Annales Univ. Sci. Budapest., Sect. Comp. 51 (2020) 173–177

ON THE ESSENTIAL UNION AND INTERSECTION OF FAMILIES OF MEASURABLE SETS

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Dedicated to the 70th birthday of Professor Antal Járai

Communicated by Péter Simon

(Received February 15, 2020; accepted February 28, 2020)

Abstract. In this short note, we demonstrate that in a σ-finite measure space the union and the intersection of arbitrary (i.e., not necessary countable) families of measurable sets can be defined in a natural manner. By examples, we also show that the σ-finiteness of the underlying measure space is only sufficient for this property but not necessary.

The notions that we discuss below turn out to be significant if one tries to extend the Radon–Nikodym Theorem for non-σ-finite measure spaces. For details, we refer to Section 1.1 and pages 65–71 of the recent monograph [1] by Fonseca and Leoni. More comments will be given at the end of this note.

Definition 1. Given a measure space $(X, \mathcal{A}, \mu)$, we say that a set $A \in \mathcal{A}$ is *μ-contained* in $B \in \mathcal{A}$ if $\mu(A \setminus B) = 0$ and we denote this fact by $A \subseteq_\mu B$. Analogously, a set $A \in \mathcal{A}$ is said to be *μ-equal* to $B \in \mathcal{A}$ if $\mu((A \setminus B) \cup (B \setminus A)) = 0$ holds and this property is denoted as $A =_\mu B$.

It is easy to see that $=_\mu$ is an equivalence relation on the σ-algebra $\mathcal{A}$, and $\subseteq_\mu$ is a partial ordering on the equivalence classes. It can also be shown that the function d_μ defined by

$$d_\mu(A, B) := \mu((A \setminus B) \cup (B \setminus A)) \qquad (A, B \in \mathcal{A})$$

Key words and phrases: Essential union and intersection of measurable sets.
2010 Mathematics Subject Classification: 28A.
The research was supported by the 2019-2.1.11-TÉT-2019-00049, EFOP-3.6.1-16-2016-00022 and EFOP-3.6.2-16-2017-00015 projects. The last two projects are co-financed by the European Union and the European Social Fund.

is a semimetric on $\mathcal{A}$ and it is a metric on the equivalence classes with respect to the equivalence relation $=_\mu$.

Definition 2. For a family $\{A_\gamma : \gamma \in \Gamma\} \subseteq \mathcal{A}$ of measurable subsets, we say that a set $\overline{A} \in \mathcal{A}$ is its *μ-essential union* if

$$A_\gamma \subseteq_\mu \overline{A} \qquad (\gamma \in \Gamma),$$

and if, for $A^* \in \mathcal{A}$,

$$A_\gamma \subseteq_\mu A^* \qquad (\gamma \in \Gamma),$$

holds, then $\overline{A} \subseteq_\mu A^*$. The notion of the *$\mu$-essential intersection* is defined completely in a similar manner: A set $\underline{A} \in \mathcal{A}$ is called the μ-essential intersection of the family $\{A_\gamma : \gamma \in \Gamma\} \subseteq \mathcal{A}$ if

$$\underline{A} \subseteq_\mu A_\gamma \qquad (\gamma \in \Gamma),$$

and if, for $A_* \in \mathcal{A}$,

$$A_* \subseteq_\mu A_\gamma \qquad (\gamma \in \Gamma),$$

is satisfied, then $A_* \subseteq_\mu \underline{A}$.

Clearly, if the sets $\overline{A}, \underline{A} \in \mathcal{A}$ exist, then they are uniquely determined up to the equivalence $=_\mu$. It is also easy to see that, for all countable subset $\Gamma_0 \subseteq \Gamma$,

$$\underline{A} \subseteq_\mu \bigcap_{\gamma\in\Gamma_0} A_\gamma, \qquad \bigcup_{\gamma\in\Gamma_0} A_\gamma \subseteq_\mu \overline{A}$$

are valid. On the other hand, in general, it is not true that $\underline{A} \subseteq_\mu \bigcap_{\gamma\in\Gamma} A_\gamma$, and $\bigcup_{\gamma\in\Gamma} A_\gamma \subseteq_\mu \overline{A}$. To see the clear difference from the standard notions of union and intersection, consider the following example. Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the standard Lebesgue measure space on $\mathbb{R}$. Then, for any non-Lebesgue measurable subset $\Gamma \subseteq \mathbb{R}$, the essential union of the family $\{\{\gamma\} : \gamma \in \Gamma\}$ is the empty set because $\lambda(\{\gamma\} \setminus \emptyset) = 0$ for all $\gamma \in \Gamma$. On the other hand, the standard union of this family is Γ, which cannot be λ-equal to the empty set because it is non-measurable.

The following theorem establishes the existence of the essential union and intersection of arbitrary families of measurable sets in σ-finite measure spaces.

Theorem. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Then any family $\{A_\gamma :$ $: \gamma \in \Gamma\} \subseteq \mathcal{A}$ of measurable sets has a μ-essential union and a μ-essential intersection.*

For the proof of the above Theorem, we shall need the following auxiliary result.

Lemma. *If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space then there exists a finite measure ν on $\mathcal{A}$ such that μ and ν are mutually absolutely continuous with respect to each other, that is, $\mu(A) = 0$ holds if and only if $\nu(A) = 0$.*

Proof. The statement is trivial if $\mu(X) < \infty$, therefore we may assume that $\mu(X) = \infty$. Let $A_1, A_2, \dots$ be pairwise disjoint measurable sets such that $X = \bigcup_{n=1}^{\infty} A_n$ and $0 < \mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Then define ν in the following way:

$$\nu(A) := \sum_{n=1}^{\infty} \frac{\mu(A \cap A_n)}{2^n \mu(A_n)} \qquad (A \in \mathcal{A}).$$

One can easily verify that ν is a measure on $\mathcal{A}$, $\nu(X) = 1$, and μ and ν are mutually absolutely continuous with respect to each other. ■

Proof of the Theorem. It is sufficient to prove the statement about the existence of the μ-essential union, because the assertion for the μ-essential intersection follows from this by switching to complements of the sets belonging to the given family. In view of the Lemma, we can assume that $\mu(X) < \infty$. We may also assume that the family $\{A_\gamma : \gamma \in \Gamma\}$ is closed under countable union.

We are going to show that, for all $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma$ such that

$$\mu(A_\gamma \setminus A_{\gamma_n}) < \frac{1}{n} \qquad (\gamma \in \Gamma).$$

Let $\gamma_{n_1} \in \Gamma$ be arbitrary. If, for all $\gamma \in \Gamma$, we have $\mu(A_\gamma \setminus A_{\gamma_{n_1}}) < \frac{1}{n}$, then the choice $\gamma_n := \gamma_{n_1}$ completes the argument. In the other case, there exists $\gamma' \in \Gamma$ such that $\mu(A_{\gamma'} \setminus A_{\gamma_{n_1}}) \geq \frac{1}{n}$. Since the family is closed under countable union, there exists $\gamma_{n_2} \in \Gamma$ such that $A_{\gamma'} \cup A_{\gamma_{n_1}} = A_{\gamma_{n_2}}$. Then

$$A_{\gamma_{n_2}} \supseteq A_{\gamma_{n_1}} \qquad \text{and} \qquad \mu(A_{\gamma_{n_2}}) \geq \mu(A_{\gamma_{n_1}}) + \frac{1}{n}.$$

If, for all $\gamma \in \Gamma$, we have $\mu(A_\gamma \setminus A_{\gamma_{n_2}}) < \frac{1}{n}$, then with $\gamma_n := \gamma_{n_2}$ we are done. Otherwise, there exists $\gamma' \in \Gamma$ such that $\mu(A_{\gamma'} \setminus A_{\gamma_{n_2}}) \geq \frac{1}{n}$. Then there exists $\gamma_{n_3} \in \Gamma$ such that $A_{\gamma'} \cup A_{\gamma_{n_2}} = A_{\gamma_{n_3}}$ and we have

$$A_{\gamma_{n_3}} \supseteq A_{\gamma_{n_2}} \qquad \text{and} \qquad \mu(A_{\gamma_{n_3}}) \geq \mu(A_{\gamma_{n_2}}) + \frac{1}{n}.$$

Continuing this procedure, it terminates after finitely many steps, because the measure space is finite and the measure of the sequence increases by $\frac{1}{n}$ in each step.

Finally, let

$$\overline{A} := \bigcup_{n=1}^{\infty} A_{\gamma_n}.$$

Then, for all $\gamma \in \Gamma$,

$$\mu(A_\gamma \setminus \overline{A}) < \frac{1}{n} \qquad (n \in \mathbb{N}),$$

therefore $\mu(A_\gamma \setminus \overline{A}) = 0$, i.e., $A_\gamma \subseteq_\mu \overline{A}$.

If $A^* \in \mathcal{A}$ has similar properties as $\overline{A}$, then it μ-contains A_{γ_n} for all $n \in \mathbb{N}$, hence it should μ-contain $\overline{A}$, too. ■

The σ-finiteness of the measure space is neither necessary or sufficient for the conclusion of the theorem. To see this, let $\mathcal{A}$ be the σ-algebra of all countable and co-countable subsets of $\mathbb{R}$ and define two measures, μ and ν as follows: let $\mu(A) := 0$ if A is countable and $\mu(A) := \infty$ if A is co-countable (i.e., its complement is countable), and let ν be the counting measure, i.e., let $\nu(A) := n$ if A is finite and contains n elements and let $\nu(A) := \infty$ if A has infinitely many elements. Obviously, the two measure spaces $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}, \nu)$ are not σ-finite.

First we show that the conclusion of the Theorem holds in $(X, \mathcal{A}, \mu)$. Let $\{A_\gamma : \gamma \in \Gamma\} \subseteq \mathcal{A}$ be a family of measurable sets. If, for each $\gamma \in \Gamma$, the set A_γ is countable, then the empty set (moreover, any countable set) can be chosen as the μ-essential union of the family. If, for some $\gamma_0 \in \Gamma$, A_{γ_0} is co-countable, then $\mathbb{R}$ (or any co-countable set) can be chosen as the μ-essential union of the family.

Finally, let $\Gamma \subseteq \mathbb{R}$ be a non-countable set such that its complement is also not countable. We show that the family $\{\{\gamma\} : \gamma \in \Gamma\}$ cannot have a ν-essential union. To see this, observe that a set $A \in \mathcal{A}$ has ν measure zero if and only if $A = \emptyset$. Thus, $A \subseteq_\nu B$ is equivalent to the standard inclusion $A \subseteq B$. Hence, the ν-essential union of the family $\{\{\gamma\} : \gamma \in \Gamma\}$ should be equal to the set Γ, which does not belong to $\mathcal{A}$, i.e., which is not measurable. (This second example was also observed by Zoltán Boros during the 16th Debrecen-Katowice Winter Seminar in Hernádvécse.)

The following result was established in [1, Theorem 108]. Here the notation $\leq_\mu$ stands for inequality which holds μ-almost everywhere.

Corollary. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Then, for any family $\{f_\gamma : X \to [-\infty, +\infty] \mid \gamma \in \Gamma\}$ of extended real-valued measurable functions, there exist measurable functions $\underline{f}, \overline{f} : X \to [-\infty, +\infty]$ such that*

$$\underline{f} \leq_\mu f_\gamma \leq_\mu \overline{f} \qquad (\gamma \in \Gamma),$$

and if $f_*, f^* : X \to [-\infty, +\infty]$ *are measurable functions such that*

$$f_* \leq_\mu f_\gamma \leq_\mu f^* \qquad (\gamma \in \Gamma),$$

then $f_* \leq_\mu \underline{f}$ *and* $\overline{f} \leq_\mu f^*$.

The μ-uniquely determined functions $\underline{f}$ and $\overline{f}$ defined in the above corollary are called the *essential infimum* and *essential supremum* of the family of measurable functions (cf. [1, Definition 106]). It is easy to see that, for all countable subset $\Gamma_0 \subseteq \Gamma$,

$$\underline{f} \leq_\mu \inf_{\gamma\in\Gamma_0} f_\gamma, \qquad \sup_{\gamma\in\Gamma_0} f_\gamma \leq_\mu \overline{f}.$$

On he other hand, in general, it is not true that $\underline{f} \leq_\mu \inf_{\gamma\in\Gamma} f_\gamma$, and $\sup_{\gamma\in\Gamma} f_\gamma \leq_\mu$ $\leq_\mu \overline{f}$.

The above result can easily be deduced from the Theorem by applying it to the epigraphs of the functions and observing that, for two extended real-valued functions $f, g : X \to [-\infty, +\infty]$ the relation $f \leq_\mu g$ (i.e., the inequality $f \leq g$ holds μ almost everywhere) if and only if $\operatorname{epi} g \subseteq_{\mu\otimes\lambda} \operatorname{epi} f$, where $\mu \otimes \lambda$ denotes the product measure of μ and the Lebesgue measure λ.

In the book [1] first the notions of μ-essential supremum and infimum of families of measurable functions is defined. The concepts of μ-essential union and intersection of families of measurable sets are then introduced via the μ-essential supremum and infimum of their characteristic functions. In this note we followed a different approach, we defined the notions μ-essential union and intersection of families of measurable sets first.

Those measure spaces where the conclusion of our Theorem remains valid are called *localizable* by the terminology of the book [1]. It turns out that the Radon–Nikodym Theorem remains valid for measures on such spaces.

References

[1] **Fonseca, I. and G. Leoni,** *Modern Methods in the Calculus of Variations: L^p Spaces*, Springer Monographs in Mathematics, Springer Verlag, 2007.

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