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ON BARYCENTRIC INTERPOLATION. III. (ON CONVERGENT TYPE PROCESSES)

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Dedicated to Professor Antal Járai on his 70th birthday

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Abstract. We define some processes for barycentric interpolation based on equidistant node-system which is convergent for *any continuous function*. As far as we know this is *the first process of this type*. We also prove an upper estimate for the rate of convergence. It turns out that the results are very similar to the ones known for the process obtained Bernstein from the classical Lagrange interpolation.

1. Classical Lagrange interpolation

1.1. Definitions

Let $C = C(I)$ denote the space of continuous functions on the interval $I := [-1, 1]$, and let $\mathcal{P}_n$ denote the set of algebraic polynomials of degree at most n, moreover $\|\cdot\|$ stands for the usual maximum norm on C. Let X be an *interpolatory matrix* (*array*), i.e.

$$X = \{x_{kn} = \cos\vartheta_{kn} \ : \ k = 1, 2, \dots, n; \ n = 1, 2, \dots\},$$

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with

$$(1.1) \qquad -1 = x_{n+1,n} \le x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} \le x_{0n} = 1$$

and $0 \le \vartheta_{kn} \le \pi$. For a function $f \in C$ consider the corresponding *Lagrange interpolation polynomial*

$$(1.2) \qquad L_n\big(f, X, x\big) := \sum_{k=1}^{n} f(x_{kn})\ell_{kn}(X, x), \quad n \in \mathbb{N}.$$

Here, for $n \in \mathbb{N}$,

$$\ell_{kn}\big(X, x\big) := \frac{\omega_n\big(X, x\big)}{\omega_n'\big(X, x_{kn}(x - x_{kn}\big)}, \quad 1 \le k \le n,$$

with

$$\omega_n\big(X, x\big) := \prod_{k=1}^{n}(x - x_{kn}),$$

are polynomials of exact degree $n-1$. They are the *fundamental polynomials*, obeying the relations $\ell_{kn}\big(X, x_{jn}\big) = \delta_{kj}$, $1 \le k, j \le n$, therefore the polynomial $L_n(f, X, x)$ of degree at most $n-1$ interpolates the function f at n points:

$$L_n\big(f, X, x_{kn}\big) = f\big(x_{kn}\big), \quad k = 1, 2, \ldots, n.$$

The main question is: *For what choices of the the interpolatory array X can we expect that* (*uniformly, pointwice, etc.*) $L_n(f, X, x) \to f$ $(n \to +\infty)$*?*

It is well known that in studying these questions, the quantities

$$\lambda_n(X, x) := \sum_{k=1}^{n}\big|\ell_{kn}(X, x)\big|, \quad x \in [-1, 1], \; n \in \mathbb{N}$$

(the *Lebesgue function*) and

$$\Lambda_n(X) := \|\lambda_n(X, x)\|, \quad n \in \mathbb{N}$$

(the *Lebesgue constant of Lagrange interpolation*) play major rules.

1.2. Divergence and convergence of Lagrange interpolation

G. Faber [6] proved in 1914, using his own estimation for any X

$$\Lambda_n(X) \ge \frac{1}{12}\log n, \quad n \ge 1$$

the following fundamental divergence-type result.

Theorem A (G. Faber [6]). *For any interpolatory matrix X there exists a function $f \in C[-1,1]$ such that the sequence of the interpolation polynomials $L_n(f,X,x)$, $n \in \mathbb{N}$ does not converge uniformly to f on $[-1,1]$.*

During the last about 100 years, there were proved many results concerning the behaviour the Lebesgue function and Lebesgue constant and applied to obtain divergence theorems the sequence $L_n(f,X)$ $(n \in \mathbb{N})$ (see e.g. the book of J. Szabados and P. Vértesi [13] and P. Vértesi [19]).*

At the same time let us remark that to every interpolatory matrix there correspond a *class of functions* for which the interpolation process *converges uniformly*. However this class is always substantially more restricted than the class $C[-1,1]$ (see e.g. [12, Chapter III] or [13, Chapter II]).

1.3. Uniformly convergent interpolation processes

We can see that the Lagrange interpolatory polynomials do not define a uniformly convergent approximating process for arbitrary continuous function. However relaxing certain conditions we can define better tools. First we recall the Fejér polynomials of degree $2n-1$ interpolating the function f on the Chebyshev nodes which do converge for any f from C.

It is natural to look for a uniformly converging interpolation procedure in which the degree of the nth polynomial and the number of nodes are close as possible to the number one. This problem was posed by S. N. Bernstein to whom is also due one of the possible solutions at the Chebyshev nodes (see [1], [2], [12, Chapter IV, §4.]). Similar statement were proved for certain Jacobi nodes in [17] and [15].

2. Barycentric interpolation

2.1. Let us introduce the notation

$$\Omega_n\big(X,x\big) := \frac{1}{\sum\limits_{s=1}^{n} \frac{(-1)^s}{x-x_{sn}}}\,.$$

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*More than 20 years after Fabers's result G. Grünwald [7] and J. Marcinkiewicz [10] obtained a very strong pointwise divergence statement for the "very good" Chebyshev matrix $\mathcal{C} := \big\{\cos\frac{2k-1}{2n}\pi;\ k=1,2,\dots,n; n=1,2,\dots\big\}$. They constructed a continuous function for which the Lagrange interpolation process diverges everywhere on $[-1,1]$ based on the nodes $\mathcal{C}$.
In 1981, P. Erdős and P. Vértesi [5] proved that for *arbitrary* interpolatory matrix X one can define a continuous f for which the Lagrange interpolatory polynomials diverge almost everywhere on $[-1,1]$.

Then we have

$$\Omega_n\big(X,x\big) := \frac{1}{\sum\limits_{s=1}^{n}\frac{(-1)^s}{x-x_{sn}}} = \frac{\omega_n\big(X,x\big)}{\omega_n\big(X,x\big)\cdot\sum\limits_{s=1}^{n}\frac{(-1)^s}{x-x_{sn}}} \qquad (2.1)$$

$$\big(x\in[-1,1],\ n\in\mathbb{N}\big).$$

Here the denominator

$$q_n(x) = -\frac{\omega_n\big(X,x\big)}{x-x_{1n}} + \frac{\omega_n\big(X,x\big)}{x-x_{2n}} + \cdots + (-1)^n\frac{\omega_n\big(X,x\big)}{x-x_{nn}}$$

is a polynomial of degree $\le n-1$. J.-P. Berrut observed that (see [3, Lemma 2.1]) it has no real root, so it has a constant sign on the whole real line. Since $q_n(x_{1n}) = -\omega_n'\big(X,x_{1n}\big) < 0$, thus $q_n(x) < 0$ for all $x\in\mathbb{R}$ and $n\in\mathbb{N}$. Consequently Ω_n is a *rational function* and it has no pole on the real line, so it is defined on $\mathbb{R}$.

From the definition of $\Omega_n(x) := \Omega_n(X,x)$ it follows that

$$\Omega_n\big(X,x\big) = 0 \quad\Longleftrightarrow\quad x = x_{kn},\quad 1\le k\le n. \qquad (2.2)$$

Since

$$\Omega_n'(x_{kn}) = \lim_{\varrho\to 0}\frac{\Omega_n(x_{kn}+\varrho)-\Omega_n(x_{kn})}{\varrho} = \lim_{\varrho\to 0}\frac{\Omega_n(x_{kn}+\varrho)}{\varrho} =$$
$$= \lim_{\varrho\to 0}\frac{1}{\varrho}\cdot\frac{1}{\frac{(-1)^k}{\varrho}+\sum\limits_{s\ne k}\frac{(-1)^s}{x_{kn}+\varrho-x_{sn}}} = (-1)^k,$$

thus we obtain that

$$\Omega_n'(X,x_{kn}) = (-1)^k,\quad 1\le k\le n. \qquad (2.3)$$

After certain considerations (see [20, p. 404]) we obtain the figure of $\Omega_n(x)$ if $x\in\big(x_{k+1,n},x_{kn}\big)$ and $k=$ odd, say. Namely, Ω_n is concave on this interval and it has unique local maximum at a certain point $u_{kn}\in\big(x_{k+1,n},x_{kn}\big)$ (see Figure 1). This means that $\Omega_n\big(X,x\big)$ is "similar" to $\omega_n\big(X,x\big)$ (cf. 1.1.).

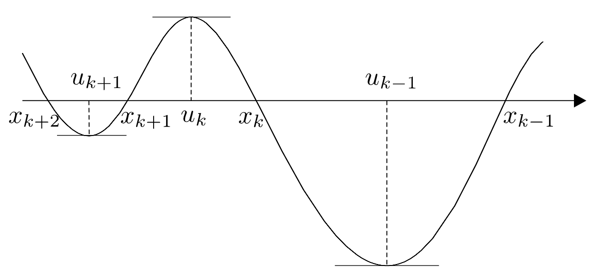

Figure 1

2.2. Barycentric interpolation. Now we define the classical *barycentric interpolation* process for $f \in C$ as follows

$$B_n\big(f, X, x\big) := \sum_{k=1}^{n} f(x_{kn})\, b_{kn}\big(X, x\big), \quad x \in \mathbb{R},\ n \in \mathbb{N},$$

where for $n \in \mathbb{N}$

$$b_{kn}\big(X, x\big) = \frac{\Omega_n\big(X, x\big)}{\Omega_n'\big(X, x_{kn}\big)(x - x_{kn})} = \frac{\frac{(-1)^k}{x - x_{kn}}}{\sum\limits_{s=1}^{n} \frac{(-1)^s}{x - x_{sn}}}, \quad 1 \le k \le n \tag{2.4}$$

are the *fundamental functions.*

The considerations of 2.1. show that $B_n\big(f, X, x\big)$ is a *rational function* which has no pole on the real line.

Since (see (2.2))

$$b_{kn}\big(X, x_{jn}\big) = \delta_{kj}, \quad 1 \le k, j \le n,\ n \in \mathbb{N}, \tag{2.5}$$

thus we obtain that the process $\big\{B_n\big\}$ has the *interpolatory property*, i.e.

$$B_n\big(f, X, x_{kn}\big) = f\big(x_{kn}\big), \quad 1 \le k \le n,\ n \in \mathbb{N}.$$

Summarizing, the advantage of the introduction of $\Omega_n(X, x)$ is the similarity between the classical Lagrange- and the barycentric interpolation (namely, $\Omega_n(X, x)$ and $b_{kn}(X, x)$ are analogous to $\omega_n(X, x)$ and $\ell_{kn}(X, x)$ (cf. 1.1.)).

2.3. Many papers deal with the barycentric interpolation (cf. [4] and its references). These results are analogous to the convergence-divergence of the classical Lagrange interpolation. For example a Faber-type statement is true for *arbitrary* nodes.

Theorem B (G. Halász [8]; see in P. Vértesi [20]). *For arbitrary system of nodes X*

$$\widetilde{\Lambda}_n(X) > \frac{\log n}{8} \quad (n \ge 3),$$

where

$$\widetilde{\Lambda}_n(X) := \left\| \sum_{k=1}^{n} \big| b_{kn}(X, \cdot) \big| \right\|.$$

Several *convergence-divergence* properties of barycentric operators were considered in [11].

3. A Bernstein-type process

Hereinafter, we shall consider the equidistant nodes

$$\mathbf{e}_n = \left\{ e_{kn} = 1 - \frac{2k-1}{n} \;\middle|\; k = 1, 2, \ldots, n \right\}, \quad n = 2, 3, \ldots \tag{3.1}$$

and for the simplicity we shall suppose that n is *even*, $n = 2m$.

For a function $f \in C[-1,1]$ we define *the barycentric Bernstein-type operators* by

$$\mathcal{B}_n(f, \mathbf{e_n}, x) = \sum_{k=1}^{m} f(e_{2k-1,n}) \big\{ b_{2k-1,n}(\mathbf{e}_n, x) + b_{2k,n}(\mathbf{e}_n, x) \big\}, \qquad x \in [-1,1], \quad n = 2, 4, \ldots . \tag{3.2}$$

From (2.5) it follows that $\mathcal{B}_n(f, \mathbf{e}_n)$ interpolates the function f at the points $e_{2k-1,n}$ $(k = 1, 2, \ldots, m)$:

$$\mathcal{B}_n(f, \mathbf{e_n}, e_{2k-1,n}) = f(e_{2k-1,n}), \qquad k = 1, 2, \ldots, n; \; n = 2, 4, \ldots .$$

Theorem 1. *For every function $f \in C[-1,1]$ we have*

$$\lim_{n \to +\infty} \big\| \mathcal{B}_n(f, \mathbf{e}_n, \cdot) - f(\cdot) \big\| = 0. \tag{3.3}$$

As far as we know the $\{\mathcal{B}_n\}$ process is the first barycentric interpolation operator sequence which is uniformly convergent for *arbitrary* continuous function f.

Theorem 2. *For every function $f \in C[-1,1]$ we have*

$$\big| f(x) - \mathcal{B}_n(f, \mathbf{e}_n, x) \big| \le 8 \sum_{i=1}^{n} \omega\left(f, \frac{i}{n}\right) \cdot \frac{1}{i^2} \qquad \big(x \in [-1,1], \; n = 2, 4, \ldots\big), \tag{3.4}$$

where $\omega(f, \cdot)$ is the modulus of continuity of f.

It is easy to get Theorem 1 using Theorem 2, too. Indeed, let $\varepsilon_n := \frac{\log n}{n}$ $(n = 2, 4, \ldots)$. Then

$$\sum_{i=1}^{n} \omega\left(f, \frac{i}{n}\right) \cdot \frac{1}{i^2} = O(1)\, \omega(f, \varepsilon_n),$$

whence we obtain (3.3).

4. Proofs

As before we shall suppose that $n = 2m$ is an *even* natural number and sometimes we use short notations, for example $\Omega_n(x) = \Omega_n(\mathbf{e}_n, x)$, $b_k(x) = b_{kn}(\mathbf{e}_n, x)$, $e_k = e_{kn}$.

4.1. Lemmas

Lemma 1. *Let n be an even natural number. Then the function $\Omega_n(\mathbf{e}_n, x)$, $x \in [-1, 1]$ (see (2.1) and (3.1)) is even and we have*

$$\big|\Omega_n(\mathbf{e}_n, x)\big| \le \frac{2}{n}, \quad x \in [-1, 1]. \tag{4.1}$$

Proof. The parity of $\Omega_n(\mathbf{e}_n)$ follows immediately from its definition (see (2.1)) using the symmetry property

$$e_s = -e_{n-s+1}, \quad s = 1, 2, \ldots, n.$$

For the proof of (4.1) (cf. [3]) we use the function

$$S_n(x) := \sum_{s=1}^{n} \frac{(-1)^s}{x - e_s}, \quad x \in [-1, 1] \setminus \mathbf{e}_n.$$

Fix an index $j = 1, 2, \ldots, n-1$ and suppose that $x \in \big(e_{j+1}, e_j\big)$. Then we have

$$S_n(x) = (-1)^{j+1}\left(\frac{1}{e_j - x} - \frac{1}{e_{j-1} - x} + \cdots + (-1)^{j+1}\frac{1}{e_1 - x}\right) + \\ +(-1)^{j+1}\left(\frac{1}{x - e_{j+1}} - \frac{1}{x - e_{j+2}} + \cdots + (-1)^{n-j-1}\frac{1}{x - e_n}\right).$$

Observe that the sums are Leibniz type (cf. [3, p. 5], say) thus

$$\big|S_n(x)\big| = \left|\sum_{s=1}^{n} \frac{(-1)^s}{x - e_s}\right| \le \frac{1}{e_j - x} + \frac{1}{x - e_{j+1}} = \frac{2}{n} \cdot \frac{1}{(e_j - x)(x - e_{j+1})}.$$

Using elementary calculations show that

$$\big|S_n(x)\big| \ge \left(\frac{1}{e_j - x} - \frac{1}{e_{j-1} - x}\right) + \left(\frac{1}{x - e_{j+1}} - \frac{1}{x - e_{j+2}}\right) = \\ = \left(\frac{1}{e_j - x} + \frac{1}{x - e_{j+1}}\right) - \left(\frac{1}{e_{j-1} - x} + \frac{1}{x - e_{j+2}}\right) \ge \\ \ge \frac{1}{2n} \cdot \frac{1}{(e_j - x)(x - e_{j+1})}.$$

Consequently, we have

$$|\Omega_n(x)| = \frac{1}{|S_n(x)|} \le 2n \cdot (e_j - x)(x - e_{j+1}) \le \frac{2}{n}, \tag{4.2}$$

if $x \in (e_n, e_1)$ and $n = 2, 4, \dots$.

If $x \in (e_1, 1]$, then

$$|S_n(x)| = \frac{1}{x - e_1} - \frac{1}{x - e_2} + \cdots - \frac{1}{x - e_n} \ge$$
$$\ge \frac{1}{x - e_1} - \frac{1}{x - e_2} = \frac{2}{n} \cdot \frac{1}{(x - e_1)(x - e_2)},$$

so

$$|\Omega_n(x)| = \frac{1}{|S_n(x)|} \le \frac{n}{2} \cdot (x - e_1)(x - e_2) \le \frac{3}{2\,n}. \tag{4.3}$$

Since $\Omega_n(x)$ is even, thus

$$|\Omega_n(x)| \le \frac{3}{2\,n}, \quad x \in [-1, e_n] \cup [e_1, 1].$$

Combining this with (4.2) we obtain (4.1). ■

In the next statement we collect some formulas with respect to the fundamental functions $b_{kn}(\mathbf{e}_n, \cdot)$ (see (2.4)).

Lemma 2. *Let n be an even natural number. Then we have*

(a) $\displaystyle\sum_{k=1}^{m} \big(b_{2k-1,n}(\mathbf{e}_n, x) + b_{2k,n}(\mathbf{e}_n, x)\big) = 1, \quad x \in [-1, 1]$;

(b) $\displaystyle\sum_{k=1}^{m} \big|b_{2k-1,n}(\mathbf{e}_n, x) + b_{2k,n}(\mathbf{e}_n, x)\big| \le 7, \quad x \in [-1, 1]$.

(c) *Let $x \in [-1, 1]$, $\delta > 0$ arbitrary and*

$$\triangle_n(x) := \big\{k \in \{1, 2, \dots, m\} \;\big|\; |x - e_{2k-1,n}| \ge \delta \text{ and } |x - e_{2k,n}| \ge \delta\big\}.$$

Then

$$\sum_{k \in \triangle_n(x)} \big|b_{2k-1,n}(\mathbf{e}_n, x) + b_{2k,n}(\mathbf{e}_n, x)\big| \le \frac{4}{n\,\delta^2}. \tag{4.4}$$

Proof. (a) The identity follows immediately from the definition of $b_{kn}(\mathbf{e}_n, x)$ (see (2.4)).

(b) Fix the index $j = 1, 2, \ldots, n-1$ and suppose that $x \in (e_{j+1}, e_j)$. Then using (2.4), (2.3), (3.1) and (4.2) we obtain that

$$\sum_{k=1}^{m} \left| b_{2k-1}(x) + b_{2k}(x) \right| = \frac{2}{n} \sum_{k=1}^{m} \left| \frac{\Omega_n(x)}{(x - e_{2k-1})(x - e_{2k})} \right| =$$

$$= \frac{2}{n} \cdot \sum_{k=1}^{m} \left| \frac{\Omega_n(x)}{(e_j - x)(x - e_{j+1})} \right| \cdot \left| \frac{(e_j - x)(x - e_{j+1})}{(x - e_{2k-1})(x - e_{2k})} \right| \leq$$

$$\leq 4 \cdot \sum_{k=1}^{m} \left| \frac{(e_j - x)(x - e_{j+1})}{(x - e_{2k-1})(x - e_{2k})} \right| =: A.$$

Suppose that $j = 2l - 1$ is an odd index, say. Then

$$A = 4 \left(\sum_{k=1}^{l-1} \ldots + 1 + \sum_{k=l+1}^{m} \ldots \right).$$

Since both sums can be handled analogously, we consider only the first one:

$$4 \sum_{k=1}^{l-1} \left| \frac{(e_j - x)(x - e_{j+1})}{(x - e_{2k-1})(x - e_{2k})} \right| \leq 4 \sum_{k=1}^{l-1} \frac{\frac{1}{n^2}}{(e_{2l-1} - e_{2k})^2} =$$

$$= 4 \sum_{k=1}^{l-1} \frac{1}{4 \left(2(l-k) - 1 \right)^2} \leq \sum_{s=1}^{+\infty} \frac{1}{(2s-1)^2} = \frac{\pi^2}{8},$$

i.e. $A \leq 4 + 2 \cdot \frac{\pi^2}{8} < 7$.
Therefore, if $x \in \left(e_{j+1}, e_j \right)$ and j is odd, then we have

$$\sum_{k=1}^{m} \left| b_{2k-1}(x) + b_{2k}(x) \right| \leq 7.$$

The same upper bound holds for even j, too.
Consequently, we proved the statement (b) for $x \in (e_n, e_1)$.

Now suppose that $x \in (e_1, 1]$. Then using (4.3) we obtain that

$$\sum_{k=1}^{m} \left| b_{2k-1}(x) + b_{2k}(x) \right| = \frac{2}{n} \sum_{k=1}^{m} \left| \frac{\Omega_n(x)}{(x - e_{2k-1})(x - e_{2k})} \right| \leq$$

$$\leq 1 + \sum_{k=2}^{m} \left| \frac{(x - e_1)(x - e_2)}{(x - e_{2k-1})(x - e_{2k})} \right| \leq 1 + \sum_{k=2}^{m} \frac{\frac{3}{n^2}}{(e_{1n} - e_{2k-1,n})^2} \leq$$

$$\leq 1 + \frac{3}{4^2} \sum_{k=2}^{+\infty} \frac{1}{(k-1)^2} < 2.$$

Since the same upper bound is true for $x \in [-1, e_n)$, too, thus the proof of (b) is complete.

(c) For $k \in \triangle_n(x)$, it follows that $\big|(x-e_{2k-1})(x-e_{2k})\big| \geq \delta^2$, i.e.

$$\frac{\big|(x-e_{2k-1})(x-e_{2k})\big|}{\delta^2} \geq 1,$$

whence using (4.1) we obtain that

$$\sum_{k\in\triangle_n(x)} \big|b_{2k-1}(x)+b_{2k}(x)\big| = \frac{2}{n}\sum_{k\in\triangle_n(x)} \left|\frac{\Omega_n(x)}{(x-e_{2k-1})(x-e_{2k})}\right| \leq$$
$$\leq \frac{2}{n}\sum_{k\in\triangle_n(x)} \frac{\big|(x-e_{2k-1})(x-e_{2k})\big|}{\delta^2}\cdot\left|\frac{\Omega_n(x)}{(x-e_{2k-1})(x-e_{2k})}\right| \leq \frac{4}{n\,\delta^2},$$

which proves (4.4). ■

4.2. Proof of Theorem 1. If $\varepsilon > 0$ is arbitrary, a number $\delta > 0$ can be found such that for $|x'' - x'| < \delta$ implies $\big|f\big(x''\big) - f\big(x'\big)\big| < \varepsilon$.

Now let $x \in [-1, 1]$ be arbitrary. Using Lemma 2 (a) we have

$$f(x) = \sum_{k=1}^{m} f(x)\big(b_{2k-1}(x)+b_{2k}(x)\big),$$

so that

$$\mathcal{B}_n\big(f,\mathbf{e_n},x\big) - f(x) = \sum_{k=1}^{m}\big(f(e_{2k-1})-f(x)\big)\big(b_{2k-1}(x)+b_{2k}(x)\big).$$

We split this sum into two parts

$$\big|\mathcal{B}_n\big(f,\mathbf{e_n},x\big) - f(x)\big| = \Big|\sum_{k\in\Gamma_n(x)}\ldots + \sum_{k\in\triangle_n(x)}\ldots\Big|,$$

where $\Gamma_n(x) := \big\{1,2,\ldots,m\big\}\setminus\triangle_n(x)$. The first sum is $\leq 7\varepsilon$ by Lemma 2 (b), the second is, by (4.4), $\leq 8M/(n\delta^2)$, where $|f(x)| \leq M$ $(x \in [-1,1])$. Therefore

$$\big|\mathcal{B}_n\big(f,\mathbf{e_n},x\big) - f(x)\big| \leq 7\varepsilon+\varepsilon = 8\varepsilon \quad \big(x\in[-1,1]\big),$$

if n is sufficiently large. This completes the proof. ■

4.3. Proof of Theorem 2. Using (3.2) and Lemma 2 (a) we obtain

$$\big|f(x) - \mathcal{B}_n(f,\mathbf{e}_n,x)\big| \leq \sum_{k=1}^{m}\big|f(x)-f(e_{2k-1})\big|\cdot\big|b_{2k-1}(x)+b_{2k}(x)\big|.$$

We consider two cases.

CASE 1. Let $x \in (e_{j+1}, e_j)$, where $j = 1, 2, \dots, n-1$ and first we suppose that $j = 2l-1$ is an odd index, i.e. $x \in (e_{2l}, e_{2l-1})$ $(l = 1, 2, \dots, m)$. We split the above sum into three parts

$$\sum_{k<l} + \sum_{k=l} + \sum_{k>l}. \tag{4.5}$$

Consider first the second term. Here, using (2.4), (2.3), (3.1) and (4.2) we obtain that

$$\begin{aligned}\sum_{k=l} &= \big|f(x) - f(e_{2l-1})\big| \cdot \big|b_{2l-1}(x) + b_{2l}(x)\big| \le \\ &\le 2\omega\big(f, \tfrac{1}{n}\big) \cdot \frac{2}{n} \cdot \frac{2n\big|(x - e_{2l-1})(x - e_{2l})\big|}{\big|(x - e_{2l-1})(x - e_{2l})\big|} = \\ &= 8\omega\big(f, \tfrac{1}{n}\big).\end{aligned}$$

The first term in (4.5) is empty if $l = 1$, so we can suppose that $l \ge 2$. Since

$$\big|x - e_{2k-1}\big| \le e_{2k-1} - e_{2l} \le 6\,\frac{l-k}{n},$$

we have

$$\big|f(x) - f(e_{2k-1})\big| \le \omega\big(f, |x - e_{2k-1}|\big) \le 6\,\omega\left(f, \frac{l-k}{n}\right).$$

Using (2.4), (2.3), (3.1) and (4.2) we obtain

$$\begin{aligned}\big|b_{2k-1}(x) + b_{2k}(x)\big| &= \frac{2}{n}\left|\frac{\Omega_n(x)}{(x - e_{2k-1})(x - e_{2k})}\right| \le \\ &\le \frac{2}{n} \cdot \frac{2n\big|(x - e_{j+1})(x - e_j)\big|}{\big|(x - e_{2k-1})(x - e_{2k})\big|} \le 4 \cdot \frac{\frac{1}{n^2}}{\big(e_{2k} - e_{2l-1}\big)^2} \le \\ &\le \frac{1}{(l-k)^2}.\end{aligned}$$

Consequently we have

$$\begin{aligned}\sum_{k<l} &= \sum_{k<l} \big|f(x) - f(e_{2k-1})\big| \cdot \big|b_{2k-1}(x) + b_{2k}(x)\big| \le \\ &\le 6 \sum_{k<1} \omega\big(f, \tfrac{l-k}{n}\big) \cdot \frac{1}{(l-k)^2} \le \\ &\le 6 \sum_{i=1}^{n} \omega\big(f, \tfrac{i}{n}\big) \cdot \frac{1}{i^2}.\end{aligned}$$

Similarly we get

$$\sum_{k>l} \leq 4 \sum_{i=1}^{n} \omega\big(f, \tfrac{i}{n}\big) \cdot \frac{1}{i^2}.$$

Collecting the above estimates we obtain

$$\big|f(x) - \mathcal{B}_n(f, \mathbf{e}_n, x)\big| \leq 8 \sum_{i=1}^{n} \omega\big(f, \tfrac{i}{n}\big) \cdot \frac{1}{i^2},$$

if $x \in (e_{j+1}, e_j)$ $(j = 1, 2, \dots, n-1)$, moreover j is odd.

The case $j =$ even is analogous.

Thus we proved that

$$\big|f(x) - \mathcal{B}_n(f, \mathbf{e}_n, x)\big| \leq 8 \sum_{i=1}^{n} \omega\left(f, \frac{i}{n}\right) \cdot \frac{1}{i^2}, \tag{4.6}$$

if $x \in (e_n, e_1)$ and $n = 2, 4, \dots$.

Case 2. Let $x \in [-1, e_n] \cup [e_1, 1]$. First we suppose that $x \in [e_1, 1]$. Then we get

$$\begin{aligned}
\big|f(x) - \mathcal{B}_n(f, \mathbf{e}_n, x)\big| &\leq \sum_{k=1}^{m} \big|f(x) - f(e_{2k-1})\big| \cdot \big|b_{2k-1}(x) + b_{2k}(x)\big| \leq \\
&\leq \sum_{k=1}^{m} 4\,\omega\big(f, \tfrac{k}{n}\big) \cdot \frac{2}{n} \cdot \left| \frac{\Omega_n(x)}{(x - e_{2k-1})(x - e_{2k})} \right| \leq \\
&\leq 4 \sum_{k=1}^{m} \omega\big(f, \tfrac{k}{n}\big) \cdot \frac{2}{n} \cdot \frac{n}{2} \cdot \frac{\big|(x - e_1)(x - e_2)\big|}{\big|(x - e_{2k-1})(x - e_{2k})\big|} \leq \\
&\leq 4 \sum_{k=1}^{m} \omega\big(f, \tfrac{k}{n}\big) \cdot \frac{\frac{3}{n^2}}{\big(4\frac{k}{n}\big)^2} \leq \frac{3}{4} \sum_{i=1}^{n} \omega\big(f, \tfrac{i}{n}\big) \cdot \frac{1}{i^2}.
\end{aligned}$$

If $x \in [-1, e_n]$ we obtain

$$\big|f(x) - \mathcal{B}_n(f, \mathbf{e}_n, x)\big| \leq 3 \sum_{i=1}^{n} \omega\big(f, \tfrac{i}{n}\big) \cdot \frac{1}{i^2}.$$

Thus for $x \in [-1, e_n] \cup [e_1, 1]$ we proved that

$$\big|f(x) - \mathcal{B}_n(f, \mathbf{e}_n, x)\big| \leq 3 \sum_{i=1}^{n} \omega\left(f, \frac{i}{n}\right) \cdot \frac{1}{i^2}, \tag{4.7}$$

From (4.6) and (4.7) we get (3.4). This completes the proof of Theorem 2. ■

5. Remarks

5.1. The papers L. Szili – P. Vértesi [18] and G. Mastroianni – J. Szabados [11] closely connected the theorems of the present paper.

5.2. Our results can be proved for the generalization of the operator (3.2). The proof gives only technical difficulties. We omit the details.

5.3. One can prove our theorems for the so called "well spaced nodes" (for the definition see J. Sidon [14, (7)–(9)]). Again, we have to consider some technicalities.

5.4. We can define the barycentric Hermite–Fejér type interpolation using $(b_{kn})^s$, where $s \geq 2$, integer. Estimations analogous to the formulas (3.3) and (3.4) can obtained. We omit the details.

5.5. The problem of the saturation is open. We conjecture that the order is $1/n$; but to get the saturation class seems to be quite difficult. We intend to deal with this problem in the near future.

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