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n),$$

Traub's-Steffensen's method:

$$(1.3) \quad x_{n+1} = x_n - [w_n, x_n; F]^{-1}F(x_n),$$

where  $x_0 \in D$ ,  $n = 0, 1, 2, \dots$ ,  $w_n = x_n + bF(x_n)$ ,  $b \in \mathbb{R}$  and  $[\cdot, \cdot; F] : D \times D \rightarrow \mathcal{L}(B_1, B_2)$  is the standard divided difference of order one [2, 12]. If  $b = 1$ , (1.3) reduces to the popular Steffensen's method [4, 11, 12, 13].

Shamanskii in [18] (when  $B_1 = B_2 = \mathbb{R}^k$ ) showed that method (1.3) is of order two if we start close enough to the solution. Notice that method (1.3) is not using derivatives (as in the case of methods (1.2) also of order two) which are expensive to compute in general. Many attempts have been made to increase the order of convergence for iterative methods [6, 7, 8, 11].

Recently, in particular the local convergence of the following methods has been studied in [5] when  $B_1 = B_2 = \mathbb{R}^k$ :

$$(1.4) \quad \begin{aligned} y_n &= x_n - A_n^{-1}F(x_n), \\ x_{n+1} &= y_n - B_n^{-1}F(y_n), \end{aligned}$$

and the Kurchatov-type method with memory

$$(1.5) \quad \begin{aligned} w_n &= x_n - C_n^{-1}F(x_n), \\ y_n &= x_n - A_n^{-1}F(x_n), \\ x_{n+1} &= y_n - B_n^{-1}F(y_n), \end{aligned}$$

where  $A_n = [w_n, x_n; F]$ ,  $B_n = [w_n, y_n; F]$  and  $C_n = [2x_n - x_{n-1}, x_{n-1}; F]$ ,  $x_0, x_{-1} \in D$ ,  $n = 0, 1, 2, \dots$

Method (1.4) is shown to be of order three using hypotheses up to the fifth derivative, whereas method (1.5) is of order fifth. But these high order derivatives are not on these methods.

Note that Kurchatov's method

$$(1.6) \quad x_{n+1} = x_n - C_n^{-1}F(x_n), \quad x_0, x_{-1} \in D, \quad n = 0, 1, 2, \dots$$

was proposed in [10]. It was later explored in [1, 2, 15, 16, 17]. Other modifications of Kurchatov's method are discussed in [3, 14].

These hypotheses limit the applicability of the methods. Indeed, consider the academic and motivational example, when  $B_1 = B_2 = \mathbb{R}$  and  $D = [-\frac{1}{2}, \frac{3}{2}]$  given by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then,  $f'''(x)$  is unbounded on  $D$ . Hence, the results in [5] do not guarantee the convergence of method (1.4) or (1.5) to  $x_* = 1$  (although these methods may converge). Other problems with the analysis in [5] include no computable bounds on  $\|x_n - x_*\|$ , results on the uniqueness of  $x_*$  in  $D$  or how the initial point  $x_0$  is chosen. Based on the above it is clear that a weaker convergence analysis is needed that also addresses all these problems. The novelty of our paper lies in the fact that we only use hypotheses on the derivative of order one. Our idea can be used to extend the usage of other methods using inverses in a similar fashion [1] – [18].

## 2. Convergence for method (1.4)

It is convenient to introduce some scalar functions and constants to be used in the local convergence. Let  $\omega_0 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  be a continuous and increasing function satisfying  $\omega_0(0, 0) = 0$  and let  $\alpha \geq 0$ . Suppose equation

$$(2.1) \quad \omega_0(\alpha t, t) = 1$$

has a minimal positive root  $\rho_1$ . Let  $\omega : [0, \rho_1) \times [0, \rho_1) \rightarrow [0, \infty)$  be a continuous and increasing function satisfying  $\omega(0, 0) = 0$ . Define functions  $g_1$  and  $g_2$  on the interval  $[0, \rho_1)$  by

$$g_1(t) = \frac{\omega((1 + \alpha)t, t)}{1 - \omega_0(\alpha t, t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

By these definitions  $h_1(0) = -1$  and  $h_1(t) \rightarrow \infty$  as  $t \rightarrow \rho_1^-$ . Denote by  $r_1$  the minimal root of equation  $h_1(t) = 0$  in  $(0, \rho_1)$  assured to exist by the intermediate value theorem.

Suppose that equation

$$(2.2) \quad \omega_0(\alpha t, g_1(t)t) = 1$$

has a minimal positive root  $\rho_2$ . Set  $\rho = \min\{\rho_1, \rho_2\}$ . Define functions  $g_2$  and  $h_2$  on the interval  $[0, \rho)$  by

$$g_2(t) = \frac{\omega(\alpha t + g_1(t)t, g_1(t)t)g_1(t)}{1 - \omega_0(\alpha t, g_1(t)t)}$$

and

$$h_2(t) = g_2(t) - 1.$$

We get again  $h_2(0) = -1$  and  $h_2(t) \rightarrow \infty$  as  $t \rightarrow \rho_2^-$ . Denote by  $r_2$  the minimal root of equation  $h_2(t) = 0$  in  $(0, \rho)$ . Define a radius of convergence  $r$  by

$$(2.3) \quad r = \min\{r_1, r_2\}.$$

Then, for all  $t \in [0, r)$  we obtain

$$(2.4) \quad 0 \leq \omega_0(\alpha t, t) < 1,$$

$$(2.5) \quad 0 \leq \omega_0(\alpha t, g_1(t)t) < 1,$$

$$(2.6) \quad 0 \leq g_1(t) < 1$$

and

$$(2.7) \quad 0 \leq g_2(t) < 1.$$

we denote by  $U(v, s)$  and  $\bar{U}(v, s)$  the open and closed balls in  $B_1$  of center  $v \in D$  and with radius  $s > 0$ .

The following conditions (A) are utilized in our analysis:

(A<sub>1</sub>)  $F : D \rightarrow B_2$  is differentiable,  $[\cdot, \cdot; F] : D \times D \rightarrow \mathcal{L}(B_1, B_2)$  is a standard divided difference of order one, and  $x_* \in D$  exists with  $F(x_*) = 0$  which is simple.

(A<sub>2</sub>)  $\omega_0 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  is a continuous and increasing function with  $\omega_0(0, 0) = 0$  such that for all  $x, y \in D$

$$\|F'(x_*)^{-1}([x, y; F] - F'(x_*))\| \leq \omega_0(\|x - x_*\|, \|y - x_*\|)$$

and

$$\|I + b[x, x_*; F]\| \leq \alpha \|x - x_*\|$$

for some  $\alpha \geq 0$ .

Set  $D_0 = D \cap U(x_*, \rho_1)$ .

(A<sub>3</sub>)  $\omega : [0, \rho_1) \times [0, \rho_1) \rightarrow [0, \infty)$  is a continuous and increasing function with  $\omega(0, 0) = 0$  such that for all  $x, y \in D_0$

$$\|F'(x_*)^{-1}([x, y; F] - [y, x_*; F])\| \leq \omega(\|x - y\|, \|y - x_*\|).$$

(A<sub>4</sub>)  $U(x_*, r_*) \subset D$ ,  $r_* = \max\{r, \alpha r\}$ ,  $\rho_1, \rho_2$  given in (2.1) and (2.2) exist and  $r$  is defined in (2.3).

(A<sub>5</sub>) There exist  $\bar{r} \geq r$  such that

$$\omega_0(0, \bar{r}) < 1 \quad \text{or} \quad \omega_0(\bar{r}, 0) < 1.$$

Set  $D_1 = D \cap \bar{U}(x_*, r_*)$ .

Next, the conditions (A) together with the preceding notation are utilized in the local convergence analysis of method (1.4).

**Theorem 2.1.** *Assume that conditions (A) hold and choose starting point  $x_0 \in U(x_*, r) - \{x_*\}$ . Then, the following items hold*

$$(2.8) \quad x_n \in U(x_*, r),$$

$$(2.9) \quad \lim_{n \rightarrow \infty} x_n = x_*,$$

$$(2.10) \quad \|w_n - x_*\| \leq \alpha \|x_n - x_*\| \leq \alpha r,$$

$$(2.11) \quad \|y_n - x_*\| \leq g_1(\|x_n - x_*\|) \|x_n - x_*\| \leq \|x_n - x_*\| \leq r,$$

$$(2.12) \quad \|x_{n+1} - x_*\| \leq g_2(\|x_n - x_*\|) \|x_n - x_*\| \leq \|x_n - x_*\|,$$

and  $x_*$  is the only solution of equation  $F(x) = 0$  in the set  $D_1$  given in (A<sub>5</sub>).

**Proof.** By (A<sub>2</sub>), (2.1), (2.3), (2.4) and choice  $x_0 \in U(x_*, r) - \{x_*\}$ , we have

$$(2.13) \quad \begin{aligned} \|w_0 - x_*\| &= \|(I + b[x_0, x_*; F])(x_0 - x_*)\| \leq \\ &\leq \|(I + b[x_0, x_*; F])\| \|x_0 - x_*\| \leq \alpha \|x_0 - x_*\| \leq \alpha r \end{aligned}$$

and

$$(2.14) \quad \|F'(x_*)^{-1}(A_0 - F'(x_*))\| \leq \omega_0(\|w_0 - x_*\|, \|x_0 - x_*\|) \leq \omega_0(\alpha r, r) < 1.$$

Consequently, by Lemma of Banach for invertible operator [9] and (2.14), we get  $A_0$  is invertible and

$$(2.15) \quad \|A_0^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\|w_0 - x_*\|, \|x_0 - x_*\|)}.$$

Hence,  $y_0$  is well defined by the method (1.4) for  $n = 0$ . Using (2.3), (A<sub>3</sub>), (2.15) and for first substep of method (1.4) for  $n = 0$ , we obtain

$$(2.16) \quad \begin{aligned} \|y_0 - x_*\| &= \|x_0 - x_* - A_0^{-1}F(x_0)\| = \\ &= \|[A_0^{-1}F'(x_*)][F'(x_*)^{-1}(A_0 - [x_0, x_*; F])]\| \|x_0 - x_*\| \leq \\ &\leq \|A_0^{-1}F'(x_*)\| \|F'(x_*)^{-1}(A_0 - [x_0, x_*; F])\| \|x_0 - x_*\| \leq \\ &\leq \frac{\omega(\|w_0 - x_*\|, \|x_0 - x_*\|) \|x_0 - x_*\|}{1 - \omega_0(\|w_0 - x_*\|, \|x_0 - x_*\|)} \leq \\ &\leq \frac{\omega((1 + \alpha)\|x_0 - x_*\|, \|x_0 - x_*\|)}{1 - \omega_0(\|w_0 - x_*\|, \|x_0 - x_*\|)} \|x_0 - x_*\| \leq \\ &\leq g_1(\|x_0 - x_*\|) \|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned}$$

so  $y_0 \in U(x_*, r)$  and (2.10),(2.11) hold for  $n = 0$ . Moreover, we have by the second substep of method (1.4), (2.3),(2.15) and (2.16) that

$$\begin{aligned}
 \|x_1 - x_*\| &= \|[B_0^{-1}F'(x_*)][F'(x_*)^{-1}(B_0 - [y_0, x_*; F])](y_0 - x_*)\| \leq \\
 &\leq \|B_0^{-1}F'(x_*)\| \|F'(x_*)^{-1}(B_0 - [y_0, x_*; F])\| \|y_0 - x_*\| \leq \\
 &\leq \frac{\omega(\|w_0 - y_0\|, \|y_0 - x_*\|)}{1 - \omega_0(\alpha\|x_0 - x_*\|, \|y_0 - x_*\|)} \|y_0 - x_*\| \leq \\
 (2.17) \quad &\leq g_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r,
 \end{aligned}$$

showing  $x_1 \in U(x_*, r)$  and (2.12) for  $n = 0$ .

These computations can be repeated with  $x_i, w_i, y_i, x_{i+1}$  replacing  $x_0, w_0, y_0, x_1$  in the preceding computations to end the induction for (2.8)–(2.12). Furthermore, in view of the estimate

$$(2.18) \quad \|x_{i+1} - x_*\| \leq c\|x_i - x_*\| < r,$$

where  $c = g_2(\|x_0 - x_*\|) \in [0, 1)$ , we conclude  $x_{i+1} \in U(x_*, r)$  and  $\lim_{i \rightarrow \infty} x_i = x_*$ .

Finally, let  $Q = [x_*, y_*; F]$  for  $y_* \in D_1$  with  $F(y_*) = 0$ . Then, using  $(A_2)$  and  $(A_5)$ , we get in turn that

$$(2.19) \quad \|F'(x_*)^{-1}(Q - F'(x_*))\| \leq \omega_0(\|x_* - x_*\|, \|y_* - x_*\|) \leq \omega(0, \bar{r}) < 1,$$

so  $Q$  is invertible. Therefore,  $x_* = y_*$  is obtained from

$$0 = F(x_*) - F(y_*) = [x_*, y_*; F](x_* - y_*) = Q(x_* - y_*). \quad \blacksquare$$

### 3. Convergence for method (1.5)

As in Section 2, suppose equation

$$(3.1) \quad \omega_0(3t, t) = 1$$

has a minimal positive solution  $\rho_0$ . Define functions  $g_0$  and  $h_0$  on the interval  $[0, \rho_0)$  by

$$g_0(t) = \frac{\omega(2t, t)}{1 - \omega_0(3t, t)}$$

and

$$h_0(t) = g_0(t) - 1.$$

Denote by  $r_0$  the minimal solution of equation  $h_0(t) = 0$  in  $(0, \rho_0)$ .

Suppose that equation

$$(3.2) \quad \omega_0(g_0(t)t, t) = 1$$

has a minimal positive solution  $\rho_1$ .

Define functions  $g_1$  and  $h_1$  on the interval  $[0, \min\{\rho_0, \rho_1\})$  by

$$g_1(t) = \frac{\omega((1 + g_0(t))t, t)}{1 - \omega_0(g_0(t)t, t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

Denote by  $r_1$  the minimal solution of equation  $h_1(t) = 0$  in  $[0, \min\{\rho_0, \rho_1\})$ . Suppose that equation

$$(3.3) \quad \omega_0(g_0(t)t, g_1(t)t) = 1$$

has a minimal positive solution  $\rho_2$ . Set  $\rho = \min\{\rho_0, \rho_1, \rho_2\}$ . Define functions  $g_2$  and  $h_2$  on interval  $[0, \rho)$  by

$$g_2(t) = \frac{\omega(g_0(t)t + g_1(t)t, g_1(t)t)g_1(t)}{1 - \omega_0(g_0(t)t, g_1(t)t)}$$

and

$$h_2(t) = g_2(t) - 1.$$

Denote by  $r_2$  the minimal solution of equation  $h_2(t) = 0$  in  $(0, \rho)$ . Define a radius of convergence  $r$  by

$$(3.4) \quad r = \min\{r_0, r_1, r_2\}.$$

Then, for all  $t \in [0, r)$

$$(3.5) \quad 0 \leq \omega_0(3t, t) < 1,$$

$$(3.6) \quad 0 \leq \omega_0(g_0(t)t, t) < 1,$$

$$(3.7) \quad 0 \leq \omega_0(g_0(t)t, g_1(t)t) < 1,$$

$$(3.8) \quad 0 \leq g_0(t) < 1,$$

$$(3.9) \quad 0 \leq g_1(t) < 1$$

and

$$(3.10) \quad 0 \leq g_2(t) < 1.$$

We shall use conditions  $(H)$ :

$$(H_1) = (A_1).$$

$(H_2)$   $\omega_0 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  is continuous and increasing function with  $\omega_0(0, 0) = 0$  such that for all  $x, y \in D$

$$\|F'(x_*)^{-1}([x, y; F] - F'(x_*))\| \leq \omega_0(\|x - x_*\|, \|y - x_*\|)$$

and

$$\|I + b[x, x_*; F]\| \leq \alpha \|x - x_*\|.$$

Set  $D_0 = D \cap U(x_*, \rho_0)$ .

$$(H_3) = (A_3).$$

$(H_4)$   $U(x_*, 3r) \subset D$ , where  $r$  is defined in (3.3) and  $\rho_0, \rho_1, \rho_2$  exist and are given by (3.1), (3.2) and (3.3), respectively.

$$(H_5) = (A_5).$$

Next, the conditions  $(H)$  and the preceding notation give the local convergence analysis of method (1.5).

**Theorem 3.1.** *Assume that conditions  $(H)$  hold and choose starting points  $x_{-1}, x_0 \in U(x_*, r) - \{x_*\}$ . Then, the following items hold*

$$x_n \in U(x_*, r),$$

$$\lim_{n \rightarrow \infty} x_n = x_*,$$

$$\begin{aligned} \|w_n - x_*\| &\leq g_0(r) \|x_n - x_*\| \leq \|x_n - x_*\| < r, \\ \|y_n - x_*\| &\leq g_1(r) \|x_n - x_*\| \leq \|x_n - x_*\| < r \end{aligned}$$

and

$$\|x_{n+1} - x_*\| \leq g_2(r) \|x_n - x_*\| \leq \|x_n - x_*\| < r,$$

and  $x_*$  is the only solution of equation  $F(x) = 0$  in the set  $D_1$  given in  $(H_5)$ .

**Proof.** As in the proof of Theorem 2.1, we use mathematical induction to obtain

$$\begin{aligned} \|w_n - x_*\| &= \|[C_n^{-1}F'(x_*)][F'(x_*)^{-1}(C_n - [x_n, x_*; F])(x_n - x_*)]\| \leq \\ &\leq \frac{\omega(\|2x_n - x_{n-1} - x_n\|, \|x_{n-1} - x_*\|)\|x_n - x_*\|}{1 - \omega_0(\|2x_n - x_{n-1} - x_n\|, \|x_{n-1} - x_*\|)} \leq \\ &\leq \frac{\omega(\|x_n - x_*\| + \|x_{n-1} - x_*\|, \|x_{n-1} - x_*\|)\|x_n - x_*\|}{1 - \omega_0(2\|x_n - x_*\| + \|x_{n-1} - x_*\|, \|x_{n-1} - x_*\|)} \leq \\ &\leq g_0(r) \|x_n - x_*\| \leq \|x_n - x_*\| < r, \end{aligned}$$



$\|2x_n - x_{n-1} - x_*\| \leq \|x_n - x_*\| + \|x_{n-1} - x_n\| \leq 2\|x_n - x_*\| + \|x_{n-1} - x_*\| \leq 3r$ ,  
 and as in (2.16), (2.17), respectively

$$\begin{aligned} \|y_n - x_*\| &\leq g_1(r)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \\ \|x_{n+1} - x_*\| &\leq g_2(r)\|x_n - x_*\| \leq \|x_n - x_*\| < r. \end{aligned} \quad \blacksquare$$

#### 4. Numerical examples

We use  $[x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta$  and  $b = 1$  in all examples.

**Example 4.1.** Define  $D = U(0, 1)$ ,  $v = (v_1, v_2, v_3)^T$ , function  $F : D \rightarrow \mathbb{R}^3$  by

$$F(v) = \left( e^{v_1} - 1, \frac{e - 1}{2}v_2^2 + v_2, v_3 \right)^T.$$

Hence, we get

$$F'(v) = \begin{pmatrix} e^{v_1} & 0 & 0 \\ 0 & (e - 1)v_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we have for  $x_* = (0, 0, 0)^T$  that  $\omega_0(s, t) = \frac{e - 1}{2}(s + t)$ ,  
 $\omega(s, t) = (e - 1)s + e^{\frac{1}{e-1}t}$  and  $\alpha = \frac{e + 3}{2}$ .

**Example 4.2.** Let us choose  $B_1 = B_2 = C_{[0,1]}$ . Set  $D = U(0, 1)$ , and define function  $F$  on  $D$  by

$$F(\Gamma)(x) = \psi(x) - \int_0^1 x\theta\Gamma(\theta)^3 d\theta$$

and

$$F'(\Gamma(\xi))(x) = \xi(x) - 3 \int_0^1 x\theta\Gamma(\theta)^2\xi(\theta)d\theta.$$

Then, we have for  $x_* = 0$  that  $\omega_0(s, t) = \frac{3}{2}(s + t)$ ,  $\omega(s, t) = 3(s + t)$  and  $\alpha = 5$ .

**Example 4.3.** For the academic problem that we considered in the introduction we have for  $x_* = 0$ ,  $\omega_0(s, t) = \omega(s, t) = \frac{96.662907}{2}(s + t)$  and  $\alpha = 52$ .

Example	Method (1.4)	Method (1.5)
4.1	7.873962171772243e-02	1.315565965112603e-01
4.2	2.973980668120351e-02	7.599937578041070e-02
4.3	1.500342940298857e-04	3.448409981435105e-03

*Table 1.* Values of convergence radii

In Table 1 there are the convergence radii of the considered methods. The Kurchatov-type method with memory (1.5) has a wider convergence region than method (1.4).

Starting points	Kurchatov's method (1.6)	Method (1.4)	Method (1.5)
(0.1, 0.1, 0.1)	4	4	3
(3, 3, 3)	9	5	5
(1, 2, 0.5)	7	6	4
(1, 5, 5)	8	9	4
(2, 10, 5)	9	19	5

*Table 2.* Count of iterations for Example 4.1

In Table 2 we give a count of iterations, which need to solve a system of nonlinear equations from Example 4.1. Results are obtained for  $\varepsilon = 10^{-8}$ . Additional starting point is chosen as  $x_{-1} = x_0 + 0.0001$ . We can see that the method (1.5) is faster than method (1.4) and the Kurchatov's method.

## 5. Conclusion

Problems from a plethora of areas can be solved by considering the corresponding equation given on a suitable abstract space by Mathematical Modeling. A sequence is then constructed by some developed method whose limit solves the equation. Looking at this direction together with the need for the development of faster methods and a wider range of initial guesses, we studied method (1.4) and (1.5) (given in [5] for  $B_1 = B_2 = \mathbb{R}^k$ ) in the more general case of a space according to Banach. Our conditions involve the first order derivative appearing in the method whereas the work in [5] used higher order derivatives not on these methods. Our analysis produces error estimates, a radius of convergence and results on the location for the solution that are computable. This was not done in [5]. Hence, we expand the usage of the methods. Our idea can be used on other methods with the same benefits. Numerical experiments testing our theory and the performance of the methods terminate this article.

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