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EXPONENTIAL SUMS RUNNING OVER PARTICULAR SETS OF POSITIVE INTEGERS

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Dedicated to the 70th birthday of Professor Antal Járai

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Abstract. In the spirit of a famous theorem of Hédi Daboussi, we examine the corresponding exponential sums where the sums run over particular subsets of the set of positive integers.

1. Introduction

According to an old result of H. Weyl [5], a sequence of real numbers $(\xi_n)_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 if and only if $\lim\limits_{x\to\infty}\dfrac{1}{x}\sum\limits_{n\le x} e(k\xi_n)=0$ for every non zero integer k. Here and thereafter, $e(y)$ stands for $\exp\{2\pi iy\}$. Weyl's criterion represents in itself an important motivation for studying *exponential sums*. These sums can be of various forms. Before we present these, let us provide some key notation. Let $T=\{z\in\mathbb{C}:|z|=1\}$ and $U=\{z\in\mathbb{C}: :|z|\le 1\}$. Let $\mathcal{M}$ stand for the set of multiplicative functions and $\mathcal{M}^*$ for the set of completely multiplicative functions. Finally, let $\mathcal{M}_U$ be the set of those $f\in\mathcal{M}$ such that $f(n)\in U$. Also, writing $\{x\}$ to denote the fractional part of

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x and given a set of N real numbers $x_1, \ldots, x_N$, we define its *discrepancy* as the quantity

$$D(x_1, \ldots, x_N) := \sup_{[a,b)\subseteq[0,1)} \left| \frac{1}{N} \sum_{\substack{n\le N \\ \{x_n\}\in[a,b)}} 1 - (b-a) \right|.$$

In 1974, Hédi Daboussi (see Daboussi and Delange [1]) proved that, for every irrational number α,

$$\sup_{f\in\mathcal{M}_U} \frac{1}{x} \left| \sum_{n\le x} f(n)e(n\alpha) \right| \to 0 \quad \text{as } x\to\infty.$$

Letting $\mathcal{A}$ stand for the set of all additive functions, Daboussi and Delange's theorem clearly implies the following result.

> Let $u \in \mathcal{A}$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and consider the corresponding sequence $(\theta_n)_{n\in\mathbb{N}}$ defined by $\theta_n = u(n) + n\alpha$, $n = 1, 2, \ldots$. Then, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is uniformly distributed modulo 1. Moreover, the discrepancy of all such sequences $(\theta_n)_{n\in\mathbb{N}}$ satisfies
>
> $$\sup_{u\in\mathcal{A}} D_N(\theta_1, \ldots, \theta_N) \to 0 \quad \text{as } N\to\infty.$$

In 1986, the second author [3] generalised the Daboussi theorem by proving the following.

Theorem A. *Let $\wp^*$ stand for a set of primes for which $\sum_{p\in\wp^*} 1/p = \infty$. Let $\mathcal{F}$ be the set of those arithmetic functions $f : \mathbb{N}\to\mathbb{C}$ for which $|f(n)| \le 1$ for all positive integers n and satisfying the condition*

$n = pm$, $(m, p) = 1$, $p \in \wp^*$ *implies that* $f(n) = f(p)f(m)$.

Further let $(a(n))_{n\in\mathbb{N}}$ be a sequence of complex numbers such that $|a(n)| \le 1$ for all $n \in \mathbb{N}$ and such that, for every $p_1, p_2 \in \wp^*$, $p_1 \ne p_2$,

$$\frac{1}{x} \sum_{n\le x} a(p_1 n)\overline{a(p_2 n)} \to 0 \quad \text{as } x\to\infty. \tag{1.1}$$

Then,

$$\sup_{f\in\mathcal{F}} \frac{1}{x} \left| \sum_{n\le x} f(n)a(n) \right| \to 0 \quad \text{as } x\to\infty. \tag{1.2}$$

Remark 1.1. Let α be an arbitrary irrational number. Since the function $a(n) := e(n\alpha)$ satisfies condition (1.1), the Daboussi theorem also applies to this function $a(n)$.

In this paper, we obtain similar results when the sums appearing in (1.2) run over particular subsets of $\mathbb{N}$.

2. Examples

Consider the following three examples.

Example 1. Let $0 < \ell < k$ be two co-prime integers. Let φ stand for Euler's totient function and consider the function

$$h(n) := \frac{1}{\varphi(k)} \sum_{\chi} \overline{\chi(\ell)} \chi(n) = \begin{cases} 1 & \text{if } n \equiv \ell \pmod{k}, \\ 0 & \text{otherwise,} \end{cases}$$

where the above sum runs over all characters modulo k. Given any arithmetic function $a(n)$, we then have that

$$\sum_{\substack{n \le x \\ n \equiv \ell \pmod{k}}} h(n)a(n) = \frac{1}{\varphi(k)} \sum_{\chi} \overline{\chi(\ell)} \sum_{n \le x} h(n)\chi(n)a(n).$$

Observe that $h\chi \in \mathcal{M}_U$ and that, in light of Theorem A and setting

$$S := \{n \in \mathbb{N} : n \equiv \ell \pmod{k}\} \quad \text{and} \quad S(x) := \#\{n \le x : n \in S\},$$

we have that

$$\sum_{f \in \mathcal{M}_U} \frac{1}{S(x)} \left| \sum_{\substack{n \le x \\ n \in S}} f(n)a(n) \right| \to 0 \quad \text{as } x \to \infty.$$

Example 2. For each $i = 1, 2, \ldots, r$, let $g_i : \mathbb{N} \to \mathbb{N}$ be a multiplicative function. Moreover, for $i = 1, \ldots, r$, let $0 < \ell_i < k_i$ be co-prime integers such that $g_i(n) \equiv \ell_i \pmod{k_i}$. Further set

$$S := \{n \in \mathbb{N} : g_i(n) \equiv \ell_i \pmod{k_i} \text{ for } i = 1, \ldots, r\}.$$

Then, assuming that there exists a number $c > 0$ such that $\lim\limits_{x \to \infty} \dfrac{S(x)}{x} \ge c$ and letting $a(n)$ be as in Theorem A, we have

$$\sup_{f \in \mathcal{M}_U} \frac{1}{S(x)} \left| \sum_{\substack{n \le x \\ n \in S}} f(n)a(n) \right| \to 0 \quad \text{as } x \to \infty.$$

Example 3. Let $J_1, \ldots, J_r \subseteq [0, 1)$ be finite unions of intervals. Let $P_1(x), \ldots, \ldots P_r(x) \in \mathbb{R}[x]$, each of positive degree, and let $Q_{m_1,\ldots,m_r}(x) := m_1 P_1(x) + + \cdots + m_r P_r(x)$, where $m_1, \ldots, m_r \in \mathbb{Z}$. Assume that $Q_{m_1,\ldots,m_r}(x) - Q_{m_1,\ldots,m_r}(0)$

has at least one irrational coefficient for each r-tuple $(m_1, \dots, m_r) \neq (0, \dots, 0)$. Further set $S := \{n \in \mathbb{N} : \{P_\ell(n)\} \in J_\ell \text{ for } \ell = 1, \dots, r\}$ and let λ stand for the Lebesgue measure. Kátai [4] proved that under these conditions,

$$\sup_{g \in \mathcal{M}_U} \left| \frac{1}{x} \sum_{\substack{n \leq x \\ n \in S}} g(n) - \frac{\lambda(J_1) \cdots \lambda(J_r)}{x} \sum_{n \leq x} g(n) \right| \to 0 \quad \text{as } x \to \infty.$$

3. A generalisation of Theorem A

Theorem 1. *Let $\wp^*$ and $a(n)$ be as in Theorem A. Let $u_1, \dots, u_r$ be additive functions, each with a continuous limit distribution. For each $\ell = 1, \dots, r$, let $\mathcal{J}_\ell := [\xi_1^{(\ell)}, \xi_2^{(\ell)})$ be intervals such that the set*

$$S := \{n \in \mathbb{N} : u_\ell(n) \in \mathcal{J}_\ell \text{ for } \ell = 1, \dots, r\}$$

has infinitely many elements and is also such that $S(x) := \#\{n \leq x : n \in S\}$ satisfies $\liminf_{x \to \infty} \frac{S(x)}{x} \geq d > 0$. Then,

$$\sup_{f \in \mathcal{M}} \frac{1}{S(x)} \left| \sum_{\substack{n \leq x \\ n \in S}} f(n) a(n) \right| \to 0 \qquad \text{as } x \to \infty.$$

Remark 3.1. Observe that according to the Erdős–Wintner theorem [2], an additive function f has a limit distribution if and only if each of the three series

$$\sum_{\substack{p \\ |f(p)|>1}} \frac{1}{p}, \qquad \sum_{\substack{p \\ |f(p)|\leq 1}} \frac{f(p)}{p}, \qquad \sum_{\substack{p \\ |f(p)|\leq 1}} \frac{f^2(p)}{p}$$

converge. It is also known that the limit distribution is continuous if and only if $\sum_{f(p) \neq 0} \frac{1}{p} = \infty$.

Proof. Given an arbitrary interval $I = [\eta_1, \eta_2)$, define the corresponding function

$$e_I(x) := \begin{cases} 1 & \text{if} \quad x \in I, \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus I. \end{cases}$$

Moreover, let M be a positive integer such that $\eta_1 + M > \eta_2$ and further let

$$L_M := \bigcup_{h=-\infty}^{\infty} (I + hM) \quad \text{so that} \quad e_{L_M}(x) = \begin{cases} 1 & \text{if} \quad x \in L_M, \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus L_M. \end{cases}$$

Since the function $e_{L_M}(x)$ is periodic modulo M, we have that

$$e_{L_M}(x) = \sum_{h=-\infty}^{\infty} c_h\, e\left(\frac{hx}{M}\right), \qquad \text{where } c_0 = \eta_2 - \eta_1.$$

Further let

$$h_{L_M}(x) := \frac{1}{\delta^2}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} e_{L_M}(x+y_1+y_2)\,dy_1\,dy_2 = \sum_{h=-\infty}^{\infty} c_h(\delta)\, e\left(\frac{hx}{M}\right),$$

where $c_0(\delta) = \eta_2 - \eta_1$ and $|c_h(\delta)| \le C\left(\frac{M}{\delta}\right)^2 \frac{1}{h^2}$ for each $h \neq 0$, for some positive constant C.

It is clear that $h_{L_M}(x) = e_{L_M}(x)$ if $x \in [\eta_1 + 2\delta, \eta_2 - 2\delta)$ and if $x \in$ $\in [-M+\eta_2, M-\eta_1] \setminus [\eta_1 - 2\delta, \eta_2 + 2\delta]$. Moreover, we have that

$$0 \le e_{L_M}(x) - h_{L_M}(x) \le 1 \qquad \text{for all } x \in \mathbb{R}.$$

Let $\varepsilon > 0$ and let M be an integer sufficiently large so that there exists a number $x_0 > 0$ for which

$$\#\{n \le x : \max_{\ell=1,\dots,r} |u_\ell(n)| \ge M/2\} < \varepsilon\, x \qquad \text{for all } x \ge x_0$$

and also

$$\max_{\ell=1,\dots,r}\left(|\xi_1^{(\ell)}| + |\xi_2^{(\ell)}|\right) \le \frac{M}{2}.$$

Observe that such an integer M must exist because each function $u_\ell(n)$ is assumed to have a limit distribution.

Now, let R be sufficiently large so that

$$\sum_{\ell=1}^{r}\sum_{|m|>R} |c_m^{(\ell)}(\delta)| < \varepsilon.$$

Then, further define

$$h_\ell^{(R)}(x) := \sum_{m=-R}^{R} c_m^{(\ell)}(\delta) e\left(\frac{mx}{\delta}\right) \qquad (\ell = 1, \dots, r)$$

and set

$$E_S(n) := \begin{cases} 1 & \text{if} \quad n \in S, \\ 0 & \text{if} \quad n \in \mathbb{N} \setminus S. \end{cases}$$

Observe that

$$\sum_{n\le x}\left|E_S(n)-h_1^{(R)}(u_1(n))\cdots h_r^{(R)}(u_r(n))\right|\le C_1\varepsilon\, x,$$

where C_1 is a positive constant depending on M and δ, only. Thus,

$$\left|\sum_{\substack{n\le x\\ n\in S}} f(n)a(n)-\sum_{|m_1|\le R,\ldots,|m_r|\le R} c_{m_1}^{(1)}(\delta)\cdots c_{m_r}^{(r)}(\delta)\,T(m_1,\ldots,m_r)\right|\le C_1\varepsilon\, x, \tag{3.1}$$

where

$$T(m_1,\ldots,m_r)=\sum_{n\le x} f(n)\,e\left(\frac{m_1u_1(n)}{M}\right)\cdots e\left(\frac{m_ru_r(n)}{M}\right)a(n). \tag{3.2}$$

Since

$$g(n):=e\left(\frac{m_1u_1(n)}{M}\right)\cdots e\left(\frac{m_ru_r(n)}{M}\right)\in\mathcal{M}_U,$$

we have that $f(n)g(n)\in\mathcal{M}_U$, implying that the expression in (3.2) is $o(x)$ as $x\to\infty$. It follows from (3.1) that

$$\limsup_{x\to\infty}\sup_{f\in\mathcal{M}_U}\frac{1}{S(x)}\left|\sum_{\substack{n\le x\\ n\in S}} f(n)a(n)\right|\le C_1\varepsilon.$$

Since this inequality holds for any $\varepsilon>0$, the proof of Theorem 1 is complete. ∎

The following two results can also be proved along the same lines.

Theorem 2. *Let $\wp^*$, $u_1,\ldots,u_r$ and S be as in Theorem 1. Let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence of real numbers for which the corresponding sequence $(\theta_n)_{n\in\mathbb{N}}$ defined by $\theta_n:=\kappa_{p_1n}-\kappa_{p_2n}$ is uniformly distributed modulo 1 for every $p_1\ne p_2$, $p_1,p_2\in\wp^*$. For each additive function v and positive integer N, consider the expression*

$$D_{N,S}(v):=\sup_{[a,b)\subseteq[0,1)}\frac{1}{S(N)}\left|\#\{n\le N: n\in S,\ v(n)+\kappa_n\in[a,b)\}-(b-a)\right|.$$

Then,

$$\sup_{v\in\mathcal{A}} D_{N,S}(v)\to 0 \qquad \textit{as } N\to\infty.$$

Theorem 3. *Let* $Q(x) \in \mathbb{R}[x]$ *be such that* $Q(0) = 0$ *and* $Q(x) \notin \mathbb{Z}[x]$. *Set* $a(n) := e(Q(n))$. *Then,*

$$\frac{1}{x} \sum_{n \le x} a(p_1 n) \overline{a(p_2 n)} \to 0 \quad \text{as } x \to \infty$$

for every $p_1 \ne p_2$, $p_1, p_2 \in \wp^*$. *Moreover, assuming that* $m_1 P_1(x) + \dots + m_r P_r(x) + Q(x) \notin \mathbb{Z}[x]$ *for every* r*-tuple* $(m_1, \dots, m_r) \in \mathbb{Z}^r$. *Then,*

$$\sup_{f \in \mathcal{M}_U} \frac{1}{S(x)} \left| \sum_{\substack{n \le x \\ n \in S}} f(n) a(n) \right| \to 0 \quad \text{as } x \to \infty.$$

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