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CAUCHY DIFFERENCES AND MEANS

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Dedicated to the 70th birthday of Professor Antal Járai

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Abstract. Basing on the four types of Cauchy differences, some general constructions of k-variable premeans and means generated by a single variable real function f defined in a real interval I is discussed, special cases are examined and open questions are proposed. In particular, if I is closed under the addition, and f is such that $F(x) := f(kx) - kf(x)$ is invertible, then the first of four considered functions $M_f : I^k \to \mathbb{R}$ is of the form

$$M_f(x_1, ..., x_k) = F^{-1}(f(x_1 + ... + x_k) - (f(x_1) + ... + f(x_k))).$$

Conditions under which M_f is a k-variable mean (referred to as quasi-Cauchy difference mean of additive type) are examined.

1. Introduction

The four types of Cauchy functional equations characterizing the elementary additive, exponential, logarithmic and power functions through their properties involving the operations of addition and multiplication (see J. Aczél [1], M. Kuczma [10]), in a natural way lead to the *Cauchy differences*, functions which are differences both sides of the respective equation (see for instance [11]).

In the present paper, basing on these Cauchy differences, and the idea coming from the construction of quasi-arithmetic mean, we propose four general

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schemes of constructions of new classes of means (for the theory of means see, for instance, [6], [2]).

To form the first two of them take: a real interval I, which is closed under addition, a function $f : I \to \mathbb{R}$, a positive integer $k \geq 2$, and write the k-variable functions

$$I^k \ni (x_1, ..., x_k) \longmapsto f(x_1 + ... + x_k) - (f(x_1) + ... + f(x_k)),$$

$$I^k \ni (x_1, ..., x_k) \longmapsto f(x_1 + ... + x_k) - f(x_1) \cdot ... \cdot f(x_k),$$

which are called *the Cauchy difference of additive type generated by* f, *and the Cauchy difference of exponential type generated by* f, respectively.

Similarly, if the interval I is closed under multiplication, two remaining k-variable functions:

$$I^k \ni (x_1, ..., x_k) \longmapsto f(x_1 \cdot ... \cdot x_k) - (f(x_1) + ... + f(x_k)),$$

$$I^k \ni (x_1, ..., x_k) \longmapsto f(x_1 \cdot ... \cdot x_k) - f(x_1) \cdot ... \cdot f(x_k),$$

are called *the Cauchy difference of logarithmic type generated by* f, *and the Cauchy difference of multiplicative type generated by* f, respectively.

Basing on these Cauchy differences, we present four general schemes of creating new kind of premeans and means.

The first scheme, Theorem 1 in Section 3, gives general conditions under which the k-variable function $M_f : I^k \to \mathbb{R}$ of the form

$$M_f(x_1, ..., x_k) = F^{-1}(f(x_1 + ... + x_k) - (f(x_1) + ... + f(x_k))),$$

where $F(x) := f(kx) - kf(x)$ (the restriction of the Cauchy difference of the additive type to the main diagonal of I^k), is a premean or a mean; and $M_f : \bigcup_{k=2}^{\infty} I^k \to I$ is referred to as the *quasi-Cauchy difference* (*premean or*) *mean of additive type generated by* f. It is clear that the additive functions $f(x) = px$ are useless here. In this section we examine the quasi-Cauchy difference means of additive type generated, respectively, by the one-parameter families of continuous multiplicative functions $f(x) = x^p$ (Proposition 1), exponential functions $f(x) = p^x$ (Proposition 2), and logarithmic functions $\log_p$ (Proposition 3). In particular, Proposition 1 gives the explicit forms of considered family of means for all parameters $p \in \mathbb{R} \setminus \{0\}$, its continuous extension to the family parameters $p \in [-\infty, \infty]$, as well as some of their properties. Proposition 2 gives the implicit formula for the general case, but only in the case $k = 2$, does it offer a complete explicit description of the considered means. In view of Proposition 3, the family of all quasi-Cauchy difference means of additive type generated by the continuous logarithmic functions reduces to a singleton family (independent of p) consisting of the mean

$\mathcal{B} : \bigcup_{k=2}^{\infty} (0,\infty)^k \to (0,\infty)$, where

$$\mathcal{B}(x_1,...,x_k) = \left(\frac{kx_1 \cdot ... \cdot x_k}{x_1 + ... + x_k}\right)^{\frac{1}{k-1}}, \qquad k \in \mathbb{N},\ k \geq 2;\quad x_1,...,x_k > 0.$$

In the remaining sections the proofs are omitted. In Section 4, Theorem 2, a general scheme allowing to determine the quasi-Cauchy difference (premeans) means of exponential type, is used, respectively, for additive functions (Proposition 4); multiplicative functions (Proposition 5), and logarithmic functions (Proposition 6).

In Section 5, Theorem 3, a general scheme of building the quasi-Cauchy difference premeans and means of logarithmic type, is used, respectively, for additive functions (Proposition 7), exponential functions (Proposition 8), and multiplicative functions (Proposition 9).

Similarly, in Section 6, we apply general construction of quasi-Cauchy difference means of multiplicative type (Theorem 4), respectively, to additive functions (Proposition 10), exponential functions (Proposition 11) and logarithmic functions (Proposition 12). In some cases the mean is given by an implicit equality. Occasionally, in the case $k = 2$, some invariant means are considered.

In each of the sections an open problem concerning equality of the considered means is proposed. In the case of quasi-Cauchy difference mean of additive type this problem reduces to a generalized Cauchy difference equation considered by Bruce Ebanks [3], [4] where, basing on the results due to Antal Járai [8], under some regularity conditions, the solutions are established.

2. Preliminaries

Let $I \subset \mathbb{R}$ be an interval. A function $M : \bigcup\limits_{k=1}^{\infty} I^k \to I$ is called a *premean in* I, if it is *reflexive*, i.e.

$$M(x,...,x) = x, \qquad x \in I,$$

which means that for every $k \in \mathbb{N}$ and all $x, x_1, ..., x_k \in I$,

$$x_1 = ... = x_k = x \Longrightarrow M(x_1,...,x_k) = x;$$

and it is called a *mean in* I, if for every $k \in \mathbb{N}$,

$$\min(x_1,...,x_k) \leq M(x_1,...,x_k) \leq \max(x_1,...,x_k), \qquad x_1,...,x_k \in I.$$

If M is a premean in I, then $M(x) = x$ for every $x \in I$. In the sequel we assume that $k \geq 2$.

A function $M : I^k \to \mathbb{R}$ is called *reflexive* if $M(x, ..., x) = x$ for all $x \in I$; and it is called a k-variable *premean* in I, if it is reflexive and $M : I^k \to I$.

If $M : \bigcup_{k=2}^{\infty} I^k \to I$ is a premean (or mean) in I, then its restriction to I^k, for simplicity denoted by $M : I^k \to I$, is called a *k-variable premean* (or a *k-variable mean*) *in* I.

A k-variable mean or premean in $M : I^k \to I$ is called:

strict if, for all $x_1, .., x_k \in I$, and $i \in \{1, ..., k\}$,

$$M(x_1, ., x_i, .., x_k) = x_i \Longrightarrow x_1 = x_2 = ... = x_k;$$

symmetric, if $M\left(x_{\sigma(1)}, ..., x_{\sigma(k)}\right) = M(x_1, ..., x_k)$ for every $(x_1, ..., x_k) \in I^k$ and every permutation σ of $\{1, ..., k\}$;

homogeneous, if $I = (0, \infty)$ and,

$$M(tx_1, ..., tx_k) = tM(x_1, ..., x_k), \qquad t, x_1, ..., x_k > 0.$$

Of course, every mean is a premean, but the converse is not true. However we have the following

Remark 1. If a function $M : I^k \to \mathbb{R}$ is reflexive and (strictly) increasing in each variable then it is a (strict) k-variable mean in I, and it is called an increasing mean.

In particular, every (strictly) increasing premean in I is a (strict) mean.

Note that if for every $k \in \mathbb{N}$, $k \geq 2$, the function $M_k : I^k \to I$ is a k-variable mean (premean) in I, then $M : \bigcup\limits_{k=1}^{\infty} I^k \to I$ defined by $M|_{I^k} := M_k$ is a mean (premean) in I.

3. Quasi-Cauchy difference means of additive type

Theorem 1. *Let $k \in \mathbb{N}$, $k \geq 2$, and an interval $I \subset (0, \infty)$ (or $I = \mathbb{R}$), closed under addition, be fixed. Assume that a function $f : I \to \mathbb{R}$ is such that function $F : I \to \mathbb{R}$ defined by*

$$F(x) := f(kx) - kf(x), \tag{1}$$

is one-to-one. Then

(i) *if the range of the Cauchy difference of additive type*

$$I^k \ni (x_1, ..., x_k) \longmapsto f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) \tag{2}$$

is contained in the range of F, then the function $M_f : I^k \to \mathbb{R}$ given by

$$M_f(x_1,...,x_k) := F^{-1}\left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) \right), \tag{3}$$

is a well defined k-variable symmetric premean in I;

(ii) *if f is continuous, and either F is strictly increasing and for every $r \in I$, the function*

$$I \ni t \longmapsto f(t+r) - f(t) \text{ is (strictly) increasing,} \tag{4}$$

or F is strictly decreasing for every $r \in I$ and the function in (4) *is* (*strictly*) *decreasing, then the function M_f given by* (3) *is a k-variable symmetric* (*strict*) *and* (*strictly*) *increasing k-variable mean in I.*

Proof. (i) The assumptions guarantee that M_f is well defined and maps I^k into I. From (1) we have, for every $x \in I$,

$$M_f(x,...,x) = F^{-1}((f(kx) - kf(x))) = x,$$

so M_f is reflexive, which proves that it is a premean in I. The symmetry of M_f is obvious.

(ii) The continuity of f implies the continuity of F. If f satisfies (4) then, clearly, for every real constant b, the function

$$I \ni t \longmapsto f(t+r) - f(t) - b \text{ is (strictly) increasing.}$$

Taking $t := x_1$, $r = \sum_{j=2}^{k} x_j$, $b = \sum_{j=2}^{k} f(x_j)$, we have

$$f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) = f(t+r) - f(t) - b,$$

which shows that function (2) is (strictly) increasing in the first variable. The symmetry of (2) implies that it is (strictly) increasing in each variable. Hence, by (1), if F is increasing, then for all $x_1,...,x_k \in I$, setting $x := \min(x_1,...,x_k)$ and $y := \max(x_1,...,x_k)$, we have

$$F(x) = f(kx) - kf(x) \le f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) \le f(ky) - kf(y) = F(y).$$

Thus, by the Darboux property of F, the range of function (2) is contained in the range of F. Now (i) implies that E_f is a k-variable premean in I. Since the composition of (strictly) increasing functions is (strictly) increasing, the result follows from Remark 1.

If F is strictly decreasing and the function (2) is (strictly) decreasing, we argue similarly, making use of the fact that composition of (strictly) decreasing functions is (strictly) increasing. ■

Remark 2. Every (strictly) convex function satisfies condition (4), that is equivalent to the (strict) Wright-convexity of f.

Remark 3. If the function f is (strictly) convex (respectively, concave), then for every $r \in I$, the function (4) is (strictly) increasing (respectively, decreasing).

Definition 1. Under the suitable conditions of Theorem 1, the function M_f is referred to as the quasi-Cauchy difference mean (premean) of additive type generated by f.

Note that the quasi-Cauchy quasi-difference mean M_f of additive type is built with the aid of the sums $f\left(\sum_{j=1}^{k} x_j\right)$ and $\sum_{j=1}^{k} f(x_j)$.

Remark 4. Since for affine functions, every quasi-Cauchy difference of additive type is constant, no affine function generates a premean of the considered type.

Let $k \in \mathbb{N}$, $k \geq 2$, be fixed, let $I \subset \mathbb{R}$ be an an interval that is closed under addition, and let $f, g : I \to \mathbb{R}$. In connection with Theorem 1 it is natural to ask for conditions which guarantee that $M_g = M_f$, i.e. such for all $x_1, ..., x_k \in I$,

$$G^{-1}\left(g\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} g(x_j)\right) = F^{-1}\left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right),$$

where

$$F(x) = f(kx) - kf(x), \quad G(x) = g(kx) - kg(x), \qquad x \in I.$$

In the case $k = 2$, setting $\varphi := G \circ F^{-1}$, this leads to the following

Problem 1. *Determine all functions f, g and φ satisfying the functional equation*

$$\text{(Eq)} \qquad \varphi(f(x+y) - f(x) - f(y)) = g(x+y) - g(x) - g(y), \qquad x, y \in I.$$

Assuming that f is continuously differentiable and locally not affine Ebanks [4] ([3]) found the general solution of this (additive type) generalized Cauchy difference equation (see also [5], where applying the results of Járai [8], a more general Pexider equation is considered).

Applying the main result of [4] we obtain the following

Remark 5. Let $I \subset (0, \infty)$ be an interval that closed under addition. Assume that $f : I \to \mathbb{R}$ is continuously differentiable and not affine on any subinterval of I, and g is continuous. Put

$$J_f := \{f(x+y) - f(x) - f(y) : x, y \in I\}.$$

If the functions f, g and $\varphi : J_f \to \mathbb{R}$ satisfy the equation (Eq), then there are $a, b, c \in \mathbb{R}$, such that

$$g(x) = af(x) + bx + c, \qquad x \in I; \qquad \text{and} \qquad \varphi(t) = at - c, \quad t \in J_f.$$

In [3], where some stronger regularity on f is assumed, the two kinds of proofs are given. Note that under some conditions, one can apply yet another argument, based on the Cauchy Mean-Value Theorem. Indeed, assume that $f, g : I \to \mathbb{R}$ are twice differentiable, $f''(x) g''(x) \neq 0$, for all $x \in I$, the function $\frac{g''}{f''}$ is locally monotonic in $I \backslash C$, where C is a nowhere dense subset. Let f, g and φ satisfy (Eq). The assumptions imply that J_f is an interval and φ is twice differentiable in J_f. Differentiating both sides of (Eq), first in x and then in y, we get

$$\varphi'(f(x+y) - f(x) - f(y))[f'(x+y) - f'(x)] = g'(x+y) - g'(x), \; x, y \in I,$$

and

$$\varphi'(f(x+y) - f(x) - f(y))[f'(x+y) - f'(y)] = g'(x+y) - g'(y), \; x, y \in I.$$

The condition $g''(x) \neq 0 \quad (x \in I)$ implies that g' is strictly monotonic; in particular, g' is one to one. These equalities imply that f' is one to one and $\varphi'(t) \neq 0$ for all $t \in J_f$. Dividing the respective sides of these two equalities we get

$$\frac{f'(x+y) - f'(x)}{f'(x+y) - f'(y)} = \frac{g'(x+y) - g'(x)}{g'(x+y) - g'(y)}, \qquad x, y \in I,$$

whence

$$\frac{g'(x+y) - g'(y)}{f'(x+y) - f'(y)} = \frac{g'(x+y) - g'(x)}{f'(x+y) - f'(x)}, \qquad x, y \in I.$$

We claim that the function $\frac{g''}{f''}$ is constant. Indeed, in the opposite case we could find an interval $I_0 \subset I \backslash C$ such that the function $\frac{g''}{f''}$ would be strictly monotonic in I_0 and, by the Cauchy Mean-Value Theorem,

$$\frac{g''}{f''}(C_{g',f'}(x+y, y)) = \frac{g''}{f''}(C_{g',f'}(x+y, x)), \qquad x, y \in I_0, \; x \neq y,$$

where $C_{g',f'} : I^2 \to I$ denotes the the Cauchy mean generated by g' and f',

$$C_{g',f'}(x, y) = \left(\frac{g''}{f''}\right)^{-1} \left(\frac{g'(x) - g'(y)}{f'(x) - f'(y)}\right), \qquad x, y \in I_0; \; x \neq y.$$

It follows that

$$C_{g',f'}(x+y,y) = C_{g',f'}(x+y,x), \quad x,y \in I_0,\ x \neq y,$$

which is a contradiction, as the Cauchy mean $C_{g',f'}$ is symmetric and strictly increasing with respect to each variable. Thus there is $a \in \mathbb{R}$, $a \neq 0$, such that

$$\frac{g''}{f''}(x) = a, \qquad x \in I.$$

Hence $g''(x) = af''(x)$ for all $x \in I$ and, consequently, there are real constant b and c such that

$$g(x) = af(x) + bx + c, \qquad x \in I.$$

It follows that, for all $x, y \in I$,

$$g(x+y) - g(x) - g(y) = a[f(x+y) - f(x) - f(y)] - c,$$

whence, setting $\varphi(t) := at + b$ for $t \in J_f$, we get the result.

Remark 6. In [3] and [4] the equation

$$\varphi(H(x,y)) = g(x+y) - g(x) - g(y), \qquad x,y \in I,$$

is considered, and in [5], under an additional assumption that $0 \in I$, its Pexider version.

3.1. Quasi-Cauchy difference means of additive type generated by multiplicative functions

Using Theorem 1 (ii) for multiplicative continuous functions f we obtain the following

Proposition 1. *For every $p \in \mathbb{R} \setminus \{0,1\}$, the function $\mathcal{M}_p : \bigcup_{k=2}^{\infty} (0,\infty)^k \to \to (0,\infty)$ defined by*

$$\mathcal{M}_p(x_1,...,x_k) = \left(\frac{(x_1 + ... + x_k)^p - (x_1^p + ... + x_k^p)}{k^p - k} \right)^{1/p}$$

is a quasi-Cauchy difference mean of additive type generated by the multiplicative function $f(x) = x^p$.

Moreover:

(i) *the functions $\mathcal{M}_0,\ \mathcal{M}_1,\ \mathcal{M}_{-\infty},\ \mathcal{M}_{\infty}$ defined by*

$$\mathcal{M}_0(x_1,...,x_k) = \left(\frac{kx_1 \cdot ... \cdot x_k}{x_1 + ... + x_k} \right)^{\frac{1}{k-1}},$$

$$\mathcal{M}_1(x_1,...,x_k)=\frac{1}{k\log k}\log\frac{(x_1+...+x_k)^{x_1+...+x_k}}{x_1^{x_1}\cdot...\cdot x_k^{x_k}},$$

$$\mathcal{M}_{-\infty}(x_1,...,x_k)=\min(x_1,...,x_k)\,,\qquad \mathcal{M}_{\infty}(x_1,...,x_k)=\frac{x_1+...+x_k}{k}$$

are means in $(0,\infty)$ *and for every* $(x_1,...,x_k)\in(0,\infty)^k$, *the function*

$$[-\infty,+\infty]\ni p\longmapsto\mathcal{M}_p(x_1,...,x_k)\ \text{ is continuous and strictly increasing;}$$

(ii) $\mathcal{M}_0=M_{\log}$, *that is* $\mathcal{M}_0$ *is quasi-Cauchy difference mean of the additive type generated by* log, *and it is the beta-type mean* ([7]); $\mathcal{M}_1=M_{\mathrm{id}\cdot\log}$, *that is* $\mathcal{M}_1$ *is a quasi-Cauchy difference mean of additive type generated by* $(\mathrm{id}\cdot\log)(x)=x\log x$;

(iii) *the means* $\mathcal{M}_\infty$ *and* $\mathcal{M}_{-\infty}$ *are not quasi-Cauchy difference means of additive type;*

(iv) *for every* $p\in(-\infty,\infty]$ *the mean* $\mathcal{M}_p$ *is strict, symmetric and homogeneous.*

Proof. Put $I=(0,\infty)$. The power function $f(x)=x^p$ for $x\in I$, with $p\in\mathbb{R}\setminus\{0,1\}$, satisfies the conditions (ii) of Theorem 1 (see Remark 3). Indeed, for every $p\in\mathbb{R}\setminus\{0,1\}$ the function f is either strictly convex or strictly concave and the respective function

$$F(x)=f(kx)-kf(x)=(k^p-k)\,x^p$$

is strictly monotonic. By Theorem 1(ii) we conclude that, for every $p\in$ $\in\mathbb{R}\setminus\{0,1\}$, the function $\mathcal{M}_p:=M_f$ is a mean, and for every $(x_1,...,x_k)\in I^p$,

$$\begin{aligned}(k^p-k)\left(\mathcal{M}_p(x_1,...,x_k)\right)^p &= F\left(\mathcal{M}_p(x_1,...,x_k)\right)=\left(\sum_{j=1}^k x_j\right)-\sum_{j=1}^k f(x_j)\\ &= (x_1+...+x_k)^p-\left(x_1^p+...+x_k^p\right),\end{aligned}$$

whence

$$\mathcal{M}_p(x_1,...,x_k)=\left(\frac{(x_1+...+x_k)^p-\left(x_1^p+...+x_k^p\right)}{k^p-k}\right)^{1/p}.$$

Since, making use of the de l'Hospital rule, for every $(x_1,...,x_k)\in I^k$,

$$\lim_{p\to0}\log\mathcal{M}_p(x_1,...,x_k)=\lim_{p\to0}\frac{\log\frac{(x_1+...+x_k)^p-\left(x_1^p+...+x_k^p\right)}{k^p-k}}{p}=$$

$$= \lim_{p\to 0}\left(\frac{\frac{\left[(x_1+...+x_k)^p \log(x_1+...+x_k) - \left(x_1^p \log x_1 + ... + x_k^p \log x_k\right)\right](k^p-k)}{(k^p-k)^2}}{\frac{(x_1+...+x_k)^p - \left(x_1^p+...+x_k^p\right)}{k^p-k}} - \right.$$

$$\left. - \frac{\frac{(x_1+...+x_k)^p - \left(x_1^p+...+x_k^p\right)}{k^p-k}}{\frac{(x_1+...+x_k)^p - \left(x_1^p+...+x_k^p\right)}{k^p-k}} \right) =$$

$$= \frac{\frac{\log(x_1+...+x_k) - (\log x_1 + \cdots + \log x_k)(1-k) - (1-k)\log k}{(1-k)^2}}{\frac{1-k}{1-k}} = \log\left(\frac{kx_1\cdot ... \cdot x_k}{x_1+...+x_k}\right)^{\frac{1}{k-1}},$$

we conclude that

$$\mathcal{M}_0(x_1, ..., x_k) := \lim_{p\to 0} \mathcal{M}_p(x_1, ..., x_k) = \left(\frac{kx_1\cdot ... \cdot x_k}{x_1+...+x_k}\right)^{\frac{1}{k-1}}.$$

On the other hand, from (3), taking $f = \log$, we have, for all positive $x_1, ..., x_k$,

$$\log(kM_{\log}) - k\log M_{\log} = \log\sum_{j=1}^{k} x_j - \sum_{j=1}^{k}\log x_j,$$

whence, after easy calculations, for all positive $x_1, ..., x_k$,

$$M_{\log}(x_1, ..., x_k) = \left(\frac{kx_1\cdot ... \cdot x_k}{x_1+...+x_k}\right)^{\frac{1}{k-1}} = \mathcal{M}_0(x_1, ..., x_k).$$

Similar calculations show that for every $(x_1, ..., x_k) \in I^{k,}$

$$\mathcal{M}_1(x_1, ..., x_k) := \lim_{p\to 1} \mathcal{M}_p(x_1, ..., x_k) = \frac{1}{k\log k}\log\frac{(x_1+...+x_k)^{x_1+...+x_k}}{x_1^{x_1}\cdot ... \cdot x_k^{x_k}}.$$

on the other hand, making use of (3) with $f(x) = x\log x$, briefly, $f = \mathrm{id}\cdot\log$, we have

$$M_{\mathrm{id}\cdot\log 1}(x_1, ..., x_k) = \frac{1}{k\log k}\log\frac{(x_1+...+x_k)^{x_1+...+x_k}}{x_1^{x_1}\cdot ... \cdot x_k^{x_k}} = \mathcal{M}_{1\mathrm{id}\cdot\log 1}(x_1, ..., x_k).$$

We omit simple calculations showing that, for all positive $x_1, ..., x_k$,

$$\mathcal{M}_\infty(x_1, ..., x_k) := \lim_{p\to\infty} \mathcal{M}_p(x_1, ..., x_k) = \frac{x_1+...+x_k}{k}.$$

To get the formula for $\mathcal{M}_{-\infty}$, assume, for the simplicity of notations, that

$k=2$, and put $x_1=x$, $x_2=y$ and $p=-q$ where $q>0$. Thus

$$\begin{aligned}\mathcal{M}_p(x,y) &= \mathcal{M}_{-q}(x,y)=\left(\frac{(x+y)^{-q}-x^{-q}-y^{-q}}{2^{-q}-2}\right)^{-1/q}= \\ &= \frac{\left(2-\frac{1}{2^q}\right)^{1/q}}{\left(\frac{1}{x^q}+\frac{1}{y^q}-\frac{1}{(x+y)^q}\right)^{1/q}}.\end{aligned}$$

Without any loss of generality we may assume that $x=\min(x,y)$. Hence we get

$$\frac{\left(2-\frac{1}{2^q}\right)^{1/q}}{\left(\frac{2}{x^q}\right)^{1/q}}\le\mathcal{M}_{-q}(x,y)\le\frac{\left(2-\frac{1}{2^q}\right)^{1/q}}{\left(\frac{1}{x^q}\right)^{1/q}},$$

whence

$$\lim_{p\to-\infty}\mathcal{M}_p(x,y)=\lim_{q\to\infty}\mathcal{M}_{-q}(x,y)=x=\min(x,y).$$

We omit similar calculations in the case when $k\ge 3$. Thus, for every $(x_1,...,x_k)\in I^k$, the function $[-\infty,\infty]\ni p\longmapsto\mathcal{M}_p(x_1,...,x_k)$ is continuous. Since the pointwise limits of means are means, the functions $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_{-\infty}$, $\mathcal{M}_\infty$ are means. Calculating the partial derivatives of $\mathcal{M}_p$ one can easily verify that for every $(x_1,...,x_k)\in I^k$, the function $[-\infty,\infty]\ni p\longmapsto\mathcal{M}_p(x_1,...,x_k)$ is strictly increasing in each of the open intervals $(-\infty,0)$, $(0,1)$, and $(1,\infty)$. The continuity of this function implies that it is strictly increasing in $[-\infty,\infty]$. This completes the proofs of (i) and (ii).

Of course, the mean $\mathcal{M}_{-\infty}$, being not strict, is not of the considered type. Assume, for the contrary, that $\mathcal{M}_\infty$, the arithmetic mean is of the form (3) in $(0,\infty)$ in the case $k=2$, i.e. that $\mathcal{M}_\infty(x,y)=\frac{x+y}{2}$ and that $\mathcal{M}_\infty=M_f$ for a function $f:(0,\infty)\to\mathbb{R}$. From (3) and (1) we get

$$f\left(2\frac{x+y}{2}\right)-2f\left(\frac{x+y}{2}\right)=f(x+y)-f(x)-f(y),\qquad x,y>0,$$

whence

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2},\qquad x,y>0,$$

that implies that $f(x)=ax+b$ for all $x>0$. Since for this function the Cauchy difference is the constant zero, formula (3) does not define a mean. This completes the proof of (iii).

Result (iv) is obvious. ■

Remark 7. In the case $k=2$, Proposition 1 gives

$$\mathcal{M}_p(x,y)=\begin{cases}\left(\frac{(x+y)^p-(x^p+y^p)}{2^p-2}\right)^{1/p} & \text{if} \quad p\in\mathbb{R}\setminus\{0,1\}\\ \frac{2xy}{x+y} & \text{if} \qquad p=0\\ \frac{1}{2\log 2}\log\frac{(x+y)^{x+y}}{x^x y^y} & \text{if} \qquad p=1.\end{cases}$$

In particular, $\mathcal{M}_0$ is the harmonic mean $\mathcal{H}(x,y)=\frac{2xy}{x+y}$ mean, $\mathcal{M}_2$ is the geometric mean $\mathcal{G}(x,y)=\sqrt{xy}$, but this is not the case if $k>2$. Moreover we have, for all $x,y>0$,

$$\mathcal{M}_{-1}(x,y)=\frac{3xy(x+y)}{2(x^2+xy+y^2)},\qquad \mathcal{M}_3(x,y)=\mathcal{A}^{1/3}(x,y)\,\mathcal{G}^{2/3}(x,y),$$

where $\mathcal{A}(x,y)=\frac{x+y}{2}$.

In the case $p=2$ Proposition 1 gives

$$\mathcal{M}_2(x_1,...,x_k)=\sqrt{\frac{2\prod\limits_{i,j=1,\dots,k,\ i\neq j}x_ix_j}{k(k-1)}}.$$

Conjecture 1. *Let $k\in\mathbb{N}$, $k\geq 2$ be fixed. Assume that $f:(0,\infty)\to\mathbb{R}$ satisfies the conditions of Theorem 1, guarantying that M_f given by (3) is a k-variable mean in $(0,\infty)$. If M_f is homogeneous then $M_f=\mathcal{M}_p$ for some $p\in\mathbb{R}$.*

3.2. Quasi-Cauchy difference means of additive type generated by exponential functions

Using Theorem 1 with $f(x)=p^x$ where $p>0$, $p\neq 1$, we obtain

Proposition 2. *For every $p>1$ the function $M_p:\bigcup\limits_{k=2}^{\infty}(0,\infty)^k\to(0,\infty)$, with the restriction to $(0,\infty)^k$ defined as the continuous solutions of the implicit equality*

$$p^{kpM_p}-kp^{pM_p}=p^{x_1+\cdots+x_k}-(p^{x_1}+\cdots+p^{x_k}),\qquad x_1,...,x_k>0,$$

is a quasi-Cauchy difference mean of additive type generated by the exponential function $f(x)=p^x$.

Moreover, if $k=2$, then for all $x,y>0$,

$$M_p(x,y)=\begin{cases}\log_p\left(1+\sqrt{(p^x-1)(p^y-1)}\right) & if \qquad p>1\\ \log_p\left(1-\sqrt{(1-p^x)(1-p^y)}\right) & if \quad 0<p<1,\end{cases}$$

$$M_1(x,y) := \lim_{p\to 1} M_p(x,y) = \sqrt{xy}, \quad M_\infty(x,y) := \lim_{p\to\infty} M_p(x,y) = \frac{x+y}{2}.$$

$$M_0(x,y) := \lim_{p\to 0} M_p(x,y) = \min(x,y),$$

and the function

$$[0,\infty] \ni p \longmapsto M_p(x,y) \quad \textit{is continuous.}$$

Proof. Take arbitrary $p>0$, $p\neq 1$, and $f(x) = p^x$ for $x\in(0,\infty)$. Since $p = e^q$ for some $q\in\mathbb{R}$, $q\neq 0$, applying Theorem 1, we have $F(x) = \exp(kqx) - k\exp(qx)$. Since $F'(x) = kq\exp(qx)[\exp((k-1)qx) - 1]$ is positive for all $x\in(0,\infty)$ and $q\in\mathbb{R}\setminus\{0\}$, so F is strictly increasing in $(0,\infty)$ for each $q\in\mathbb{R}\setminus\{0\}$. Similarly, for each $j=1,...,k$, and every $q\in\mathbb{R}\setminus\{0\}$, the function

$$(0,\infty) \ni x_j \longmapsto \exp p(x_1+...+x_k) - (\exp px_1 + ... + \exp px_k)$$

is increasing in $(0,\infty)$. Therefore, in view of Theorem 1 (ii) and Remark 1, for every $p>0$, the function $M_p := M_f$ is a well defined strict and symmetric mean in $(0,\infty)$.

For $k=2$, setting $x_1 = x$, $x_2 = y$, $x,y>0$ in the implicit equality, we get

$$\left[p^{M_p(x,y)}\right]^2 - 2p^{M_p(x,y)} = p^{x+y} - (p^x + p^y).$$

Hence, after easy calculations,

$$M_p(x,y) = \log_p\left(1 + \sqrt{(p^x-1)(p^y-1)}\right) \quad \text{if} \quad p>1,$$

and

$$M_p(x,y) = \log_p\left(1 - \sqrt{(1-p^x)(1-p^y)}\right) \quad \text{if} \quad 0<p<1.$$

Assume that $p>1$. Since $p^x = \exp(x\log p)$, setting $q = \log p$ we have

$$p^x = \exp(qx).$$

Note that

$$\lim_{q\to 0}\frac{\exp(qx)-1}{q} = x, \quad \lim_{q\to 0}\frac{\exp(qy)-1}{q} = y, \qquad x,y>0,$$

and

$$M_q(x,y) = \frac{\log\left(1+\sqrt{(\exp(qx)-1)(\exp(qy)-1)}\right)}{q}.$$

Hence, applying the de l'Hospital rule, we get for all $x,y>0$,

$$\lim_{p\to 1+} M_p(x,y) = \lim_{q\to 0+} M_q(x,y) =$$

$$= \lim_{q\to 0+} \frac{1}{q} \left(\log \left(1 + \sqrt{\left(\exp(qx) - 1\right)\left(\exp(qy) - 1\right)} \right) \right) =$$

$$= \lim_{q\to 0+} \frac{x\left(\exp(qx)\right)\left(\exp(qy) - 1\right) + y\left(\exp(qy)\right)\left(\exp(qx) - 1\right)}{2\sqrt{\left(\exp(qx) - 1\right)\left(\exp(qy) - 1\right)}\left(1 + \sqrt{\left(\exp(qx) - 1\right)\left(\exp(qy) - 1\right)}\right)} =$$

$$= \lim_{q\to 0+} \frac{x\left(\exp(qx)\right) \cdot q \cdot \frac{\exp(qy)-1}{q} + y\left(\exp(qx)\right) \cdot q \cdot \frac{\exp(qx)-1}{q}}{2 \cdot q \cdot \sqrt{\frac{\exp(qx)-1}{q} \cdot \frac{\exp(qy)-1}{q}}\left(1 + \sqrt{\left(\exp(qx) - 1\right)\left(\exp(qy) - 1\right)}\right)} =$$

$$= \frac{x \cdot y + y \cdot x}{2 \cdot \sqrt{x \cdot y}} = \sqrt{x\,y}.$$

Similarly, making use of the formula for M_p if $0 < p < 1$, we can show, that

$$\lim_{p\to 1-} M_p\left(x, y\right) = \sqrt{xy}, \qquad x, y > 0,$$

so M_1 is well defined.

We omit similar calculations for the remaining results. ■

Remark 8. A counterpart of Proposition 2 holds true for the interval $\left(-\infty, 0\right)$.

Problem 2. *Under the assumptions of Proposition* 2*, determine* M_1 *and* M_∞ *for arbitrary* $k \geq 3$.

3.3. Quasi-Cauchy difference mean of additive type generated by logarithmic functions

Theorem 1 with $f = \log_p$, where $p > 0$, $p \neq 1$, gives

Proposition 3. *For every* $p > 0$, $p \neq 1$, *the quasi-Cauchy difference function of additive type* $M_f : \bigcup_{k=2}^{\infty} \left(0, \infty\right)^k \to \left(0, \infty\right)$ *generated by logarithmic function* $f = \log_p$, *is a mean, and* $M_f = \mathcal{B}$, *where*

$$\mathcal{B}\left(x_1, ..., x_k\right) = \left(\frac{kx_1 \cdot ... \cdot x_k}{x_1 + ... + x_k}\right)^{\frac{1}{k-1}}, \qquad k \in \mathbb{N},\ k \geq 2; \quad x_1, ..., x_k > 0;$$

so it does not depend on p.

Proof. It is easy to verify that $f = \log_p$ satisfies the conditions of Theorem 1 in the interval $\left(0, \infty\right)$. Since $F\left(t\right) = \log_b\left(kt\right) - k\log_b$, formula (3) implies that

$$\log_p\left(kM_{\log_p}\right) - k\log_p\left(M_{\log_p}\right) = \log_b\left(x_1 + ... + x_k\right) - \left[\log_b\left(x_1\right) + ... + \log_b\left(x_k\right)\right],$$

whence, after simple calculations we get the result. ■

Remark 9. The mean $\mathcal{B}$ has recently appeared in [7] where it is shown that $\mathcal{B}$ is the only mean that is of the beta-type, which means that it is of the form $\frac{f(x+y)}{f(x)f(y)}$, just like the Euler Beta is of the form $\frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)}$.

4. Quasi-Cauchy difference means of exponential type

Similarly to Theorem 1 we obtain

Theorem 2. *Let $k \in \mathbb{N}$, $k \geq 2$, and an interval $I \subset (0,\infty)$ (or $I = \mathbb{R}$), closed under addition, be fixed. Assume that $f : I \to (0,\infty)$ is such that function $F : I \to \mathbb{R}$ defined by*

$$F(x) := f(kx) - [f(x)]^k, \tag{5}$$

is one-to-one. Then

(i) *if the range of the Cauchy difference of exponential type*

$$I^k \ni (x_1, ..., x_k) \longmapsto f\left(\sum_{j=1}^{k} x_j\right) - \prod_{j=1}^{k} f(x_j) \tag{6}$$

is contained in the range of F, then the function $M_f : I^k \to (0,\infty)$ defined by

$$E_f(x_1, ..., x_k) := F^{-1}\left(f\left(\sum_{j=1}^{k} x_j\right) - \prod_{j=1}^{k} f(x_j)\right), \tag{7}$$

is a well defined k-variable symmetric premean in I;

(ii) *if f is continuous and either F is strictly increasing and*

$$I \ni t \longmapsto f\left(t + \sum_{j=2}^{k} x_j\right) - \left(\prod_{j=2}^{k} f(x_j)\right) f(t) \text{ is (strictly) increasing,} \tag{8}$$

or F is strictly decreasing and the function in (8) is (strictly) decreasing, then the function E_f given by (7) is a k-variable symmetric (strict) mean in I.

Definition 2. Under the suitable conditions of Theorem 2, the function E_f can be referred to as the quasi-Cauchy difference mean (premean) of the exponential type generated by f.

Remark 10. Since for every exponential function, the Cauchy difference of exponential type is constant, no exponential function generates a premean of that type.

Let $k \in \mathbb{N}$, $k \geq 2$, an interval $I \subset \mathbb{R}$, closed under addition, and let $f, g : I \to \mathbb{R}$. In connection with Theorem 2, one can ask for conditions guaranteeing the equality $M_g = M_f$, i.e. such that, for all $x_1, ..., x_k \in I$,

$$G^{-1}\left(g\left(\sum_{j=1}^{k} x_j\right) - \prod_{j=1}^{k} g(x_j)\right) = F^{-1}\left(f\left(\sum_{j=1}^{k} x_j\right) - \prod_{j=1}^{k} f(x_j)\right),$$

where

$$F(x) = f(kx) - [f(x)]^k, \quad G(x) = g(kx) - [g(x)]^k, \qquad x \in I.$$

In the case $k = 2$, setting $\varphi := G \circ F^{-1}$, this leads to the following open

Problem 3. *Determine all functions f, g and φ satisfying the functional equation*

$$\varphi(f(x+y) - f(x)f(y)) = g(x+y) - g(x)g(y), \qquad x, y \in I.$$

This equation is a special case of the generalized Cauchy difference equation of the exponential type

$$\varphi(H(x,y)) = g(x+y) - g(x)g(y),$$

where g, H and φ are the unknown functions.

4.1. Quasi-Cauchy difference means of exponential type generated by additive functions

Using Theorem 2 with the linear functions $f(x) = px$ we obtain

Proposition 4. *Let $k \in \mathbb{N}$, $k \geq 2$ be fixed. For $p > 0$ put $I = \left(\frac{1}{p}, \infty\right)$ or $I = \left(0, \frac{1}{p}\right)$. For every $p > 0$ there is a unique continuous function $\mathcal{E}_p : I^k \to \mathbb{R}$ satisfying the implicit equality*

$$pk\mathcal{E}_p - (p\mathcal{E}_p)^k = p(x_1 + ... + x_k) - p^k x_1 \cdot ... \cdot x_k$$

and it is the quasi-Cauchy difference mean of exponential type generated by the function $f(x) = px$.

Moreover, in the case $k = 2$,

if $I = \left(\frac{1}{p}, \infty\right)$ then

$$\mathcal{E}_p(x,y) = \frac{1}{p} + \sqrt{\left(x - \frac{1}{p}\right)\left(y - \frac{1}{p}\right)}, \qquad x, y \in \left(\frac{1}{p}, \infty\right),$$

and

$$\lim_{p\to\infty} E_p(x,y) = \sqrt{xy}, \qquad x,y \in (0,\infty);$$

if $I = \left(\frac{1}{p}, \infty\right)$ *then*

$$\mathcal{E}_p(x,y) = \frac{1}{p} - \sqrt{\left(\frac{1}{p} - x\right)\left(\frac{1}{p} - x\right)}, \qquad x,y \in \left(0, \frac{1}{p}\right),$$

and

$$\lim_{p\to 0} E_p(x,y) = \frac{x+y}{2}, \qquad x,y \in (0,\infty).$$

4.2. Quasi-Cauchy difference means of exponential type generated by multiplicative functions

Using Theorem 2 for power functions $f(x) = x^p$ we obtain

Proposition 5. *Let* $k \in \mathbb{N}$, $k \geq 2$ *be fixed. For every* $p > 0$ *there is a unique continuous function* $E_p : \left(k^{\frac{p-1}{(k-1)p}}, \infty\right)^k \to \left(k^{\frac{p-1}{(k-1)p}}, \infty\right)$ *satisfying the equality*

$$(kE_p)^p - \left(E_p^k\right)^p = (x_1 + ... + x_k)^p - (x_1 \cdot ... \cdot x_k)^p,$$

and E_p *is the quasi-Cauchy difference mean of exponential type generated by the multiplicative function* $f(x) = x^p$.

Moreover, if $k = 2$ *then*

$$E_p(x,y) = \left(2^{p-1} + \sqrt{(xy)^p - (x+y)^p + 4^{p-1}}\right)^{1/p}, \qquad x,y \in \left(2^{\frac{p-1}{p}}, \infty\right);$$

in particular

$$E_1(x,y) = 1 + \sqrt{(x-1)(y-1)}, \qquad x,y \in (1,\infty);$$

and

$$E_0(x,y) := \lim_{p\to 0} E_p(x,y) = \frac{2xy}{x+y}, \qquad x,y > 0,$$

$$E_\infty(x,y) := \lim_{p\to\infty} E_p(x,y) = \sqrt{xy}, \qquad x,y > 2 = \sup\left\{2^{\frac{p-1}{p}} : p > 0\right\},$$

and for all $x, y > 2$, *the function*

$$[0,\infty] \ni p \longmapsto E_p(x,y) \quad \textit{is continuous.}$$

4.3. Quasi-Cauchy difference means of exponential type generated by logarithmic functions

For the logarithmic functions $f = \log_p$ we have less satisfactory

Proposition 6. *For every $p > 1$ and $k \in \mathbb{N}$, $k \geq 2$, there is an interval $I = (c(p), \infty) \subset (0, \infty)$ and a unique continuous function $E_p : I^k \to I$ satisfying the implicit equality*

$$\log_p(kE_p) - \left[\log_p(E_p)\right]^k = \log_p(x_1 + ... + x_k) - \left(\log_p(x_1)\right) \cdot ... \cdot \left(\log_p(x_k)\right);$$

it is a quasi-Cauchy difference mean of exponential type generated by function $f = \log_p$.

Moreover, if $k = 2$ then,

$$\log_p E_p(x, y) = \frac{1}{2}\left(1 + \sqrt{1 + 4\left[\left(\log_p x\right)\left(\log_p y\right) - \log_p(x + y) + \log_p 2\right]}\right), \quad x, y \in I$$

and

$$E_\infty(x, y) := \lim_{p\to\infty} E_p(x, y) = \frac{x + y}{2}, \qquad x, y \in I.$$

5. Quasi-Cauchy difference means of logarithmic type

Theorem 3. *Let $k \in \mathbb{N}$, $k \geq 2$, and an interval $I \subset (0, \infty)$ closed under multiplication, be fixed. Assume that $f : I \to \mathbb{R}$ is such that $F : I \to \mathbb{R}$ defined by*

$$F(x) := f\left(x^k\right) - kf(x), \tag{9}$$

is one-to-one. Then

(i) *if the range Cauchy difference of the logarithmic type*

$$I^k \ni (x_1, ..., x_k) \longmapsto f\left(\prod_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) \tag{10}$$

is contained in the range of F, then the function $L_f : I^k \to (0, \infty)$ defined by

$$L_f(x_1, ..., x_k) := F^{-1}\left(f\left(\prod_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right), \tag{11}$$

is a well defined k-variable symmetric premean in I;

(ii) *if f is continuous and either F is strictly increasing and*

$$I \ni t \longmapsto f\left(t\left(\prod_{j=2}^{k} x_j\right)\right) - \left(f(t) + \sum_{j=2}^{k} f(x_j)\right) \tag{12}$$

is (strictly) increasing,

or F is strictly decreasing and the function in (12) *is (strictly) decreasing, then the function L_f given by* (11) *is a k-variable strict symmetric mean in I.*

Let us note the following

Remark 11. The function f satisfies condition (12) if for all $x, y, z \in I$,

$$x < z \Longrightarrow f(zy) - f(xy) < z - x.$$

Definition 3. Under the suitable conditions of Theorem 2, the function L_f can be referred to as a quasi-Cauchy difference mean (premean) of logarithmic type generated by f.

Remark 12. Since for the logarithmic functions, the Cauchy differences of logarithmic type are constant zero, no logarithmic function generates a premean of that type.

Let $k \in \mathbb{N}$, $k \geq 2$, an interval $I \subset (0, \infty)$, closed under multiplication, and let $f, g : I \to \mathbb{R}$. In connection with Theorem 3, one can ask for conditions guaranteeing the equality $M_g = M_f$, i.e. such that, for all $x_1, ..., x_k \in I$,

$$G^{-1}\left(g\left(\prod_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} g(x_j)\right) = F^{-1}\left(f\left(\prod_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right),$$

where

$$F(x) = f\left(x^k\right) - kf(x), \quad G(x) = g\left(x^k\right) - kg(x), \qquad x \in I.$$

In the case $k = 2$, setting $\varphi := G \circ F^{-1}$, this leads to the following

Problem 4. *Determine all functions f, g and φ satisfying the functional equation*

$$\varphi(f(xy) - f(x) - f(y)) = g(xy) - g(x) - g(y), \qquad x, y \in I.$$

This equation is a special case of the generalized Cauchy difference equation of logarithmic type

$$\varphi(H(x, y)) = g(xy) - g(x) - g(y),$$

where g, H and φ are the unknown functions.

5.1. Quasi-Cauchy difference means of logarithmic type generated by additive functions

Taking $f(x) = px$ in Theorem 3 we obtain

Proposition 7. *Let $k \in \mathbb{N}$, $k \geq 2$ be fixed, and $I = (0,1)$ or $I = (1,\infty)$. Then there is a unique continuous function $\mathcal{L} : I^k \to I$ such that*

$$\mathcal{L}^k - k\mathcal{L} = \prod_{j=1}^{k} x_j - \sum_{j=1}^{k} x_j, \qquad x_1, ..., x_k \in I;$$

and it is (independent on p) the quasi-Cauchy difference mean of logarithmic type generated by the function $f(x) = px$.

Moreover, in the case $k = 2$:

if $I = (0,1)$, then

$$\mathcal{L}(x,y) = 1 - \sqrt{(1-x)(1-y)}, \quad x, y \in (0,1);$$

if $I = (1,\infty)$, then

$$\mathcal{L}(x,y) = 1 + \sqrt{(x-1)(y-1)}, \quad x, y \in (1,\infty).$$

5.2. Quasi-Cauchy difference means of logarithmic type generated by exponential functions

Applying Theorem 3 to the exponential functions we obtain the following

Proposition 8. *For every $p > 1$, there is a strict symmetric mean $L_p : \bigcup_{k=2}^{\infty} (1,\infty)^k \to (1,\infty)$ satisfying the implicit equality*

$$p^{(L_p)^k} - kp^{L_p} = p^{x_1 \cdot ... \cdot x_k} - (p^{x_1} + ... + p^{x_k}),$$

that is the quasi-Cauchy difference mean of logarithmic type generated by the function $f(x) = p^x$.

Remark 13. We do not know the effective formula of the mean L_p even if $k = 2$, or if $L_1 := \lim_{p\to 1} L_p$.

5.3. Quasi-Cauchy difference means of logarithmic type generated by multiplicative functions

Applying Theorem 3 to the multiplicative function $f(x) = x^p$ we obtain

Proposition 9. *For every $p > 0$ there is a strict symmetric mean $\mathcal{L}_p$: $: \bigcup_{k=2}^{\infty} (1,\infty)^k \to (1,\infty)$ such that*

$$(\mathcal{L}_p)^{pk} - k(\mathcal{L}_p)^p = (x_1 \cdot ... \cdot x_k)^p - (x_1^p + ... + x_k^p), \qquad x_2, ..., x_k > 1,$$

and it is the quasi-Cauchy difference mean of logarithmic type generated by the power function $f(x) = x^p$.

If $k = 2$ then, for all $x, y > 1$,

$$L_p(x,y) = \left(1 + \sqrt{(x^p - 1)(y^p - 1)}\right)^{1/p},$$

$$\mathcal{L}_0(x,y) := \lim_{p \to 0} \mathcal{L}_p(x,y) = \sqrt{xy}, \qquad \mathcal{L}_\infty(x,y) := \lim_{p \to \infty} \mathcal{L}_p(x,y) = \max(x,y).$$

Remark 14. For every $p < 0$ the function F is strictly increasing and the function in condition (12) is strictly decreasing (as the partial derivative of the respective function is negative). It follows that the function L_f is a premean in $(1,\infty)$, and in the case $k = 2$,

$$L_f(x,y) = \left(1 - \sqrt{(1 - x^p)(1 - y^p)}\right)^{1/p}.$$

In particular, in case $k = 2$, L_f is not "complementary" to $\mathcal{L}_{-p}$ with respect to the geometric mean $G = \mathcal{L}_0$ (see [12]).

Remark 15. In case $k = 2$, setting for every $p > 0$,

$$\mathcal{L}_{-p}(x,y) := \frac{xy}{\mathcal{L}_p(x,y)}, \qquad x, y > 1,$$

we obtain a one-parameter family of means $\{\mathcal{L}_p : p \in \mathbb{R}\}$ that is complete in the sense of "complementariness", namely, for every $p > 0$ the mean $\mathcal{L}_0$ is invariant with respect to mean-type mapping $(\mathcal{L}_p, \mathcal{L}_{-p}) : (1,\infty)^2 \to (1,\infty)^2$, that is

$$\mathcal{L}_0 \circ (\mathcal{L}_p, \mathcal{L}_{-p}) = \mathcal{L}_0.$$

This fact guarantees, that the sequence ($(\mathcal{L}_p, \mathcal{L}_{-p})^n : n \in \mathbb{N}$) of iterates of the mapping $(\mathcal{L}_p, \mathcal{L}_{-p})$ converges (uniformly on compact subsets) to the mean-type mapping $(\mathcal{L}_0, \mathcal{L}_0)$ (see [8]). The mean $\mathcal{L}_{-p}$ for $p > 0$ is not a logarithmic type Cauchy quasi-difference mean of a multiplicative generator. This is a disadvantageous property of this extension of the family $\{\mathcal{L}_p : p \geq 0\}$.

6. Quasi-Cauchy difference means of multiplicative type

Since the continuous multiplicative functions are power functions, we use letter P to denote the respective means.

Theorem 4. *Let $k \in \mathbb{N}$, $k \geq 2$, and an interval $I \subset (0, \infty)$ closed under multiplication, be fixed. Assume that a function $f : I \to \mathbb{R}$ is such that function $F : I \to \mathbb{R}$ defined by*

$$F(x) := f\left(x^k\right) - [f(x)]^k, \tag{13}$$

is one-to-one. Then

(i) *if the range of the Cauchy difference of exponential type*

$$I^k \ni (x_1, ..., x_k) \longmapsto f\left(\prod_{j=1}^{k} x_j\right) - \prod_{j=1}^{k} f(x_j) \tag{14}$$

is contained in the range of F, then the function $P_f : I^k \to (0, \infty)$ defined by

$$P_f(x_1, ..., x_k) := F^{-1}\left(f\left(\prod_{j=1}^{k} x_j\right) - \prod_{j=1}^{k} f(x_j)\right), \tag{15}$$

is a well defined k-variable symmetric premean in I;

(ii) *if f is continuous and either F is strictly increasing and*

$$I \ni t \longmapsto f\left(t \prod_{j=2}^{k} x_j\right) - f(t) \prod_{j=2}^{k} f(x_j) \text{ is (strictly) increasing,} \tag{16}$$

or F is strictly decreasing and the function in (16) *is (strictly) decreasing, then the function P_f given by* (15) *is a k-variable symmetric (strict) mean in I.*

Let $k \in \mathbb{N}$, $k \geq 2$, an interval $I \subset (0, \infty)$, closed under multiplication, and let $f, g : I \to (0, \infty)$. Similarly as in the previous section, we can ask for conditions guaranteeing the equality $M_g = M_f$, i.e. such that, for all $x_1, ..., x_k \in I$,

$$G^{-1}\left(g\left(\prod_{j=1}^{k} x_j\right) - \prod_{j=1}^{k} g(x_j)\right) = F^{-1}\left(f\left(\prod_{j=1}^{k} x_j\right) - \prod_{j=1}^{k} f(x_j)\right),$$

where

$$F(x) = f\left(x^k\right) - [f(x)]^k, \quad G(x) = g\left(x^k\right) - [g(x)]^k, \qquad x \in I.$$

In the case $k = 2$, setting $\varphi := G \circ F^{-1}$, this leads to the following

Problem 5. *Determine all functions f, g and φ satisfying the functional equation*

$$\varphi\left(f\left(xy\right)-f\left(x\right)f\left(y\right)\right)=g\left(xy\right)-g\left(x\right)g\left(y\right),\qquad x,y\in I.$$

This equation is a special case of the generalized Cauchy difference equation of multiplicative type

$$\varphi\left(H\left(x,y\right)\right)=g\left(xy\right)-g\left(x\right)g\left(y\right),$$

where g, H and φ are the unknown functions.

6.1. Quasi-Cauchy difference means of multiplicative type generated by additive functions

Applying Theorem 4 to the continuous additive functions we obtain

Proposition 10. *The geometric mean* $\mathcal{G}:\bigcup_{k=2}^{\infty}\left(0,\infty\right)^{k}\to\left(0,\infty\right),$

$$G\left(x_1,...,x_k\right)=\sqrt[k]{x_1\cdot...\cdot x_k},\quad x_1,...,x_k\in\left(0,\infty\right),$$

is a quasi-Cauchy difference mean of multiplicative type generated by linear functions; more precisely: if $p>0$, $p\neq 1$, and $f(x)=px$ for $x\in(0,\infty)$, then

$$P_f=\mathcal{G}.$$

Remark 16. If $f(x)=x$ then F is zero constant function, so it is not invertible. Thus, in the above proposition, the assumption $p\neq 1$ is essential.

6.2. Quasi-Cauchy difference means of multiplicative type generated by exponential functions

Applying Theorem 4 to the continuous exponential functions we obtain

Proposition 11. *Let $p>1$ and let $f(x)=p^x$ for $x\in(1,\infty)$. Then there is a unique multiplicative type Cauchy quasi-difference mean* $P_f:\bigcup_{k=2}^{\infty}\left(k^{\frac{1}{k-1}},\infty\right)^{k}\to$ $\to(1,\infty)$ *generated by f; moreover P_f satisfies the following implicit equality*

$$p^{\left(P_f\left(x_1,\ldots,x_k\right)\right)^k}-p^{kP_f\left(x_1,\ldots,x_k\right)}=p^{x_1\cdot\ldots\cdot x_k}-p^{x_1+\ldots+x_k},\quad x_1,...,x_k\in\left(k^{\frac{1}{k-1}},\infty\right).$$

6.3. Quasi-Cauchy difference means of multiplicative type generated by logarithmic functions

Now we apply Theorem 4 to continuous logarithmic functions.

Proposition 12. *Let $p > 1$; let $I = (p, \infty)$ or $I = (1, p)$, and let $f = \log_p$. There is a unique multiplicative type Cauchy quasi-difference mean $P_f : \bigcup_{k=2}^{\infty} I^k \to I$ generated by f in the interval I and $\mathcal{P}_p := P_f$ satisfies the following implicit equality*

$$k \log_p \mathcal{P}_p (x_1, ..., x_k) - \left[\log_p \mathcal{P}_p (x_1, ..., x_k)\right]^k = \\ = \log_p (x_1 \cdot ... \cdot x_k) - \left(\log_p x_1\right) \cdot ... \cdot \left(\log_p (x_k)\right).$$

Moreover, in the case $k = 2$,

if $I = (p, \infty)$, then

$$\mathcal{P}_p (x, y) = p^{1+\sqrt{\left(\log_p x - 1\right)\left(\log_p y - 1\right)}},$$

and

$$\mathcal{P}_1 (x, y) := \lim_{p \to 1} \mathcal{P}_p (x, y) = \exp \sqrt{(\log x)(\log x)}, \qquad x, y > 1;$$

if $I = (1, p)$,

$$\mathcal{P}_p (x, y) = p^{1-\sqrt{\left(1 - \log_p x\right)\left(1 - \log_p y\right)}}$$

and

$$\mathcal{P}_\infty (x, y) := \lim_{p \to \infty} \mathcal{P}_p (x, y) = \frac{x + y}{2}, \qquad x, y > 1.$$

Remark 17. This proposition and the relation $\log_p = -\log_{\frac{1}{p}}$ allow to formulate the suitable result for $p \in (0, 1)$.

7. Final remark

The question when the direct Cauchy difference of additive-, exponential-, logarithmic- or multiplicative-type of a generator g is a premean, leads, respectively, to the iterative functional equation:

$$g(2x) - 2g(x) = x; \qquad g(2x) - [g(x)]^2 = x;$$
$$g\left(x^2\right) - 2g(x) = x; \qquad g\left(x^2\right) - [g(x)]^2 = x,$$

solving of which requires some special methods (see M. Kuczma [9]). Contrary to this, result (i) in each of our four theorems shows that the relevant k-variable function is a pre-mean. Taking into account Remark 1, the author would welcome the examples of the respectively constructed premeans which are not increasing in each of the variable.

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