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REMARKS ON INFINITE HYPERGROUP JOINS

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Dedicated to Professor Antal Járai on his 70th birthday

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Abstract. In the paper [2] we dealt with some basic functional equations on an infinite hypergroup join. The special property of that join is that every non-identity element is of infinite order, i.e. no power of it is the identity, but there is no nonzero additive function on this hypergroup. In this paper we extend the ideas to construct an infinite join starting from arbitrary finite hypergroups and we describe some basic function classes on this hypergroup.

1. Introduction

A comprehensive monograph on hypergroups is [1] and a detailed study on functional equations on hypergroups can be found in [4]. Notation and terminology here will be used according to these works. By $\mathbb{C}$ we denote the set of complex numbers, $\mathbb{N}$ is the set of all non-negative integers.

A particular hypergroup which played a basic role in the paper [2] is the two-element hypergroup $D(\theta)$ (see [1, 4]). For the sake of completeness, we recall here the definition.

The two-element hypergroup $D(\theta)$ on the set $\{o, a\}$ with $a \neq o$ and θ in $(0, 1]$ is defined as follows: o is the identity element, involution is the identity

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mapping, and convolution is defined by the equations

$$\begin{aligned}\delta_o * \delta_o &= \delta_o\\ \delta_o * \delta_a &= \delta_a * \delta_o = \delta_a\\ \delta_a * \delta_a &= \theta\delta_o + (1-\theta)\delta_a.\end{aligned}$$

We note that in the case $\theta = 1$, $D(\theta)$ is the two-element group $\mathbb{Z}_2$.

It is easy to check that the normalized Haar measure on $D(\theta)$ is

$$\omega_{D(\theta)} = \frac{\theta}{\theta+1}\delta_o + \frac{1}{\theta+1}\delta_a.$$

We note that on discrete hypergroups we shall use the Haar measure which is normalized in the way that the singleton $\{o\}$ has measure 1. This will be called the *unit-normalized* Haar measure.

We also recall the definition of hypergroup joins (see [3]). Let $(C, *_C)$ be a compact hypergroup with the normalized Haar measure ω_C and $(D, *_D)$ be a discrete hypergroup. Assume that $C \cap D = \{e\}$, where e denotes the identity of both hypergroups. We write D_e for the set $D\backslash\{e\}$. The *hypergroup join* $C \vee D$ is the set $C \cup D$ with the unique topology for which both C and D are closed subspaces. Involution on $C \cup D$ is defined in the way that its restriction to C and to D, respectively, coincide with the involution on C and on D, respectively.

Convolution on $C \vee D$ is defined in the following way (see [1]):

(i) for x, y in C the convolution of δ_x and δ_y is $\delta_x *_C \delta_y$;

(ii) for x, y in D satisfying $x \neq \check{y}$ the convolution of δ_x and δ_y is $\delta_x *_D \delta_y$;

(iii) for x in C and y in D with $y \neq e$ the convolution of δ_x and δ_y, and also the convolution of δ_y and δ_x is δ_y;

(iv) for y in D with $y \neq e$ we have the unique representation

$$\delta_y *_D \delta_{\check{y}} = \sum_{w\in D} c_w\delta_w = \sum_{w\in D_e} c_w\delta_w + c_e\delta_e$$

with some complex numbers c_w with w in D. Then the convolution of δ_y and $\delta_{\check{y}}$ and also the convolution of $\delta_{\check{y}}$ and δ_y is

$$\sum_{w\in D_e} c_w\delta_w + c_e\omega_C = \delta_y *_D \delta_{\check{y}} + c_e(\omega_C - \delta_e).$$

The construction of an infinite join is due to I. Jewett in [3] and it was used by H. Zeuner in [8] who studied random walk on $\mathbb{N}$ with the convolution structure of the infinite hypergroup join which we shall present below (see [2]*).

Let $K_0 = \{0\}$ and $J_n = \{0, n\}$ for n in $\mathbb{N}$. Suppose that a sequence (θ_n) in $[0, 1]$ is given with $\theta_0 = 0$ and $\theta_n > 0$ for $n > 0$. Put

$$K_{n+1} = K_n \vee J_{n+1}, \quad n = 0, 1, 2, \ldots$$

Here J_n is the two-point hypergroup $D(\theta_n)$, i.e. the hypergroup with convolution defined as

$$\delta_n * \delta_n = \theta_n \delta_0 + (1 - \theta_n)\delta_n.$$

Let $K = \bigcup_{n\in\mathbb{N}} K_n = \{\delta_0, \delta_1, \delta_2, \ldots\}$. Of course, we identify n with δ_n and K with the set $\mathbb{N}$ of natural numbers. On K the hypergroup structure is defined in the following way: for $m \neq n$ we have

$$\delta_m * \delta_n = \delta_{\max(m,n)},$$

and for an arbitrary $n \geq 1$

$$\delta_n * \delta_n = \sum_{k=0}^{n-1} \frac{\theta_{k+1}\theta_{k+2}\cdots\theta_n}{(\theta_k + 1)(\theta_{k+1} + 1)\cdots(\theta_{n-1} + 1)}\delta_k + (1 - \theta_n)\delta_n,$$

where we write $\theta_0 = 0$. This is easy to check by the definition of the convolution on hypergroup joins. Then K_n is a subhypergroup of K. In particular, $K_1 = K_0 \vee D(\theta_1) = D(\theta_1)$. We note that the translation operator τ_n, defined for each n in $\mathbb{N}$ by

$$\tau_n f(k) = f(k * n) = \delta_n * f(k)$$

has the property that $\tau_n f(k) = f(n)$ for $k = 0, 1, \ldots, n-1$, $\tau_n f(k) = f(k)$ for $k = n+1, n+2, \ldots$, and

$$\tau_n f(n) = \sum_{k=0}^{n-1} \frac{\theta_{k+1}\theta_{k+2}\cdots\theta_n}{(\theta_k + 1)(\theta_{k+1} + 1)\cdots(\theta_{n-1} + 1)} f(k) + (1 - \theta_n) f(n).$$

In [2], we proved the following theorems.

Theorem 1.1. *Every generalized exponential monomial on K is a constant multiple of an exponential.*

Theorem 1.2. *Every exponential on K is a character.*

Corollary 1.1. *Every exponential polynomial on K is a trigonometric polynomial.*

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*Unfortunately, in the final published version of paper [2] two pages are missing – here we present the content of those pages.

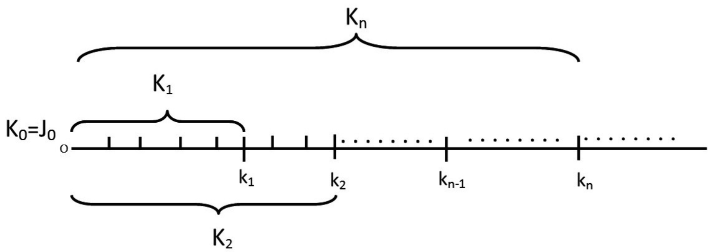
In this paper we generalize the above construction by replacing the J_n hypergroups with arbitrary finite hypergroups.

2. Infinite join of finite hypergroups

Let J_n be a sequence of finite hypergroups with $J_0 = \{0\}$ and $J_n \cap J_k = \{0\}$ for each n, k in $\mathbb{N}$ with $n \neq k$. Our purpose is to construct an infinite join from the "blocks" J_n. For the sake of simplicity we identify the sets J_n with subsets of $\mathbb{N}$ in the following manner. Let $k_0 = 0 < k_1 < \cdots < k_n < \ldots$ be a strictly increasing sequence of natural numbers and we asume that the hypergroup J_n $(n = 1, 2, \ldots)$ is defined on the set $\{0, k_{n-1}+1, k_{n-1}+2, \ldots, k_n\}$. In other words, $J_0 = \{0\}$, $J_1 = \{0, 1, 2, \ldots, k_1\}$, $J_2 = \{0, k_1+1, k_1+2, \ldots, k_2\}$, etc.

Let $K_0 = J_0 = \{0\}$ and $K_{n+1} = K_n \vee J_{n+1}$ for $n = 0, 1, \ldots$. Here we consider K_n as the compact part and J_{n+1} as the discrete part of the join. It is clear that, for $k \leq n$, the hypergroup K_k is a subhypergroup of K_n. It follows that if we let $K = \mathbb{N} = \bigcup_{n=0}^{\infty} K_n$, then there is a unique hypergroup structure such that each K_n is a subhypergroup of K. We note that in the paper [2] we considered the special case where $k_n = n$.

We can illustrate the set K by the picture below:



In the rest of this paper we consider only commutative joins. Hence we assume that all the finite hypergroups J_n above are commutative. We shall use the following theorems about the description of exponentials and exponential monomials on hypergroup joins (see [7]).

Theorem 2.1. *On a compact commutative hypergroup every nonzero generalized exponential monomial is of degree zero, that is, a constant multiple of an exponential.* (See [7, Theorem 3]).

Theorem 2.2. *Let C be a compact hypergroup and D a discrete hypergroup and let $C \vee D$ denote the corresponding hypergroup join (see Section 1). The continuous function $m : C \cup D \to \mathbb{C}$ is an exponential on the hypergroup join $C \vee D$ if and only if one of the the following possibilities holds:*

i) $m\big|_C \neq 1$ *is an exponential on* C *and* $m\big|_{D_e}$ *is identically zero;*

ii) $m\big|_C$ *is identically* 1 *and* $m\big|_D$ *is an exponential on* D. (See [7, Theorem 6]).

Theorem 2.3. *Let C be a compact hypergroup and D a discrete hypergroup, d a natural number, and let $C \vee D$ denote the corresponding hypergroup join (see Section 1). The continuous function $f : C \cup D \to \mathbb{C}$ is a generalized exponential monomial of degree at most d on the hypergroup join $C \vee D$ if and only if one of the the following possibilities holds:*

i) $f\big|_C \neq 1$ *is a generalized exponential monomial of degree at most d associated with an exponential $m_C \neq 1$ on C, and and $f\big|_{D_e}$ is identically zero;*

ii) $f\big|_C$ *is constant, and $f\big|_D$ is a generalized exponential monomial of degree at most d on D.* (See [7, Theorem 7]).

3. Exponential monomials and polynomials on the infinite join

In this section K denotes the hypergroup defined as in Section 2. Let $M : K \to \mathbb{C}$ be an exponential. We note that every exponential is 1 at the identity element. Then, clearly, the restriction $M\big|_{K_n}$ is an exponential on K_n. The following theorem describes all exponentials on K.

Theorem 3.1. *Let $M : K \to \mathbb{C}$ be an exponential. Then we have the following possibilities:*

1. $M \equiv 1$*;*

2. *There exists a unique natural number n such that $M(x) = 1$ for x in K_n, $M\big|_{J_{n+1}}$ is an exponential, which is not identically 1, and $M(x) = 0$ for x not in K_{n+1}.*

Proof. Suppose that $M \not\equiv 1$, then let x be in K such that $M(x) \neq 1$, then $x \neq 0$. Let n be the greatest positive integer such that x is not in K_n but x is in K_{n+1}. Then, by Theorem 2.2, when applied to the hypergroup join $K_{n+1} = K_n \vee J_{n+1}$, we have that $M\big|_{K_n}$ is identically 1, hence $M\big|_{J_{n+1}}$ is an exponential on J_{n+1}. On the other hand, $M\big|_{K_{n+1}}$ is an exponential on K_{n+1}, and it is not identically 1, hence by Theorem 2.2, when applied to the hypergroup join $K_{n+2} = K_{n+1} \vee J_{n+2}$, we infer that $M\big|_{J_{n+2,o}}$ is identically zero, where $J_{n+2,o} = J_{n+2} \backslash \{o\}$. It follows that there is a y in J_{n+2} such that $M(y) = 0$. Let z be not in K_{n+1}, then, by the definition of the convolution on the join, we have $\delta_z = \delta_y * \delta_z$, hence, by the multiplicative property of the exponential M we have

$$M(z) = M(\delta_z) = M(\delta_y * \delta_z) = M(\delta_y)M(\delta_z) = M(y)M(z) = 0,$$

which proves our theorem. ■

We establish that every exponential M on K arises in the following way: there exists a unique positive integer n and a unique exponential m_M on K_n such that

$$M(x) = \begin{cases} 1 & \text{for } x \leq k_{n-1} \\ m_M(x) & \text{for } k_{n-1} < x \leq k_n \\ 0 & \text{for } x > k_n. \end{cases} \tag{3.1}$$

In other words, the mapping $M \leftrightarrow m_M$ is a one-to-one correspondence between the set of all exponentials of K and the union of the sets of exponentials of the hypergroups J_n for $n = 0, 1, \ldots$: every exponential on K is the extension of a unique exponential of some of the J_n's.

Now it is easy to describe generalized exponential monomials and polynomials on the infinite join K. For more details about exponential monomials and polynomials on general commutative hypergroups see [5] and [6].

Let $f : K \to \mathbb{C}$ be a nonzero generalized exponential monomial associated with the exponential M on K. Then $f\big|_{K_n}$, the restriction of f to K_n, is a generalized exponential monomial associated with the exponential $M\big|_{K_n}$, the restriction of M to K_n. We may choose n as large as necessary so that f is nonzero, e.g. $n \geq N$. Then, by Theorem 2.1, $f\big|_{K_n} = c_n M\big|_{K_n}$ with some complex number c_n. Clearly, for $n \geq k \geq N$ we have $c_n = c_k$. It follows that $f(x) = cM(x)$ for each x.

We can summarize our results in the following theorem.

Theorem 3.2. *Let the finite hypergroups J_n $(n = 0, 1, 2, \ldots)$ be commutative, and let K denote the infinite join constructed above. For each exponential M on K, the function $f : K \to \mathbb{C}$ is a generalized exponential monomial associated with M if and only if $f = cM$ with some complex number c. Hence every generalized exponential polynomial on K is a linear combination of exponentials. In particular, every generalized exponential monomial (polynomial) on K is an exponential monomial (polynomial).*

As the function cM – with c is an arbitrary complex number – is obviously an exponential monomial associated with the exponential M on K, hence the previous theorem completely characterizes (generalized) exponential polynomials on K.

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