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SOME RESULTS ON SIMULTANEOUS NUMBER SYSTEMS IN THE RING OF INTEGERS OF IMAGINARY QUADRATIC FIELDS

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Dedicated to Professor Antal Járai on his 70th birthday

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Abstract. Simultaneous number systems were analysed in the ring of the Gaussian and the Eisenstein integers in [5, 3, 7, 8]. In this paper we present some results for further characterisations in the rings of integers of general imaginary quadratic fields. We show that in each ring there are infinitely many simultaneous number systems, and we give an efficient algorithm for the decision problem.

1. Introduction

Let M be an $n \times n$ integer linear operator. Let furthermore D be a finite subset of $\mathbb{Z}^n$ containing 0. The system $(\mathbb{Z}^n, M, D)$ is a *number system* if each element x of $\mathbb{Z}^n$ has a unique, finite representation of the form

$$x = \sum_{i=0}^{m} M^i d_i \,,$$

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where $d_i \in D$, $m \in \mathbb{N}$. Here M is called the *base* or *radix*, and D is the *digit set* or *alphabet*. Necessary conditions for this unique representation property are (1) the expansiveness of the base, (2) the full residue system property of the digit set, and (3) the unit condition $\det(M-I) \neq \pm 1$ [6]. The congruence relation here means that two elements are congruent if they belong to the same coset of the factor group $\mathbb{Z}^n/M\mathbb{Z}^n$. When the first two conditions hold then we speak about *radix systems*.

Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$, $x \overset{\varphi}{\mapsto} M^{-1}(x-d)$ for the unique $d \in D$ satisfying $x \equiv d \pmod{M}$. Since M^{-1} is contractive and D is finite, there exists a vector norm $\|.\|$ on $\mathbb{R}^n$ such that for the corresponding operator norm $\left\|M^{-1}\right\| < 1$ holds, and there is a constant C such that the orbit of every $x \in \mathbb{Z}^n$ eventually enters the finite set (so-called *testing set*) $T = \{x \in \mathbb{Z}^n \mid \|x\| < C\}$ for the repeated application of φ, and after entering the orbit never leaves it. This means that the sequence $x, \varphi(x), \varphi^2(x), \ldots$ is eventually periodic for all $x \in \mathbb{Z}^n$. A point p is called *periodic* if $\varphi^k(p) = p$ for some $k > 0$. Clearly, a system is a number system if the orbit of each point goes to zero, hence, the only periodic point is the zero. We note that such a norm exists with continuum cardinality.

Since the testing set is finite, the finite representation property can algorithmically be decided (the "decision problem"). Unfortunately, depending on the eigenvalue spectrum of M, the testing set can be enormous huge [4].

There are two main types of structured digit sets which are studied and applied extensively in the research: the arithmetic type (including the canonical and symmetric alphabets) and the dense one. Dense alphabets contain elements with minimal norm from each residue class. A special dense alphabet is the adjoint one, where $D = \{0 = d_1, \ldots, d_t\}$, $t = |\det(M)|$, and each coordinate of $M^{adj}d_i$ belong to the set $|\det(M)| \times (-1/2, 1/2]$. Dense alphabets are basic blocks for constructing number systems. Suppose that the spectral radius of M^{-1} is smaller that $1/2$. Then the system $(\mathbb{Z}^n, M, D)$ is always a number system with the dense digit set.

There is an important connection between lattice-based number systems and number systems in the ring of integers in algebraic number fields. Consider any non-zero monic polynomial $f(x) \in \mathbb{Z}[x]$. Then, the factor ring $\Lambda = \mathbb{Z}[x]/(f)$ is a lattice and all the number expansion related problems can be formulated in $\mathbb{Z}^n$ where the operator M is the companion of f. If $f(x)$ is irreducible then Λ is isomorphic with $\mathbb{Z}[\theta]$ (where $f(\theta) = 0$ in an appropriate extension of $\mathbb{Q}$). If K is a number field with degree n and $\mathcal{O}_K$ is the ring of its integers then there always exist a $\mathbb{Z}$-basis of $\mathcal{O}_K$. Hence, the columns of M can be expressed with this basis as well (e.g. with the power basis in monogenic fields).

Block diagonal systems were introduced in [5] in the following way: let the radix systems $(\mathbb{Z}^{n_i}, M_i, D_i)$ be given $(1 \leq i \leq k \geq 2)$. Consider the direct product of the lattices $\mathbb{Z}^n = \mathbb{Z}^{\sum_i n_i}$ and the direct sum $M = M_1 \oplus \cdots \oplus M_k$

of the operators. For simplicity, we denote the components of any $z \in \mathbb{Z}^n$ by $z = z_1 \bullet z_2 \bullet \ldots \bullet z_k$ where $z_i \in \mathbb{Z}^{n_i}$. Based on the alphabets D_i there are different ways for producing a complete residue system D mod M. The simplest one is the homomorphic construction [5].

In the following we concentrate to block diagonal systems with special alphabets.

2. Simultaneous systems

Definition 2.1. A block diagonal system $(\mathbb{Z}^{kn}, M_1 \oplus M_2 \oplus \cdots \oplus\ M_k, D)$ is called an (n,k)-*simultaneous system* if $M_i : \mathbb{Z}^n \to \mathbb{Z}^n$ and all the digits $d_j \in D$ have the form $v \bullet v \bullet \cdots \bullet v$, $v \in \mathbb{Z}^n$.

Kátai et. al. investigated the $(1,k)$ cases [2] where $N_1, N_2, \ldots, N_k$ are mutually coprime integers, none of them are ± 1, and $D = \{\delta e\}$, $e = 1 \bullet \cdots \bullet 1$, $\delta = 1, 2, \ldots |N_1 N_2 \cdots N_k| - 1$. They showed that the system $(\mathbb{Z}^k, N_1 \oplus \cdots \oplus$ $\oplus N_k, D)$ is a number system iff $k = 2$ and $N_2 = N_1 + 1 < 1$.

Regarding simultaneous systems one of the main research problem is to determine digit sets allowing number system constructions. As a first attempt, Nagy [8] applied canonical digit sets for the blocks in the lattice of Gaussian integers. He proved that in this case simultaneous number system constructions are not possible. Then, in the same lattice, the following construction was applied: for two blocks M_1 and M_2 with digit sets D_1 mod M_1 and D_2 mod M_2 let us consider the set

$$D = \{d_1 + M_1 d_2\}, \text{ where } d_1 \in D_1 \text{ and } d_2 \in D_2 \,. \tag{2.1}$$

It is easy to check that D is a full residue system. We note that for more than two blocks the construction can be made recursively.

In a series of papers $(2,2)$-simultaneous constructions were analysed in Gaussian and the Eisenstein rings [5, 3, 7, 8]. Nagy and Krutki investigated the $(2,3)$-simultaneous systems numerically with (2.1) type alphabets [9] where the blocks are integers from the ring $\mathbb{Q}[\sqrt{5}]$. We remark that since the rings of integers of algebraic number fields are commutative, the possible simultaneous constructions are narrowed, namely, the pairwise difference of the bases must be units.

Consider the $(2,2)$-simultaneous system $(\mathbb{Z}^4, M, D)$ with digit set D of form (2.1) where $M = M_1 \oplus M_2$. In the following we investigate that in which circumstances $(\mathbb{Z}^4, M_1, M_2, D)$ can be simultaneous number systems.

Lemma 2.1. *Let $S_1 = (\mathbb{Z}^2, M_1, D_1)$, $S_2 = (\mathbb{Z}^2, M_2, D_2)$ be two radix systems with some D_1, D_2. If the block diagonal system $S = (\mathbb{Z}^4, M_1, M_2, D)$ with the* *alphabet*

$$D = \{d \bullet d \ : d = d_1 + M_1 d_2, \ \text{where } d_1 \in D_1 \text{ and } d_2 \in D_2\} \tag{2.2}$$

is a $(2,2)$-simultaneous number system then S_1 and S_2 are both number systems.

Proof. Consider the function $f : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^4$,

$$f(z_1, z_2) = (c_m \bullet c_m, c_{m-1} \bullet c_{m-1}, \dots, c_1 \bullet c_1, c_0 \bullet c_0)_M = x \bullet y \,,$$

where $c_i = a_i + M_1 b_i$, $z_1 = \sum_i^m M_1^i a_i, z_2 = \sum_i^m M_2^i b_i$, $a_i \in D_1, b_i \in D_2$. The function f operates on those points which have finite expansions in S_1 and S_2, respectively. Observe that f is injective but not necessary surjective. Let $x \in \mathbb{Z}^2$ be any point chosen and let us examine the expansions of the points $x \bullet y \in \mathbb{Z}^4$. These points have all finite expansions in S. Exactly one of them is $f(z_1, 0) = \sum_i M_1^i a_i \bullet \sum_i M_2^i a_i$, which shows that x has a finite expansion in S_1. Similarly, let $y \in \mathbb{Z}^2$ any point. Consider the (necessarily finite) expansions of the points $x \bullet M_1 y$. Exactly one of them is $f(0, z_2) = \sum_i M_1^{i+1} b_i \bullet M_1 \sum_i M_2^i b_i$ showing that y has finite expansion in S_2. ■

Remark 2.1. The number system property of S_1 and S_2 are not sufficient for S being a number system. For example

$(\mathbb{Z}^2, \left(\begin{smallmatrix} 2 & -1 \\ 1 & 2 \end{smallmatrix}\right), D_1 = \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)\})$ and

$(\mathbb{Z}^2, \left(\begin{smallmatrix} 3 & -1 \\ 1 & 3 \end{smallmatrix}\right), D_1 \cup \{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)\})$

are number systems due to [1], but their simultaneous construction is not, having 9 periodic witnesses.

In this paper we examine $(2,2)$-simultaneous systems in arbitrary rings of integers of imaginary quadratic fields.

3. Simultaneous systems in the ring of integers of imaginary quadratic fields

Let $F \geq 2$ be a square-free integer. Let $\mathbb{Q}(i\sqrt{F})$ be an imaginary quadratic extension of $\mathbb{Q}$. It is known that if $F \not\equiv 3 \pmod 4$ then $\{1, \delta\}$ while for $F \equiv 3 \pmod 4$ $\{1, \omega\}$ is an integer basis in the set of integers of $\mathbb{Q}(i\sqrt{F})$ ($\delta = i\sqrt{F}$ and $\omega = \frac{1+i\sqrt{F}}{2}$). The lattice generated by the basis $\{1, \delta\}$ will be called as δ-lattice and the lattice generated by $\{1, \omega\}$ will be called as ω-lattice.

As it was mentioned, full characterisation of simultaneous number system constructions is known in case of the Gaussian ring ($F = 1$), a partial one in the Eisenstein ring ($F = 3$). Due to Dirichlet unit theorem, the group of units in the remaining cases is always $\{-1, 1\}$.

Let $(\mathbb{Z}^2, M_1, D_1)$, $(\mathbb{Z}^2, M_2, D_2)$ be radix systems with dense alphabets D_1 and D_2 respectively. Consider the operators

$$M_1 = \begin{pmatrix} a & -F\,b \\ b & a \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} a+1 & -F\,b \\ b & a+1 \end{pmatrix} \tag{3.1}$$

in the δ-lattice and the operators

$$M_1 = \begin{pmatrix} a & -G\,b \\ b & a+b \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} a+1 & -G\,b \\ b & a+b+1 \end{pmatrix} \tag{3.2}$$

in the ω-lattice, where $G = \frac{F+1}{4}$, $a, b \in \mathbb{Z}$ and $b \neq 0$.

In the following we apply the 2-norm for further investigations. We start with some notations. Let us denote the sublattice of points $v \bullet v \in \mathbb{Z}^2 \otimes \mathbb{Z}^2$ by W. Let furthermore $r_m = \min\left\{\left\|M_1^{-1}\right\|, \left\|M_2^{-1}\right\|\right\}$, $r_M = \max\left\{\left\|M_1^{-1}\right\|, \left\|M_2^{-1}\right\|\right\}$, $K^* = \max\{\|d^*\| : d^* \bullet d^* \in D\}$, $K = \max\{\|d\| : d \in D\}$, $L^* = K^* \frac{r_M}{1-r_M}$, $L = K \frac{r_M}{1-r_M}$, $R = \max\{\|M_1\|, \|M_2\|\}$, and κ_i the condition numbers of M_i, respectively, $\kappa = \max\{\kappa_1, \kappa_2\}$. In the rest of the paper we suppose that D_i are adjoint alphabets modulo M_i respectively.

Lemma 3.1. $\|M_2^{-1} - M_1^{-1}\| = \|M_1^{-1}\|\,\|M_2^{-1}\|$.

The proof is straightforward.

Lemma 3.2. $K^* \leq \frac{1}{\sqrt{2}} \|M_1\| \left(\|M_2\| + 1\right)$.

Proof. Since

$$\max_{d \in D_i} \|d\| = \max_{b \in [-1/2, 1/2)^2} \|M_i^*\, b\| \leq \frac{\|M_i\|}{\sqrt{2}}$$

for $i = 1, 2$ therefore

$$K^* = \max_{d \bullet d \in D} \|d\| = \max_{d_1 \in D_1, d_2 \in D_2} \|M_1\; d_2 + d_1\| \leq \frac{\|M_1\| \left(\|M_2\| + 1\right)}{\sqrt{2}}. \qquad \blacksquare$$

Theorem 3.1. *There is a constant $\gamma \in \mathbb{R}$ depending on M_1 and M_2 such that if*

(1) $\frac{\gamma}{1-r_m} < 1$ *and all the points in $W \cap L \setminus \{0\}$ are non-periodic or*

(2) $\frac{\gamma}{1-r_m} \geq 1$ *and all the points $0 \neq z = z_1 \bullet z_2 \in \mathbb{Z}^4$ for which*

$$\|z\| \leq L,\ \|z_1 - z_2\| < \frac{\gamma}{1 - r_m} \tag{3.3}$$

are non-periodic, then $(\mathbb{Z}^4, M_1, M_2, D)$ is a simultaneous number system.

Proof. Let $z = z_1 \bullet z_2 \in \mathbb{Z}^4$. Since

$$\begin{aligned}
&\|\varphi_1(z_1) - \varphi_2(z_2)\| = \\
&\quad = \left\|M_1^{-1}(z_1 - d) - M_2^{-1}(z_2 - d)\right\| = \\
&\quad = \left\|M_1^{-1}(z_1 - d) - M_1^{-1}(z_2 - d) + M_1^{-1}(z_2 - d) - M_2^{-1}(z_2 - d)\right\| \le \\
&\quad \le \left\|M_1^{-1}((z_1 - d) - (z_2 - d))\right\| + \left\|(M_1^{-1} - M_2^{-1})(z_2 - d)\right\| \le \\
&\quad \le \left\|M_1^{-1}\right\| \|z_1 - z_2\| + \left\|M_1^{-1} - M_2^{-1}\right\| \|z_2 - d\| \le \\
&\quad \le r_m \|z_1 - z_2\| + \|M_1^{-1}\|\|M_2^{-1}\|(L^* + K^*) \le \\
&\quad \le r_m \|z_1 - z_2\| + \frac{r_M\ r_m\ K^*}{1 - r_M} \le \\
&\quad \le r_m \|z_1 - z_2\| + \underbrace{\frac{r_m\ r_M \|M_1\|(\|M_2\| + 1)}{\sqrt{2}(1 - r_M)}}_{\gamma},
\end{aligned}$$

by which the theorem follows. ■

Let us investigate the limits of the condition numbers. In the δ-lattice

$$\begin{aligned}
\kappa &= \frac{F^2b^2 + 2a^2 + b^2 + \sqrt{b^2(F-1)^2((F+1)^2b^2 + 4a^2)}}{2a^2 + 2Fb^2} = \\
&= 1 + \frac{(F-1)^2b^2 + |b|(F-1)\sqrt{(F+1)^2b^2 + 4a^2}}{2a^2 + 2Fb^2},
\end{aligned}$$

by which $\lim_{a\to\infty} \kappa = 1, \lim_{b\to\infty} \kappa = F, \lim_{F\to\infty} \kappa = \infty$. Clearly, if $F = 1$ then $\kappa = 1$. In the ω-lattice

$$\begin{aligned}
\kappa = 1 &+ \frac{b^2(G^2 - 2G + 2)}{2(Gb^2 + a^2 + ab)} + \\
&+ \frac{\sqrt{b^2(G^2 - 2G + 2)(4(a^2 + Gb^2 + ab) + b^2(G^2 - 2G + 2))}}{2(Gb^2 + a^2 + ab)}
\end{aligned}$$

by which $\lim_{a\to\infty} \kappa = 1, \lim_{b\to\infty} \kappa = 1 + \frac{G^2 - 2G + 2 + \sqrt{G^4 + 4}}{2G}, \lim_{G\to\infty} \kappa = \infty$.

Theorem 3.2. *Consider the ring of integers in any imaginary quadratic field and the operators of form* (3.1) *and* (3.2) *acting on them. Then for each* $0 \ne b \in \mathbb{Z}$ *there can be infinitely many simultaneous number systems constructed.*

Proof. Let us examine when in Theorem 3.1 the condition $\frac{\gamma}{1-r_m} < A = 1$ is satisfied:

(3.4)
$$\frac{\gamma}{1-r_m} = \frac{r_m\, r_M\, \|M_1\|(\|M_2\|+1)}{\sqrt{2}(1-r_m)\,(1-r_M)} \le \frac{r_M^2 R(R+1)}{\sqrt{2}(1-r_M)^2} \le \frac{\kappa(r_M+\kappa)}{\sqrt{2}(1-r_M)^2} < A\,,$$

by which
$$\sqrt{2}r_M^2 - (2\sqrt{2}+\kappa)r_M + \sqrt{2} - \kappa^2 > 0\,.$$

The solutions for $0 < r_M < 1$ can be given by
$$\frac{\sqrt{2}+\kappa-\sqrt{\kappa^2+4\sqrt{2}\kappa(\kappa+1)}}{2\sqrt{2}} > 0\,.$$

The above inequality holds iff $\kappa^2 < \sqrt{2}$ ($\kappa < 1.1892$) which happens in infinitely many cases for given b and F.

We need yet to control the fulfilment of the condition that $W \cap L\setminus\{0\}$ having only non-periodic elements. Suppose the contrary. Then $M_1^{-1}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = M_2^{-1}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ for some $x, y \in \mathbb{Z}$. However, it is possible only for $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = 0$. The proof is finished. ∎

We note that following the thread of Theorem 3.2 for $A = 2$ in (3.4) we get that $\kappa < 2^{3/4}$ in which case
$$r_M < -1/8\,2^{3/4} + 1 - 1/8\sqrt{32 + 2\sqrt{2} - 16\,2^{3/4}} \sim 0.438\,.$$

Similarly, for the case $A = 4$ the inequality $\kappa < 2^{5/4}$ follows and in this case $r_M < 0.6694$ holds, approx. We draw the reader's attention to the fact that the small norm of the inverse does not mean that the condition number is small. Figure 1 shows how the values $\frac{\gamma}{1-r_m}$ changes in the neighbourhood of the origin for $F = 2$ and $F = 7$.

In the following we examine how many periodic elements may exist for a given simultaneous system.

Consider a circle in $\mathbb{R}^2$ centered at the origin with radius $s \ge 0$. Let us denote the number of integer point inside the circle by $N(s)$.

Theorem 3.3. *Given a simultaneous radix system $(\mathbb{Z}^4, M_1, M_2, D)$ in a ring of integers of an imaginary quadratic field, where the operators are of form (3.1) or (3.2) with digits (2.2). Suppose that $A \in \mathbb{R}$ is calculated by (3.4). Then, for the possible periodic elements $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ the following conditions hold:*
$$\|(\pi_1,\pi_2) - (\pi_3,\pi_4)\| < A\,, \quad \|\pi\| \le A\sqrt{2}(1/r_m - 1), \quad \pi \in T\,,$$
where T is an effectively computable set having $N(A)^2$ elements.

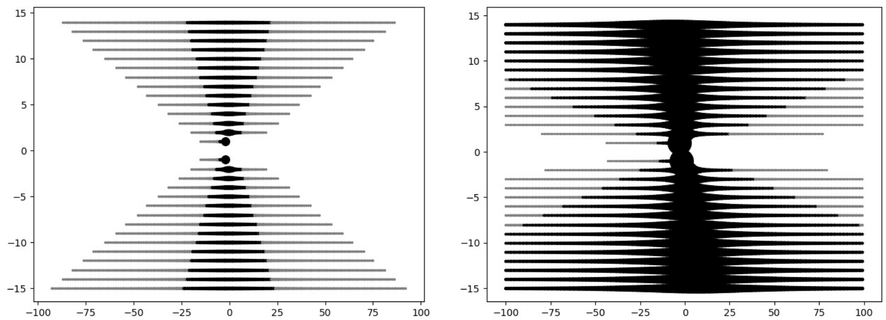


Figure 1: The x axis represents a, the y axis represents the b values. The points represent the values $\frac{\gamma}{1-r_m}$ for $F = 2$ (left) and $F = 7$ (right) in the neighbourhood of the origin. Darker points represent higher values.

Proof. If π is periodic, then clearly

$$\|\pi\| \le L = \frac{Kr_M}{1-r_M} \le \frac{r_M\|M_1\|(\|M_2\|+1)}{1-r_M}\,,$$

hence

$$\frac{Lr_m}{\sqrt{2}(1-r_m)} \le \frac{r_M r_m\|M_1\|(\|M_2\|+1)}{\sqrt{2}(1-r_m)(1-r_M)} < A$$

by which $\|\pi\| \le L < A\sqrt{2}(1/r_m - 1)$. This bound can easily be computed.

Now we construct the set T. If $\pi' = (\pi'_1, \pi'_2, \pi'_3, \pi'_4)$ is periodic then

$$\|(\pi'_1, \pi'_2) - (\pi'_3, \pi'_4)\| < A$$

and

$$\|(\pi'_1 - d_1, \pi'_2 - d_2) - (\pi'_3 - d_1, \pi'_4 - d_2)\| < A$$

for any digit $(d_1, d_2, d_1, d_2) \in D$. For the appropriate congruent digit d let $x_1 = = \pi'_1 - d_1$, $x_2 = \pi'_2 - d_2$, $x_3 = \pi'_3 - d_1$, $x_4 = \pi'_4 - d_2$. Hence, $M_1(\pi_1, \pi_2) = (x_1, x_2)$, $M_2(\pi_3, \pi_4) = (x_3, x_4)$ for some $(\pi_1, \pi_2, \pi_3, \pi_4) \in \mathbb{Z}^4$, where

$$\|(\pi_1, \pi_2) - (\pi_3, \pi_4)\| < A\ .$$

Rewriting the previous equations in the δ-lattice we get that

$$-Fb(\pi_2-\pi_4)+a(\pi_1-\pi_3)-\pi_3 = x_1-x_3,\ a(\pi_2-\pi_4)+b(\pi_1-\pi_3)-\pi_4 = x_2-x_4\,,$$

while in the ω-lattice

$$-Gb(\pi_2-\pi_4)+a(\pi_1-\pi_3)-\pi_3 = x_1-x_3,\ (a+b)(\pi_2-\pi_4)+b(\pi_1-\pi_3)-\pi_4 = x_2-x_4\,.$$

Solving the previous equations for π we get that

$$\pi_1 = \pi_3 + \frac{a(x_1 - x_3 + \pi_3) + Fb(x_2 - x_4 + \pi_4)}{Fb^2 + a^2},$$
$$\pi_2 = \pi_4 + \frac{a(x_2 - x_4 + \pi_4) - b(x_1 - x_3 + \pi_3)}{Fb^2 + a^2}$$

in the δ-lattice, and

$$\pi_1 = \pi_3 + \frac{(a+b)(x_1 - x_3 + \pi_3) + Gb(x_2 - x_4 + \pi_4)}{Gb^2 + a^2 + ab},$$
$$\pi_2 = \pi_4 + \frac{a(x_2 - x_4 + \pi_4) - b(x_1 - x_3 + \pi_3)}{Gb^2 + a^2 + ab}$$

in the ω-lattice. Simplifying the equations above we have

(3.5)
$$\pi_1 = FYb - (a+1)X - \xi_1,\ \pi_2 = -bX - (a+1)Y - \xi_2, \pi_3 = \pi_1 + X,\ \pi_4 = \pi_2 + Y$$

in the δ-lattice, and

(3.6)
$$\pi_1 = GYb - (a+1)X - \xi_1,\ \pi_2 = -bX - (a+b+1)Y - \xi_2, \pi_3 = \pi_1 + X,\ \pi_4 = \pi_2 + Y$$

in the ω-lattice, where $X^2 + Y^2 < A^2$, $\xi_1^2 + \xi_2^2 < A^2$, $(\xi_1 = x_1 - x_3, \xi_2 = x_3 - x_4)$.

Searching for all solutions we have $N(A)^2$ equations for the periodic candidates, exactly what we stated. ■

Let us examine the case $A = 2$.

In case of a δ-lattice the formula (3.5) has the following form:

$$\begin{cases} -1 \le (a+1)X - FbY + \pi_1 \le 1 \\ -1 \le bX + (a+1)Y + \pi_2 \le 1 \\ X, Y \in \{-1, 0, 1\} \end{cases} \tag{3.7}$$

Table 1 and Table 2 contain the possible periodic elements in the δ and ω lattices. A row in Table 1 can be interpreted in the way that $\pi = (\pi_1, \pi_2, \pi_1 + X, \pi_2 + Y)$ where X, Y, π_1 and π_2 are from (3.7). For example, if $X = 1$, $Y = 0$ then $\pi_1 \in \{-2-a, -1-a, -a\}$ and $\pi_2 \in \{-1-b, -b, 1-b\}$, therefore e.g. $(-2-a, -1-b, -1-a, -1-b) \in T$, a periodic candidate. We remark that in case of algorithmic search we have to check the fulfilness of $\|\pi\| \le L$ for each candidate π. The rows in Table 2 can be interpreted similarly.

In case of ω-lattice the solution (3.6) has the following form:

$$\begin{cases} -1 \le (a+1)X - GbY + \pi_1 \le 1 \\ -1 \le bX + (a+b+1)Y + \pi_2 \le 1 \\ X, Y \in \{-1, 0, 1\} \end{cases} \tag{3.8}$$

X	Y	π_1	π_2
0	0	$\{-1,0,1\}$	$\{-1,0,1\}$
1	0	$\{-2-a,-1-a,-a\}$	$\{-1-b,-b,1-b\}$
-1	0	$\{a,a+1,a+2\}$	$\{b-1,b,b+1\}$
0	1	$\{-1+Fb,Fb,1+Fb\}$	$\{-2-a,-1-a,-a\}$
0	-1	$\{-1-Fb,-Fb,1-Fb\}$	$\{a,a+1,a+2\}$
1	1	$\{Fb-a-2,Fb-a-1,Fb-a\}$	$\{-2-a-b,-1-a-b,-a-b\}$
-1	-1	$\{a-Fb,a-Fb+1,a-Fb+2\}$	$\{a+b,a+b+1,a+b+2\}$
1	-1	$\{-2-a-Fb,-1-a-Fb,-a-Fb\}$	$\{a-b,a-b+1,a-b+2\}$
-1	1	$\{a+Fb,a+Fb+1,a+Fb+2\}$	$\{b-a-2,b-a-1,b-a\}$

Table 1: The table contains the solutions of (3.7). From the rows one can build the elements of T.

X	Y	π_1	π_2
0	0	$\{-1,0,1\}$	$\{-1,0,1\}$
1	0	$\{-2-a,-1-a,-a\}$	$\{-1-b,-b,1-b\}$
-1	0	$\{a,a+1,a+2\}$	$\{b-1,b,b+1\}$
0	1	$\{-1+Gb,Gb,1+Gb\}$	$\{-2-a-b,-1-a-b,-a-b\}$
0	-1	$\{-1-Gb,-Gb,1-Gb\}$	$\{a+b,a+b+1,a+b+2\}$
1	1	$\{Gb-a-2,Gb-a-1,Gb-a\}$	$\{-2-a-2b,-1-a-2b,-a-2b\}$
-1	-1	$\{a-Gb,a-Gb+1,a-Gb+2\}$	$\{a+2b,a+2b+1,a+2b+2\}$
1	-1	$\{-2-a-Gb,-1-a-Gb,-a-Gb\}$	$\{a,a+1,a+2\}$
-1	1	$\{a+Gb,a+Gb+1,a+Gb+2\}$	$\{-a-2,-a-1,-a\}$

Table 2: The table contains the solutions of (3.8). From the rows one can build the elements of T.

Example 3.4. Let $F=2$ and $b=2020$ be fixed. Table 3 describes that for $a<-11297$ or $a>11296$ the inequality $A<1$ holds. If $-11297\le a<-5053$ and $5052<a\le 11296$ then $1\le A<\sqrt{2}$, etc. We get the highest value for A when $a=0$.

a	-11297	-5053	-2575	0	2574	5052	11296
A	1	$\sqrt{2}$	2	2.84	2	$\sqrt{2}$	1
L	15110.95	9761.59	8387.18	8086.01	8387.18	9761.59	15110.95
$N(A)^2$	1	25	81	625	81	25	1

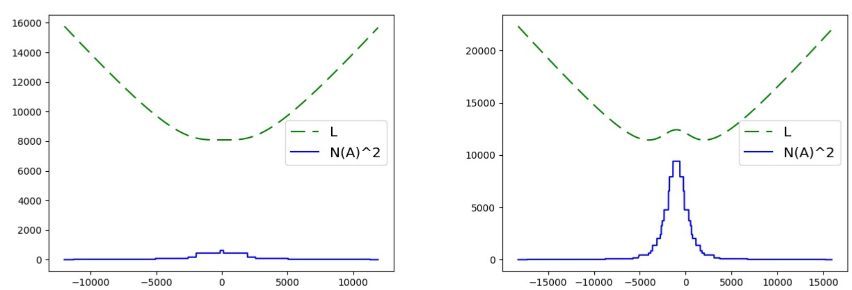
Table 3: The table shows the simultaneous case of $F=2, b=2020$ for some boundary values of a, the corresponding radius L and the number of points $N(A)^2$ to be checked. L is calculated by the formula from Theorem 3.3.

Example 3.5. Let now $F=7$ and $b=2020$ be fixed. We did similar computations as before. From Table 4 we can see that for $a<-17267$ or $a>15246$ the inequality $A<1$ holds. The highest value for A happens when $a=-1011$.

a	-1011	-141	0	748	1791	3725	6769	15246
A	5.49	5	4.85	4	3	2	$\sqrt{2}$	1
L	12420.87	12181.89	12111.53	11730	11443.89	11858.72	13817.4	21365
$N(A)^2$	9409	4761	4761	2025	625	81	25	1
a	-1011	-1880	-2769	-3812	-5746	-8771	-17267	
A	5.49	5	4	3	2	$\sqrt{2}$	1	
L	12420.87	12181.89	11730	11443.89	11858.72	13817.4	21365	
$N(A)^2$	9409	4761	2025	625	81	25	1	

Table 4: The case of $F = 7$ and $b = 2020$ for some boundary values of a, the corresponding radius L and the number of points $N(A)^2$ to be checked. L is calculated by the formula from Theorem 3.3.

Figure 2 shows how L and $N(A)^2$ are changes in the function of a in the cases $F = 2, b = 2020$ and $F = 7, b = 2020$. We note that for a given L the size of the testing set (integer points in the 4-dimensional ball with radius L) can be enormous large. Our method reduces the points to be examined significantly (see Tables 3 and 4).



(a) $F = 2$ and $b = 2020$ (b) $F = 7$ and $b = 2020$

Figure 2: The x axis represents a, the y axis represents the L (dashed line) and the $N(A)^2$ values.

4. Further works

Continuing this research we plan to develop an algorithm by which the simultaneous number system concept can be examined in the rings of integers of imaginary quadratic fields in some finite regions.

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