

A NOTE ON THE MAXIMAL FUNCTION

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Dedicated to Professor Antal Járαι on his 70th birthday

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Abstract. We investigate the Hardy–Littlewood maximal function. A local condition for non-integrability is formulated. Moreover, a generalization of the Hardy–Littlewood’s and Stein’s integrability theorem is given. As special case, we get the classical statements.

1. Introduction

If $[a, b]$ is a compact interval on the real line and $f \in L^1[a, b]$ (i.e. the real valuable function f is defined on $[a, b]$ and it is integrable in Lebesgue’s sense), then Mf (the *Hardy–Littlewood maximal function* of f) is defined as follows:

$$Mf(x) := \sup_I \frac{1}{|I|} \int_I |f| \quad (x \in [a, b]),$$

where the supremum is taken over all $x \in I \subset [a, b]$ intervals (and $|A|$ stands for the Lebesgue measure of a set $A \subset \mathbb{R}$). It is well-known [2] that for the measurable function $Mf : [a, b] \rightarrow [0, +\infty]$ (in other words for the maximal operator M) the next properties hold: there is an absolute constant $C > 0$ such that

$$|\{Mf > y\}| \leq \frac{C}{y} \cdot \|f\|_1 \quad (f \in L^1[a, b], y > 0)$$

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(the operator M is of *weak type* $(1, 1)$). Moreover, the "strong" property

$$|\{Mf > y\}| \leq \frac{C}{y} \cdot \int_{\{|f|>y\}} |f| \quad (f \in L^1[a, b], y > 0)$$

is also true. From this it is clear that

$$|\{Mf = +\infty\}| \leq |\{Mf > y\}| \leq \frac{C}{y} \cdot \|f\|_1 \rightarrow 0 \quad (y \rightarrow +\infty),$$

i.e. $|\{Mf = +\infty\}| = 0$ and thus the maximal function is finite a.e. Furthermore [2], for all $1 < p \leq +\infty$ with a suitable constant $C_p > 0$ (depending only on p)

$$\|Mf\|_p \leq C_p \cdot \|f\|_p \quad (f \in L^p[a, b])$$

(the operator M is of *type* (p, p)). Since $|f| \leq Mf$ a.e., we can say that for these exponents p the equivalence

$$\|Mf\|_p < +\infty \iff \|f\|_p < +\infty$$

holds. However, there is a function $f \in L^1[a, b]$ for which $\|Mf\|_1 = +\infty$ (for example see the function f_0 in the next section).

2. Non-integrability

If $f \in L^1[a, b]$ is given then let

$$F(t) := \int_a^t |f| \quad (t \in [a, b]).$$

It is obvious that for all $a \leq d < y < b$

$$Mf(x) \geq \frac{1}{x-d} \cdot \int_d^x |f| \geq \frac{F(y) - F(d)}{x-d} \quad (y \leq x \leq b).$$

Therefore

$$\begin{aligned} \|Mf\|_1 &= \int_a^b Mf \geq \int_y^b Mf \geq (F(y) - F(d)) \cdot \int_y^b \frac{dx}{x-d} = \\ &= (F(y) - F(d)) \cdot (\ln(b-d) - \ln(y-d)). \end{aligned}$$

If we take into account that $F(y) - F(d) \rightarrow 0$ ($y \rightarrow d$), it follows

$$\|Mf\|_1 \geq \limsup_{y \rightarrow d+0} ((F(y) - F(d)) \cdot |\ln(y - d)|).$$

So we get the next

Proposition 1. *If $f \in L^1[a, b]$ and for some points $a \leq d < y < b$*

$$\limsup_{y \rightarrow d+0} ((F(y) - F(d)) \cdot |\ln(y - d)|) = +\infty,$$

then $\|Mf\|_1 = +\infty$.

Analogously, if $a < e \leq b$, then

$$\limsup_{y \rightarrow e-0} ((F(e) - F(y)) \cdot |\ln(e - y)|) = +\infty \implies \|Mf\|_1 = +\infty,$$

or with an arbitrary point $z \in [a, b]$:

$$\limsup_{y \rightarrow z} (|F(y) - F(z)| \cdot |\ln(|y - z|)|) = +\infty \implies \|Mf\|_1 = +\infty.$$

(We see later that in this connection the assumption $\limsup \dots > 0$ is also enough.)

Since

$$\begin{aligned} (F(y) - F(d)) \cdot |\ln(y - d)| &= (y - d) \cdot |\ln(y - d)| \cdot \frac{1}{y - d} \cdot \int_d^y |f| \leq \\ &\leq Mf(d) \cdot (y - d) \cdot |\ln(y - d)| \end{aligned}$$

and

$$(y - d) \cdot \ln(y - d) \rightarrow 0 \quad (y \rightarrow d + 0),$$

it follows (assumed $Mf(d) < +\infty$) that

$$\lim_{y \rightarrow d+0} ((F(y) - F(d)) \cdot \ln(y - d)) = 0,$$

or

$$\lim_{y \rightarrow e-0} ((F(e) - F(y)) \cdot \ln(e - y)) = 0,$$

if $Mf(e) < +\infty$. We know that Mf is finite a.e., therefore

$$\lim_{y \rightarrow z} ((F(y) - F(z)) \cdot \ln(|y - z|)) = 0 \quad (\text{a.e. } z \in [a, b]).$$

In other words for such points $z \in [a, b]$ we have

$$(1) \quad \lim_{n \rightarrow \infty} |I_n| = 0 \implies \lim_{n \rightarrow \infty} \left(\ln(|I_n|) \cdot \int_{I_n} |f| \right) = 0$$

with every sequence (I_n) of intervals $z \in I_n \subset [a, b]$ ($n \in \mathbb{N}$).

We recall that F is differentiable a.e. and if this is true for a point z then

$$\frac{1}{y-z} \cdot \int_z^y |f| = \frac{F(y) - F(z)}{y-z} \rightarrow F'(z) = |f(z)| \quad (y \rightarrow z)$$

and (1) follows.

For example, we consider the function

$$f(x) := f_0(x) := \begin{cases} \frac{1}{(x-a) \cdot \ln^2(x-a)} & (a < x \leq c) \\ 0 & (x \in [a, b] \setminus (a, c)) \end{cases}$$

(where $c \in (a, b)$ and $c - a < 1$),. It is not hard to see that $f_0 \in L^1[a, b]$ and

$$F(y) \cdot |\ln(y-a)| = 1 \quad (y \in (a, c)).$$

On the other hand, if

$$f(x) := \begin{cases} \frac{1}{(x-a) \cdot |\ln(x-a)|^{3/2}} & (a < x \leq c) \\ 0 & (x \in [a, b] \setminus (a, c]), \end{cases}$$

then $f \in L^1[a, b]$, but

$$F(y) \cdot |\ln(y-a)| = 2 \cdot \sqrt{|\ln(y-a)|} \rightarrow +\infty \quad (y \rightarrow a).$$

The first statement on the integrability of Mf reads as follows [2]:

$$f \in L \log^+ L[a, b] \implies \|Mf\|_1 < \infty.$$

This means by Proposition 1 that for all $z \in [a, b]$ the implication

$$(2) \quad f \in L \log^+ L[a, b] \implies \limsup_{y \rightarrow z} (|F(y) - F(z)| \cdot |\ln(|y-z|)|) < +\infty$$

is true. Moreover, it is easy to show

Proposition 2. *Let $f \in L \log^+ L[a, b]$, then for all $z \in [a, b]$ we have*

$$\limsup_{y \rightarrow z} (|F(y) - F(z)| \cdot |\ln(|y - z|)|) = 0.$$

Proof. For example, take the points $a < z = d < y < b$ and $y - d < 1$. Then

$$\begin{aligned} F(y) - F(d) &= \int_d^y |f| = \int_{[d,y] \cap \{|f| < 1/\sqrt{y-d}\}} |f| + \int_{[d,y] \cap \{|f| \geq 1/\sqrt{y-d}\}} |f| \leq \\ &\leq \sqrt{y-d} + \int_{[d,y] \cap \{f^2 \geq 1/(y-d)\}} |f|. \end{aligned}$$

Therefore

$$\begin{aligned} &(F(y) - F(d)) \cdot |\ln(y-d)| \leq \\ &\leq \sqrt{y-d} \cdot |\ln(y-d)| + 2 \cdot \int_{[d,y] \cap \{f^2 \geq 1/(y-d)\}} |f(x)| \cdot \ln(|f(x)|) dx \leq \\ &\leq \sqrt{y-d} \cdot |\ln(y-d)| + 2 \cdot \int_d^y |f| \cdot (\log^+ \circ |f|) \end{aligned}$$

and here

$$\sqrt{y-d} \cdot |\ln(y-d)| \rightarrow 0 \quad (y \rightarrow d+0).$$

If $f \in L \log^+ L[a, b]$, then

$$\int_d^y |f| \cdot (\log^+ \circ |f|) \rightarrow 0 \quad (y \rightarrow d+0),$$

which leads to

$$(F(y) - F(d)) \cdot |\ln(y-d)| \rightarrow 0 \quad (y \rightarrow d+0). \quad \blacksquare$$

The function f_0 shows that the converse of (2) is not true, in contrast to the above statement on the integrability of Mf . Namely, Stein [3] proved that for a function $f \in L^1[a, b]$, we get

$$\|Mf\|_1 < +\infty \implies f \in L \log^+ L[a, b].$$

From this it follows immediately that for all $f \in L^1[a, b]$ the next equivalence is true:

$$(3) \quad \|Mf\|_1 < +\infty \iff f \in L \log^+ L[a, b]$$

and (see Proposition 2)

$$\|Mf\|_1 < +\infty \implies \lim_{y \rightarrow z} ((F(y) - F(z)) \cdot \ln(|y - z|)) = 0 \quad (z \in [a, b]).$$

In other words the next consequence holds:

Proposition 3. *If $f \in L^1[a, b]$ and there is a point $z \in [a, b]$ such that*

$$\limsup_{y \rightarrow z} (|F(y) - F(z)| \cdot |\ln(|y - z|)|) > 0,$$

then $\|Mf\|_1 = +\infty$.

Now, we consider an example which shows that the assumption $\|Mf\|_1 = +\infty$ is not enough for

$$\limsup_{y \rightarrow z} (|F(y) - F(z)| \cdot |\ln(|y - z|)|) > 0.$$

Proposition 4. *There exists $f \in L^1[a, b]$ such that $\|Mf\|_1 = +\infty$, but*

$$\limsup_{y \rightarrow z} (|F(y) - F(z)| \cdot |\ln(|y - z|)|) = 0$$

for all $z \in [a, b]$.

Proof. Indeed, let

$$c_n := q_n \cdot 2^n \quad (n \in \mathbb{N}),$$

where the numbers $q_n > 0$ ($n \in \mathbb{N}$) satisfy the conditions

$$\sum_{n=0}^{\infty} q_n < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} nq_n = +\infty$$

and

$$n \cdot \sum_{k=n}^{\infty} q_k \rightarrow 0 \quad (n \rightarrow \infty).$$

For example the numbers

$$q_n := \frac{1}{(n+1)^2 \cdot \ln(n+2)} \quad (n \in \mathbb{N})$$

fulfil these requirements. By means of c_n 's we define the function

$$f = f_1 : [0, 1] \rightarrow (0, +\infty)$$

as follows:

$$f_1(x) := c_n \quad (x \in [2^{-n-1}, 2^{-n}], n \in \mathbb{N}).$$

Then

$$\int_0^1 |f_1| = \sum_{n=0}^{\infty} c_n \cdot 2^{-n-1} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} q_n < +\infty,$$

i.e. $f_1 \in L^1[0, 1]$. However, with a suitable $N \in \mathbb{N}$ it follows that

$$\begin{aligned} \int_0^1 |f_1| \cdot (\log^+ \circ |f|) &= \frac{1}{2} \cdot \sum_{n=0}^{\infty} q_n \cdot \log^+(c_n) = \frac{1}{2} \cdot \sum_{n=N}^{\infty} q_n \cdot \ln c_n = \\ &= \frac{1}{2} \cdot \sum_{n=N}^{\infty} q_n (n \cdot \ln 2 - (2 \cdot \ln(n+1) + \ln(\ln(n+2))). \end{aligned}$$

Here we can assume also that

$$2 \cdot \ln(n+1) + \ln(\ln(n+2)) < \frac{n \cdot \ln 2}{2} \quad (N \leq n \in \mathbb{N}).$$

So

$$\int_0^1 |f_1| \cdot (\log^+ \circ |f|) \geq \frac{\ln 2}{4} \cdot \sum_{n=N}^{\infty} n q_n = +\infty.$$

This means that $f_1 \notin L \log^+ L[0, 1]$, i.e. $\|Mf\|_1 = +\infty$. Furthermore, let $n \in \mathbb{N}$ be given, then

$$\int_0^{2^{-n}} |f_1| = \sum_{k=n}^{\infty} c_k \cdot 2^{-k-1} = \frac{1}{2} \cdot \sum_{k=n}^{\infty} q_k$$

and we get

$$|\ln 2^{-n}| \cdot \int_0^{2^{-n}} |f_1| = \frac{n \cdot \ln 2}{2} \cdot \sum_{k=n}^{\infty} q_k \rightarrow 0 \quad (n \rightarrow \infty).$$

Finally, if $0 < z \in [0, 1]$, then f_1 is bounded on $[z/2, 1]$ and this implies

$$\lim_{n \rightarrow \infty} |I_n| = 0 \implies \lim_{n \rightarrow \infty} \left(\ln(|I_n|) \cdot \int_{I_n} |f| \right) = 0$$

for all intervals $z \in I_n \subset [0, 1]$ ($n \in \mathbb{N}$). ■

Thus for the set

$$S[a, b] := \{f \in L^1[a, b] : \lim_{y \rightarrow z} ((F(y) - F(z)) \cdot \ln(|y - z|)) = 0 \quad (z \in [a, b])\}$$

the inclusions

$$L \log^+ L[a, b] \subset S[a, b] \subset L^1[a, b]$$

are strong:

$$f_1 \in S[a, b] \setminus L \log^+ L[a, b] \quad \text{and} \quad f_0 \in L^1[a, b] \setminus S[a, b].$$

3. Integrability

In connection of the equivalence (3) Stein [4, p. 23] says in his famous book (without proof) that for all functions $f \in L^1[a, b]$ and for all numbers $\rho > 0$

$$\int_a^b Mf \cdot (\log^+ \circ (Mf))^\rho < +\infty \iff \int_a^b |f| \cdot (\log^+ \circ |f|)^{\rho+1} < +\infty$$

holds. We remark, that a proof for \Leftarrow can be found in the books of Weisz and Bennet–Sharpley [5], [1], resp. Next we shall generalize this Stein's equivalence. As special case, a common proof for the (L^p, L^p) and $(L \log^+ L, L^1)$ -boundedness follows as well.

To this end let the functions

$$\psi, \Lambda : [0, +\infty] \rightarrow [0, +\infty]$$

be given and assume that the next properties hold: ψ, Λ are absolute continuous on all compact intervals $I \subset [0, +\infty)$, are nondecreasing,

$$\lim_{x \rightarrow +\infty} \psi(x) = \lim_{x \rightarrow +\infty} \Lambda(x) = +\infty$$

and $\psi(0) = \Lambda(0) = 0$. We consider the functions

$$\Phi(y) := y \cdot \psi(y) \quad (y \geq 0)$$

and

$$\Theta(y) := y \cdot \Lambda(y) \quad (y \geq 0).$$

It will be assumed also that there are numbers $r, c > 0$ such that

$$c \cdot y \cdot \psi'(y) \leq \int_0^y t^{-1} \cdot \psi(t) dt \leq \Lambda(y) + y \cdot \psi'(y) \quad (\text{a.e. } y > r)$$

and

$$\Theta'(y) = \int_0^y t^{-1} \cdot \Phi'(t) dt \quad (\text{a.e. } y > 0).$$

Theorem. For all $f \in L^1[a, b]$ we have

$$\int_a^b \Phi \circ (Mf) < +\infty \iff \int_a^b \Theta \circ |f| < +\infty.$$

It is clear that in the case $\Theta \leq \Phi$ the part \implies follows simple from $|f| \leq Mf$ a.e.

Proof of Theorem. First of all, we remark that by the assumptions there is $\alpha > r$ such that $\psi(\alpha) > 0$ and $\Lambda(\alpha) > 0$. If $g : [0, +\infty] \rightarrow [0, +\infty]$ is Lebesgue-measurable and a.e. finite, then

$$\int_a^b g \cdot (\psi \circ g) = \int_{\{g \leq \alpha\}} g \cdot (\psi \circ g) + \int_{\{g > \alpha\}} g \cdot (\psi \circ g),$$

where

$$\int_{\{g \leq \alpha\}} g \cdot (\psi \circ g) \leq (b-a) \cdot \alpha \cdot \psi(\alpha) < +\infty.$$

Furthermore

$$\begin{aligned} \int_{\{g > \alpha\}} g \cdot (\psi \circ g) &= \int_a^b (\Phi \circ g) \cdot \chi_{\{g > \alpha\}} = \int_0^{+\infty} |\{(\Phi \circ g) \cdot \chi_{\{g > \alpha\}} > y\}| dy = \\ &= \int_0^{\Phi(\alpha)} |\{(\Phi \circ g) \cdot \chi_{\{g > \alpha\}} > y\}| dy + \int_{\Phi(\alpha)}^{+\infty} |\{(\Phi \circ g) \cdot \chi_{\{g > \alpha\}} > y\}| dy =: A + B. \end{aligned}$$

Here

$$A = \int_0^{\Phi(\alpha)} |\{(\Phi \circ g) \cdot \chi_{\{g > \alpha\}} > y\}| dy \leq (b-a) \cdot \Phi(\alpha) < +\infty$$

and (taking into account that Φ is increasing on $[\alpha, +\infty)$ and absolute continuous on $[\alpha, z]$ for all $z > \alpha$)

$$\begin{aligned} B &= \int_{\Phi(\alpha)}^{+\infty} |\{(\Phi \circ g) \cdot \chi_{\{g > \alpha\}} > y\}| dy = \\ &= \int_{\alpha}^{+\infty} |\{(\Phi \circ g) \cdot \chi_{\{g > \alpha\}} > \Phi(y)\}| \cdot \Phi'(y) dy = \int_{\alpha}^{+\infty} |\{g > y\}| \cdot \Phi'(y) dy. \end{aligned}$$

Therefore

$$\int_a^b g \cdot (\psi \circ g) < +\infty \iff \int_\alpha^{+\infty} |\{g > y\}| \cdot \Phi'(y) dy < +\infty.$$

Now we assume that

$$\int_a^b \Theta \circ |f| = \int_a^b |f| \cdot (\Lambda \circ |f|) < +\infty.$$

Then

$$\begin{aligned} \int_\alpha^{+\infty} |\{Mf > y\}| \cdot \Phi'(y) dy &\leq C \cdot \int_\alpha^{+\infty} y^{-1} \cdot \left(\int_{\{|f|>y\}} |f| \right) \cdot \Phi'(y) dy = \\ &= C \cdot \int_\alpha^{+\infty} y^{-1} \cdot \left(\int_a^b |f| \cdot \chi_{\{|f|>y\}} \right) \cdot \Phi'(y) dy = \\ &= C \cdot \int_\alpha^{+\infty} y^{-1} \cdot \left(\int_0^{+\infty} |\{|f| \cdot \chi_{\{|f|>y\}} > t\}| dt \right) \cdot \Phi'(y) dy = \\ &= C \cdot \left(\int_\alpha^{+\infty} y^{-1} \cdot \left(\int_0^y |\{|f| > y\}| dt + \int_y^{+\infty} |\{|f| > t\}| dt \right) \cdot \Phi'(y) dy \right) = \\ &= C \cdot \left(\int_\alpha^{+\infty} |\{|f| > y\}| \cdot \Phi'(y) dy + \int_\alpha^{+\infty} y^{-1} \cdot \left(\int_y^{+\infty} |\{|f| > t\}| dt \right) \cdot \Phi'(y) dy \right). \end{aligned}$$

We obtain

$$\begin{aligned} &\int_\alpha^{+\infty} y^{-1} \cdot \left(\int_y^{+\infty} |\{|f| > t\}| dt \right) \cdot \Phi'(y) dy = \\ &= \int_\alpha^{+\infty} y^{-1} \cdot \left(\int_\alpha^{+\infty} |\{|f| > t\}| \cdot \chi_{[y,+\infty)}(t) dt \right) \cdot \Phi'(y) dy = \\ &= \int_\alpha^{+\infty} |\{|f| > t\}| \cdot \left(\int_\alpha^t y^{-1} \cdot \Phi'(y) dy \right) dt = \int_\alpha^{+\infty} |\{|f| > t\}| \cdot \Theta'(t) dt. \end{aligned}$$

So by our previous remark (with $|f|$ and φ instead of g and ψ , resp.) the boundedness

$$\int_a^b |f| \cdot (\Lambda \circ |f|) < +\infty$$

implies

$$\int_{\alpha}^{+\infty} |\{|f| > t\}| \cdot \Theta'(t) dt < +\infty.$$

By

$$\begin{aligned} c \cdot \Phi'(y) &= c \cdot (\psi(y) + y\psi'(y)) \leq \Theta'(y) = \\ &= \int_0^y z^{-1} \cdot \Phi'(z) dz = \int_0^y (\psi'(z) + z^{-1} \cdot \psi(z)) dz = \\ &= \psi(y) + \int_0^y z^{-1} \cdot \psi(z) dz \quad (\text{a.e. } y > \alpha) \end{aligned}$$

we get

$$\int_{\alpha}^{+\infty} |\{|f| > y\}| \cdot \Phi'(y) dy \leq c^{-1} \cdot \int_{\alpha}^{+\infty} |\{|f| > y\}| \cdot \Theta'(y) dy < +\infty.$$

Summarizing the above facts it follows that

$$\int_a^b \Phi \circ (Mf) = \int_a^b Mf \cdot (\psi \circ (Mf)) < +\infty.$$

Next we assume the last boundedness. Then

$$\begin{aligned} \int_a^b |f| \cdot (\Lambda \circ |f|) &= \int_{\{|f| \leq \alpha\}} |f| \cdot (\Lambda \circ |f|) + \int_{\{|f| > \alpha\}} |f| \cdot (\Lambda \circ |f|) \leq \\ &\leq (b-a) \cdot \alpha \cdot \Lambda(\alpha) + \int_{\{|f| > \alpha\}} |f| \cdot (\Lambda \circ |f|) =: C + D. \end{aligned}$$

Since $C < +\infty$, it is enough to investigate the second expression:

$$D = \int_{\{|f| > \alpha\}} |f| \cdot (\Lambda \circ |f|) = \int_{\{|f| > \alpha\}} |f(x)| \cdot \left(\int_0^{|f(x)|} \varphi(y) dy \right),$$

where the function φ is defined by means of the equality

$$\Lambda(y) = \int_0^y \varphi(z) dz \quad (y \geq 0).$$

Thus

$$\begin{aligned} \int_{\{|f|>\alpha\}} |f| \cdot (\Lambda \circ |f|) &= \int_{\{|f|>\alpha\}} |f(x)| \cdot \left(\int_0^{+\infty} \varphi(y) \cdot \chi_{[0,|f(x)|)}(y) dy \right) dx = \\ &= \int_0^{+\infty} \varphi(y) \cdot \left(\int_{\{|f|>\alpha\}} |f(x)| \cdot \chi_{[0,|f(x)|)}(y) dx \right) dy = \\ &= \int_0^\alpha \varphi(y) \cdot \left(\int_{\{|f|>\alpha\}} |f| \right) dy + \int_\alpha^{+\infty} \varphi(y) \cdot \left(\int_{\{|f|>y\}} |f| \right) dy \leq \\ &\leq \|f\|_1 \cdot \int_0^\alpha \varphi(y) dy + \int_\alpha^{+\infty} \varphi(y) \cdot \left(\int_{\{|f|>y\}} |f| \right) dy, \end{aligned}$$

where

$$\|f\|_1 \cdot \int_0^\alpha \varphi(y) dy = \|f\|_1 \cdot \Lambda(\alpha) < +\infty.$$

We recall [3] that with a suitable $y_0 > 0$

$$\int_{\{|f|>y\}} |f| \leq 2y \cdot |\{Mf > y\}| \quad (y \geq y_0).$$

Let $\beta := \max\{y_0, \alpha\}$, then

$$\int_\alpha^\beta \varphi(y) \cdot \left(\int_{\{|f|>y\}} |f| \right) dy \leq \|f\|_1 \cdot \int_\alpha^\beta \varphi(y) dy = \|f\|_1 \cdot (\Lambda(\beta) - \Lambda(\alpha)) < +\infty$$

and

$$\int_\beta^{+\infty} \varphi(y) \cdot \left(\int_{\{|f|>y\}} |f| \right) dy \leq 2 \cdot \int_\beta^{+\infty} |\{Mf > y\}| \cdot y \cdot \varphi(y) dy.$$

From assumptions on ψ, Λ it follows that

$$\begin{aligned} y \cdot \varphi(y) &= y \cdot \Lambda'(y) = \Theta'(y) - \Lambda(y) = \int_0^y t^{-1} \cdot \Phi'(t) dt - \Lambda(y) = \\ &= \psi(y) + \int_0^y t^{-1} \cdot \psi(t) dt - \Lambda(y) \leq \psi(y) + y \cdot \psi'(y) = \Phi'(y) \quad (\text{a.e. } y > \alpha). \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\beta}^{+\infty} |\{Mf > y\}| \cdot y \cdot \varphi(y) dy &\leq \int_{\alpha}^{+\infty} |\{Mf > y\}| \cdot \Phi'(y) dy \leq \\ &\leq \int_a^b Mf \cdot (\psi \circ (Mf)) < +\infty, \end{aligned}$$

which proves the boundedness

$$\int_a^b \Theta \circ |f| = \int_a^b |f| \cdot (\Lambda \circ |f|) < +\infty. \quad \blacksquare$$

4. Special cases

a) First let $\rho > 0$ and

$$\psi(x) := (\log^+ x)^\rho \quad (x \geq 0).$$

Then

$$\psi'(y) = \begin{cases} 0 & (0 \leq y < 1) \\ \rho \cdot y^{-1} \cdot (\ln y)^{\rho-1} & (y > 1) \end{cases}$$

and

$$\int_0^y t^{-1} \cdot \psi(t) dt = \begin{cases} 0 & (0 \leq y < 1) \\ (\rho + 1)^{-1} \cdot (\ln y)^{\rho+1} & (y > 1). \end{cases}$$

If $r := e^{\sqrt{\rho(\rho+1)}}$, then

$$y \cdot \psi'(y) = \rho \cdot (\ln y)^{\rho-1} \leq \int_1^y t^{-1} \cdot \psi(t) dt = \frac{1}{\rho+1} \cdot (\ln y)^{\rho+1} \quad (y \geq r).$$

It can be simply verified that with the function

$$\Lambda(y) := \frac{1}{\rho + 1} \cdot (\log^+ y)^{\rho+1} \quad (y \geq 0),$$

we get

$$\Theta'(y) = \begin{cases} 0 & (0 \leq y < 1) \\ (\rho + 1)^{-1} \cdot (\ln y)^{\rho+1} + (\ln y)^\rho & (y > 1) \end{cases}$$

and

$$\begin{aligned} & \int_0^y t^{-1} \cdot \psi'(t) dt = \\ & = \int_0^y t^{-1} \cdot \psi(t) dt + \psi(y) = \begin{cases} 0 & (0 \leq y < 1) \\ (\rho + 1)^{-1} \cdot (\ln y)^{\rho+1} + (\ln y)^\rho & (y > 1). \end{cases} \end{aligned}$$

Since

$$\int_0^y t^{-1} \cdot \psi(t) dt \leq \Lambda(y) \quad (y \geq 0),$$

the second assumption for ψ is also fulfilled. Therefore for all $f \in L^1[a, b]$ the equivalence

$$\int_a^b Mf \cdot (\log^+ \circ (Mf))^\rho < +\infty \iff \int_a^b |f| \cdot (\log^+ \circ |f|)^{\rho+1} < +\infty$$

holds, which is the above mentioned Stein's equivalence.

b) Now, let $s > 0$ be given and define the function ψ as follows:

$$\psi(x) := x^s \quad (x \geq 0).$$

Then

$$\Phi(x) := x^{s+1} \quad (x \geq 0)$$

and

$$y \cdot \psi'(y) = s \cdot y^s \quad (y \geq 0).$$

Furthermore

$$\int_0^y t^{-1} \cdot \psi(t) dt = \frac{y^s}{s} \quad (y \geq 0)$$

and

$$\Theta'(y) = (s+1) \cdot \int_0^y t^{s-1} dt = \frac{s+1}{s} \cdot y^s \quad (y \geq 0).$$

In other words

$$\Theta(y) = \frac{y^{s+1}}{s} \quad (y \geq 0)$$

and

$$\Lambda(y) = \frac{y^s}{s} \quad (y \geq 0)$$

and our assumptions for ψ (with the constant $c := 1/s^2$) are true. Thus Theorem gives the next equivalence for all $1 < p < +\infty$:

$$\|Mf\|_p < +\infty \iff \|f\|_p < +\infty,$$

which is the Hardy–Littlewood’s (L^p, L^p) -boundedness.

c) Finally, we take $s > 0$ and

$$\psi(x) := x^s \cdot \log^+ x \quad (x \geq 0).$$

Then

$$\begin{aligned} cy \cdot \psi'(y) &= c \cdot (sy^s \cdot \ln y + y^s) \leq \int_1^y t^{-1} \cdot \psi(t) dt = \\ &= \frac{1}{s} \cdot y^s \cdot \ln y - \frac{1}{s^2} \cdot y^s + \frac{1}{s^2} \quad (y \geq r) \end{aligned}$$

with $c := 1/(2s^2)$ and with a suitable $r > 1$. Furthermore

$$\Theta'(y) = \frac{s+1}{s} \cdot y^s \cdot \ln y - \frac{1}{s^2} \cdot y^s + \frac{1}{s^2} \quad (y \geq 1).$$

Therefore let

$$\Theta(y) := \begin{cases} 0 & (0 \leq y < 1) \\ \frac{1}{s} \cdot y^{s+1} \cdot \ln y - \frac{1}{s^2} \cdot y^{s+1} + \frac{1}{s^2} \cdot y & (y \geq 1), \end{cases}$$

and the second assumption on ψ is automatically fulfilled. Thus

$$\int_a^b \Phi \circ (Mf) < +\infty \iff \int_a^b \Theta \circ |f| < +\infty.$$

It is easy to see that

$$\int_a^b \Theta \circ |f| < +\infty \iff \int_a^b |f|^{s+1} \cdot (\log^+ \circ |f|) < +\infty,$$

which implies the next equivalence for all $1 < p < +\infty$:

$$\int_a^b (Mf)^p \cdot (\log^+ \circ (Mf)) < +\infty \iff \int_a^b |f|^p \cdot (\log^+ \circ |f|) < +\infty.$$

(where \implies follows from $|f| \leq Mf$ a.e. as well also for $p = +\infty$).

We remark that the last equivalence does not hold in the case $p = 1$. To this end we consider the function $f \in L^1[0, 1]$ such that

$$f(x) := \frac{2^n}{(n+1)^3} \quad (n \in \mathbb{N}, 2^{-n-1} < x \leq 2^{-n}).$$

Then

$$\int_0^1 |f| \cdot (\log^+ \circ |f|) < \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

Since for $x \in (2^{-n-1}, 2^{-n})$ the inequality

$$Mf(x) \geq 2^n \cdot \int_0^{2^{-n}} f = 2^{n-1} \cdot \sum_{k=n}^{\infty} \frac{1}{(k+1)^3} > \frac{2^{n-2}}{(n+1)^2} \quad (11 \leq n \in \mathbb{N})$$

is true, so we get

$$\int_0^1 Mf \cdot (\log^+ \circ (Mf)) \geq \sum_{n=10}^{\infty} \int_{2^{-n-1}}^{2^{-n}} Mf \cdot (\log^+ \circ (Mf)) \geq \frac{1}{16} \cdot \sum_{n=11}^{\infty} \frac{1}{n+1} = +\infty.$$

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