Annales Univ. Sci. Budapest., Sect. Comp. 51 (2020) 39–50

ON SOME PENTADIAGONAL MATRICES: THEIR DETERMINANTS AND INVERSES

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Dedicated to the 70th birthday of Professor Antal Járai

Communicated by László Szili

(Received February 7, 2020; accepted June 2, 2020)

Abstract. In this paper we consider pentadiagonal $(n+1)\times(n+1)$ matrices with two super-diagonals and two sub-diagonals at distances k and $2k$ from the main diagonal. We give an explicit formula for their determinants provided that $(n+1)/3 \leq k \leq n/2$. We consider the Toeplitz and "imperfect" Toeplitz versions of such matrices, and give explicit formulas for their determinants. We also show that the inverse matrix can be obtained as the product of an upper triangle matrix with two super-diagonals, a diagonal and a lower triangle matrix with two sub-diagonals.

1. Introduction

Let n, k, ℓ be given positive integers with $1 \leq k < \ell \leq n$ and denote by $\mathcal{M}_n$ the set of $n \times n$ complex matrices. Consider the pentadiagonal matrix $A_0 = (a_{ij}) \in \mathcal{M}_{n+1}$ where for $i, j = 0, 1, \ldots, n$,

$$a_{ij} = \begin{cases} L_j & \text{if } j-i=-\ell, \\ l_j & \text{if } j-i=-k, \\ d_i & \text{if } j-i=0, \\ r_i & \text{if } j-i=k, \\ R_i & \text{if } j-i=\ell, \\ 0 & \text{otherwise.} \end{cases}$$

Key words and phrases: Toeplitz matrix, pentadiagonal, tridiagonal matrices.
2010 Mathematics Subject Classification: Primary 15A15, 15B9, Secondary 65F40, 15A42.

Notice that the numbering of entries starts with zero. We also denote the matrix A_0 by $A_{n+1,k,\ell}(\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R})$ and denote by $D_0 = D_{n+1,k,\ell}(\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R})$ its determinant where the diagonal vectors $\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R}$ are defined by

$$\begin{array}{rclrcl} \mathbf{L} &=& (L_0,\dots,L_{n-\ell}), & \mathbf{l} &=& (l_0,\dots,l_{n-k}),\\ \mathbf{d} &=& (d_0,\dots,d_n), && & \\ \mathbf{R} &=& (R_0,\dots,R_{n-\ell}), & \mathbf{r} &=& (r_0,\dots,r_{n-k}). \end{array}$$

We will refer to the diagonal vectors $\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R}$ as ℓth sub-diagonal, kth sub-diagonal, (main) diagonal, kth super-diagonal, ℓth super-diagonal, respectively.

If $\mathbf{L},\mathbf{R}$ are zero vectors then our matrix becomes a tridiagonal one denoted by $A_{n+1,k}(\mathbf{l},\mathbf{d},\mathbf{r})$ and its determinant by $D_{n+1,k}(\mathbf{l},\mathbf{d},\mathbf{r})$.

If $1 \le k = \ell \le n$, then the two super-diagonals and the two sub-diagonals "slip together" and our matrix becomes a tridiagonal matrix

$$A_{n+1,k,k}(\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R}) = A_{n+1,k}(\mathbf{L}+\mathbf{l},\mathbf{d},\mathbf{r}+\mathbf{R})$$

and its determinant $D_0 = D_{n+1,k}(\mathbf{L}+\mathbf{l},\mathbf{d},\mathbf{r}+\mathbf{R})$.

We shall call $A_{n+1,k,\ell}(\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R})$ a (general) k,ℓ-pentadiagonal matrix while $A_{n+1,k}(\mathbf{l},\mathbf{d},\mathbf{r})$ will be termed as k-tridiagonal.

In [6] we developed a method to reduce the determinant of k,ℓ-pentadiagonal matrices to tridiagonal determinants provided that $k+\ell \ge n+1$. If $k \ge$ $\ge (n+1)/3$, then by suitable matrix multiplications we can reduce a $k,2k$-pentadiagonal matrix to a tridiagonal one. A similar reduction of tridagonals leads to diagonal matrices. In this way we get an explicit formula for the determinant of a general $k,2k$-pentadiagonal matrix.

In case of Toeplitz pentadiagonal matrices the diagonal vectors are constant vectors, i.e., $\mathbf{L} = (L,\dots,L)$, $\mathbf{R} = (R,\dots,R) \in \mathbb{R}^{n+1-\ell}$, $\mathbf{l} = (l,\dots,l)$, $\mathbf{r} = (r,\dots,r) \in \mathbb{R}^{n+1-k}$, $\mathbf{d} = (d,\dots,d) \in \mathbb{R}^{n+1}$ and for the matrix and its determinant the notations $A_{n+1,k,\ell}(L,l,d,r,R)$ and $D_{n+1,k,\ell}(L,l,d,r,R)$ will be used.

For "imperfect" Toeplitz matrices (the term used by Marr and Vineyard [9]) the main diagonal is

$$(\underbrace{d-\alpha,\dots,d-\alpha}_{k},\underbrace{d,\dots,d}_{n+1-2k},\underbrace{d-\beta,\dots,d-\beta}_{k}) \in \mathbb{R}^{n+1}$$

while the other diagonals are the same as above. To indicate the "imperfectness" we add a superscript $^{(\alpha,\beta)}$ to the notation of Toeplitz pentadiagonal matrices. Thus an "imperfect" $k,2k$-pentadiagonal matrix $A^{(\alpha,\beta)}_{n+1,k,2k}(L,l,d,r,R)$ has the form

$$\left(\begin{array}{ccc|ccc|ccc}
d-\alpha & & & r & & & & R & \\
& d-\alpha & & & r & & & & \ddots \\
& & \ddots & & & \ddots & & & R \\
& & d-\alpha & & & & & & \\ \hline
l & & & d & & & \ddots & & \\
& l & & & d & & & & \\
& & \ddots & & & \ddots & & \ddots & \\
& & & & & d & & & r \\ \hline
& & & \ddots & & & d-\beta & & \\
L & & & & & & & d-\beta & \\
& \ddots & & & \ddots & & & \ddots & \\
& & L & & & l & & & d-\beta
\end{array}\right)$$

As far as we can tell, the first particular k,ℓ-pentadiagonal Toeplitz matrix was studied in 1928 by Jenő Egerváry and Otto Szász [3]. One of the matrices they considered was $A_{n+1,k,\ell}(\bar{\delta},\bar{\epsilon},-\lambda,\epsilon,\delta)$ with $k+\ell=n+1$ and $|\epsilon|=|\delta|=1$. Their result has been largely ignored in the pure and applied matrix theory as we can see e.g. in [2, 10]. For some other papers on the topic, the reader is refereed to [1, 6, 11, 12]. An interesting graph theoretical approach can be found in [7, 5]. An exhaustive list of recent references is given in the survey [4].

Our paper was inspired by Marr and Vineyard [9] who have shown that the product of two 1-tridiagonal Toeplitz matrices is an imperfect Toeplitz matrix $A^{(\alpha,\beta)}_{n+1,1,2}$ which is related to the corresponding Toeplitz matrix by a two-step recursion. In this way they succeeded to express the determinant $D_{n+1,1,2}$ of $1,2$-pentadiagonal Toeplitz matrix in terms of Chebyshev polynomials of second kind. A similar approach was used in [13] to find a formula for the inverse of a $1,2$-pentadiagonal Toeplitz matrix.

2. Reduction of general $k,2k$-pentadiagonal determinants to tridiagonal ones

Our starting point is the matrix $A_0=A_{n+1,k,2k}(\mathbf{L},\mathbf{l},\mathbf{d},\mathbf{r},\mathbf{R})$ where $1\le k<$ $<2k\le n$ and $k+2k\le n+1$, i.e., $(n+1)/3\le k\le n/2<(n+1)/2$. Let us define the matrices $B=(b_{ij}),C=(c_{ij})$,$F=(f_{ij}),G=(g_{ij})$ of $\mathcal{M}_{n+1}$ by

$$b_{ij}=\begin{cases} 1 & \text{if } i=j,\\ -\dfrac{l_j}{d_j} & \text{if } i=j+k, j=0,\dots,k-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$c_{ij}=\begin{cases} 1 & \text{if } i=j,\\ -\dfrac{r_i}{d_i} & \text{if } i=0,\dots,k-1, j=i+k,\\ 0 & \text{otherwise,}\end{cases}$$

$$f_{ij}=\begin{cases} 1 & \text{if } i=j,\\ -\dfrac{L_j}{d_j} & \text{if } i=j+2k, j=0,\dots,n-2k,\\ 0 & \text{otherwise,}\end{cases}$$

$$g_{ij}=\begin{cases} 1 & \text{if } i=j,\\ -\dfrac{R_i}{d_i} & \text{if } i=0,\dots,n-2k, j=i+2k,\\ 0 & \text{otherwise.}\end{cases}$$

Performing four matrix multiplications (for some details see [6], Theorem 3, (i)) we get the matrix $A_1 = FBA_0CG = E_1 \bigoplus A_1^*$ where $E_1 \in \mathcal{M}_k$ is a diagonal matrix with diagonal elements $d_0,\dots,d_{k-1}$ and

$$A_1^*=A_{n+1-k,k,k}(\mathbf{L}^{(1)},\mathbf{l}^{(1)},\mathbf{d}^{(1)},\mathbf{r}^{(1)},R^{(1)})=A_{n+1-k,k}(\mathbf{L}^{(1)}+\mathbf{l}^{(1)},\mathbf{d}^{(1)},\mathbf{r}^{(1)}+\mathbf{R}^{(1)})$$

is tridiagonal, where the iterated diagonal vectors are

$$\mathbf{L}^{(1)} = \left(-\frac{L_0 r_0}{d_0},\dots,-\frac{L_{n-2k}r_{n-2k}}{d_{n-2k}}\right),$$

$$\mathbf{l}^{(1)} = (l_k,\dots,l_{n-k}),$$

$$\mathbf{d}^{(1)} = \left(d_k-\frac{l_0 r_0}{d_0},\dots,d_{2k-1}-\frac{l_{k-1}r_{k-1}}{d_{k-1}},d_{2k}-\frac{L_0 R_0}{d_0},\dots,d_n-\frac{L_{n-2k}R_{n-2k}}{d_{n-2k}}\right),$$

$$\mathbf{r}^{(1)} = (r_k,\dots,r_{n-k}),$$

$$\mathbf{R}^{(1)} = \left(-\frac{R_0 l_0}{d_0},\dots,-\frac{R_{n-2k}l_{n-2k}}{d_{n-2k}}\right).$$

3. Further reduction to diagonal matrix

The matrix A_1 can be reduced to a diagonal matrix with two matrix multiplications. Define the matrices $U = (u_{ij}), V = (v_{ij}) \in \mathcal{M}_{n+1}$ by

$$u_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{l_j - L_{j-k} r_{j-k}/d_{j-k}}{d_j - l_{j-k} r_{j-k}/d_{j-k}} & \text{if } i = j+k,\ j = k, \ldots, n-k, \\ 0 & \text{otherwise,} \end{cases}$$

$$v_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{r_i - R_{i-k} l_{i-k}/d_{i-k}}{d_j - l_{i-k} r_{i-k}/d_{i-k}} & \text{if } i = k, \ldots, n-k,\ j = i+k, \\ 0 & \text{otherwise.} \end{cases}$$

Multiplying A_1 from the left by U, then from the right by V we get a diagonal matrix $A_2 = UA_1V$ with diagonal

(3.1)

$$d_0, \ldots, d_{k-1},$$

$$d_k - \frac{l_0 r_0}{d_0}, \ldots, d_{2k-1} - \frac{l_{k-1} r_{k-1}}{d_{k-1}},$$

$$d_{2k} - \frac{L_0 R_0}{d_0} - \frac{\left(l_k - \frac{L_0 r_0}{d_0}\right)\left(r_k - \frac{R_0 l_0}{d_0}\right)}{d_k - \frac{l_0 r_0}{d_0}},$$

$$\ldots, d_n - \frac{L_{n-2k} R_{n-2k}}{d_{n-2k}} - \frac{\left(l_{n-k} - \frac{L_{n-2k} r_{n-2k}}{d_{n-2k}}\right)\left(r_{n-k} - \frac{R_{n-2k} l_{n-2k}}{d_{n-2k}}\right)}{d_{n-k} - \frac{l_{n-2k} r_{n-2k}}{d_{n-2k}}},$$

where in the above lines $k, k, n+1-2k$ elements are listed. Because the determinants of the matrices B, C, F, G, U, V all are $1'$s, the determinant of our general pentadiagonal matrix $A_{n+1,k,2k}$ is the product of the diagonal elements of A_2. To calculate this product we multiply the elements of the first two groups pairwise, to obtain

$$d_0 d_k - l_0 r_0, \ldots, d_{k-1} d_{2k-1} - l_{k-1} r_{k-1}.$$

Simplifying the general term of the third group we get, for $j = 0, \ldots, n-2k$,

$$d_{2k+j} - \frac{L_j R_j}{d_j} - \frac{\left(l_{k+j} - \frac{L_j r_j}{d_j}\right)\left(r_{k+j} - \dfrac{R_j l_j}{d_j}\right)}{d_{k+j} - \dfrac{l_j r_j}{d_j}} =$$

$$= \frac{d_j d_{j+k} d_{j+2k} - d_j l_{j+k} r_{j+k} - L_j R_j d_{j+k} - l_j r_j d_{j+2k} + l_j R_j l_{j+k} + r_j L_j r_{j+k}}{d_j d_{j+k} - l_j r_j}.$$

Thus we have proved

Theorem 3.1. *Assuming* $(n+1)/3 \leq k \leq n/2$ *the determinant*

$$D_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})$$

of the general $k, 2k$*-pentadiagonal matrix is*

(3.2)

$$\prod_{j=0}^{n-2k} (d_j d_{j+k} d_{j+2k} - d_j l_{j+k} r_{j+k} - L_j R_j d_{j+k} - l_j r_j d_{j+2k} + l_j R_j l_{j+k} + r_j L_j r_{j+k}) \cdot \\ \cdot \prod_{j=n+1-2k}^{k-1} (d_j d_{j+k} - l_j r_j)$$

Remark 3.1. During the calculations above we have to assume that numbers appearing in any denominators are not zero. Unfortunately it seems quite complicated to describe these assumptions in terms of the entries of the matrix. However, formula (3.2) is valid without these assumptions. Namely if any number say d_j appearing in any denominator is zero, then replace it by $d'_j \neq 0$ and continue the calculation, to arrive to the formula corresponding to (3.2). Taking the limit $d'_j \to 0 = d_j$ and then continuity arguments, we arrive to (3.2), in which $d_j = 0$ has to be substituted.

4. $k, 2k$-pentadiagonal Toeplitz and "imperfect" Toeplitz determinants

In the case of $k, 2k$-pentadiagonal Toeplitz matrices the diagonal vectors are constant vectors. Hence the determinant of such matrices can be obtained by substituting $L_j = L, l_j = l, d_j = d, r_j = r, R_j = R$, (for all possible values of the indices j) into (3.2) which gives

Theorem 4.1. *Assuming* $(n+1)/3 \leq k \leq n/2$ *the determinant of the* $k, 2k$*-pentadiagonal Toeplitz matrix* $A_{n+1,k,2k}(L, l, d, r, R)$ *is*

$$\left(d^3 - (LR + 2lr)d + Rl^2 + Lr^2\right)^{n+1-2k} \left(d^2 - lr\right)^{3k-(n+1)}. \tag{4.1}$$

The case of "imperfect" Toeplitz determinants is a little bit more complicated as here $d_0 = \cdots = d_{k-1} = d - \alpha$, $d_k = \cdots = d_{n-k} = d$, $d_{n+1-k} = \cdots = d_n = d - \beta$, $L_j = L$, $R_j = R$, $(j = 0, \ldots, n - 2k)$, $l_j = l$, $r_j = r$, $(j = 0, \ldots, n - k)$. Thus in the first product of (3.2) we have to substitute $d_j = d - \alpha, d_{j+k} = d, d_{j+2k} = d - \beta$. In the second product of (3.2) we have to substitute $d_j = d - \alpha, d_{j+k} = d - \beta$. With this we have proved

Theorem 4.2. *Assuming $(n+1)/3 \le k \le n/2$ the determinant of the "imperfect" Toeplitz $k, 2k$-pentadiagonal matrix $A^{(\alpha,\beta)}_{n+1,k,2k}(L, l, d, r, R)$ is*

$$\begin{aligned}(4.2)\qquad &\left(d^3-(\alpha+\beta)d^2-(LR+2lr-\alpha\beta)d+(\alpha+\beta)lr+Rl^2+Lr^2\right)^{n+1-2k}\cdot\\ &\qquad\cdot\left(d^2-(\alpha+\beta)d+\alpha\beta-lr\right)^{3k-(n+1)}.\end{aligned}$$

Denote by $\lambda_1, \lambda_2, \lambda_3$ and λ_4, λ_5 the zeros d of the first (cubic) and second (quadratic) factor of (4.2) respectively. Then the eigenvalues of the matrix $A^{(\alpha,\beta)}_{n+1,k,2k}(L, l, 0, r, R)$ are $-\lambda_1, -\lambda_2, -\lambda_3$ with multiplicities $n + 1 - 2k$ and $-\lambda_4, -\lambda_5$ with multiplicities $3k - (n + 1)$.

5. Another form of $k, 2k$-pentadiagonal Toeplitz and "imperfect" Toeplitz determinants

We can express the determinants (4.1)–(4.2) in terms of Chebyshev polynomials. It is enough to deal with (4.2) as (4.1) can be obtained from it by $\alpha = \beta = 0$.

For "imperfect" Toeplitz matrices the lower and upper vectors of the tridiagonal matrix A^*_1 are

$$\begin{aligned}\mathbf{L^{(1)}} + \mathbf{l^{(1)}} &= \left(l-\frac{Lr}{d-\alpha}, \ldots, l-\frac{Lr}{d-\alpha}\right) \in \mathbb{R}^{n+1-2k},\\ \mathbf{R^{(1)}} + \mathbf{r^{(1)}} &= \left(r-\frac{Rl}{d-\alpha}, \ldots, r-\frac{Rl}{d-\alpha}\right) \in \mathbb{R}^{n+1-2k}.\end{aligned}$$

The main diagonal $\mathbf{d^{(1)}} \in \mathbb{R}^{\mathbf{n+1-k}}$ is

$$\Big(\underbrace{d-\frac{lr}{d-\alpha},\ldots,d-\frac{lr}{d-\alpha}}_{n+1-2k},\underbrace{d-\beta-\frac{lr}{d-\alpha},\ldots,d-\beta-\frac{lr}{d-\alpha}}_{3k-(n+1)},\underbrace{d-\beta-\frac{LR}{d-\alpha},\ldots,d-\beta-\frac{LR}{d-\alpha}}_{n+1-2k}\Big).$$

If $k = (n+1)/3$ then the middle part is empty, if $k > (n+1)/3$ the the middle part contains at least one element.

The matrix A_1^* (with $n+1=11, k=4$ is a 7×7 matrix) has the form

$$\left(\begin{array}{ccc|c|ccc}
d-\frac{lr}{d-\alpha} & & & & r-\frac{Rl}{d-\alpha} & & \\
& d-\frac{lr}{d-\alpha} & & & & r-\frac{Rl}{d-\alpha} & \\
& & d-\frac{lr}{d-\alpha} & & & & r-\frac{Rl}{d-\alpha} \\
\hline
& & & d-\beta-\frac{lr}{d-\alpha} & & & \\
\hline
l-\frac{Lr}{d-\alpha} & & & & d-\beta-\frac{LR}{d-\alpha} & & \\
& l-\frac{Lr}{d-\alpha} & & & & d-\beta-\frac{LR}{d-\alpha} & \\
& & l-\frac{Lr}{d-\alpha} & & & & d-\beta-\frac{LR}{d-\alpha}
\end{array}\right).$$

Evaluating the determinant D_1^* of this matrix by using Theorems 1, 2 of [8] we get that

$$D^{(\alpha,\beta)}_{n+1,k,2k}(L,l,d,r,R) = (d-\alpha)^k D_1^* = (d-\alpha)^k D^p_{q,1} D^{k-p}_{q-1,1} \tag{5.1}$$

where for $n+1-k = kq+p$, $(0\le p<k)$ and for $q = 0,1,\dots$

$$D_{q,1} = \sigma^{q+1}\left[U_{q+1}(\frac{v}{2\sigma}) + \frac{a+b}{\sigma}U_q(\frac{v}{2\sigma}) + \frac{ab}{\sigma^2}U_{q-1}(\frac{v}{2\sigma})\right] =$$

$$= \frac{\sigma^{q+1}}{\sin\vartheta}\left[\sin(q+2)\vartheta + \frac{a+b}{\sigma}\sin(q+1)\vartheta + \frac{ab}{\sigma^2}\sin(q\vartheta)\right].$$

with $s = l - \dfrac{Lr}{d-\alpha}, t = r - \dfrac{Rl}{d-\alpha}, \sigma = \sqrt{st}, v = d-\beta-\dfrac{lr}{d-\alpha}, a = \beta$, $b = \dfrac{lr-LR}{d-\alpha}$. U_j is the jth Chebyshev polynomial of the second kind, $\vartheta\in\mathbb{C}$ is such that $v = 2\sigma\cos\vartheta\,(\sin\vartheta\ne 0)$, and $st\ne 0$, $v\ne\pm 2\sigma$ are assumed. If $v=\pm 2\sigma$ then we have to take the limit as $\vartheta\to m\pi$ with $\vartheta = (-1)^m 2\sigma, m\in\mathbb{Z}$. Due to our assumptions on n,k either (i) $(n+1)/3 = k, q = 2, p = 0$ or (ii) $(n+1)/3 < k, q = 1, p = 0$. In these cases the exponents of $D_{q,1}$ and $D_{q-1,1}$ in (5.1) are $p = 0 = (n+1)-3k, k-p = k = (n+1)-2k$ in case (i) and $p = (n+1)-2k, k-p = k = 3k-(n+1)$ in case (ii).

Hence the new form of the determinant in (5.1) is in both cases (i), (ii)

$$\begin{aligned}(d-\alpha)^k\sigma^{(n+1)-k} &\left[U_2\left(\frac{v}{2\sigma}\right) + \frac{a+b}{\sigma}U_1\left(\frac{v}{2\sigma}\right) + \frac{ab}{\sigma^2}U_0\left(\frac{v}{2\sigma}\right)\right]^{(n+1)-2k}\cdot \\ \cdot &\left[U_1\left(\frac{v}{2\sigma}\right) + \frac{a+b}{\sigma}U_0\left(\frac{v}{2\sigma}\right)\right]^{3k-(n+1)}\end{aligned} \tag{5.2}$$

Substituting the values $s,t,\sigma,a,b,v,\vartheta,q$ into e.g. the trigonometric form of

$D_{q,1}$ we get after some simplifications that

$$\begin{aligned}
D_{0,1} &= v + a + b = \frac{d^2 - (\alpha+\beta)d + \alpha\beta - lr}{d-\alpha}, \\
D_{1,1} &= v^2 - \sigma^2 + (a+b)v + ab = \\
&= \frac{d^3 - (\alpha+\beta)d^2 - (LR + 2lr - \alpha\beta)d + (\alpha+\beta)lr + Rl^2 + Lr^2}{d-\alpha}.
\end{aligned}$$

Using these and (5.1) we get the same result for the determinant as in (4.2).

6. Remarks on the inverse of $A_0 = A_{n+1,k,2k}$

We have seen that $UFBA_0CGV = D$ where D is a diagonal matrix which is invertible exactly if none of its diagonal elements listed in (3.1) are zero. Assuming this A_0 is also invertible and

$$A_0^{-1} = CGVD^{-1}UFB. \tag{6.1}$$

All matrices (except D^{-1}) appearing in (6.1) have determinant 1, and writing them in forms $B = E + B_1, C = E + C_1, F = E + F_1, G = E + G_1$, $U = E + U_1, V = E + V_1$, where $E \in \mathcal{M}_{n+1}$ is the unit matrix, their inverses are $E - B_1, E - C_1, E - F_1, E - G_1, E - U_1, E - V_1$ respectively, moreover the product matrices

$$U_1F_1B_1,\ U_1F_1,\ F_1B_1,\ \ C_1G_1V_1,\ C_1G_1,\ G_1F_1 \tag{6.2}$$

are all zero matrices, while U_1B_1 contains only one (nonzero) diagonal below the main one and C_1V_1 also contains only one (nonzero) diagonal above the main diagonal. Hence we get for the inverse of A_0:

$$\begin{aligned}
A_0^{-1} &= (E+C_1)(E+G_1))(E+V_1)D^{-1}(E+U_1)(E+F_1))(E+B_1) \\
&= (E+C_1+G_1+V_1+C_1V_1)\, D^{-1}\, (E+U_1+F_1+B_1+U_1B_1)\,.
\end{aligned} \tag{6.3}$$

Using the definitions of $B_1, C_1, F_1, G_1, U_1, V_1$ we obtain that the factor matrices $S = (s_{ij})$ and $T = (t_{ij})$ on the left and right side of D^{-1} in (6.3) are

$$s_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{r_i}{d_i} & \text{if } i = 0, \dots, k-1,\ j = i+k, \\ -\dfrac{r_i - \dfrac{R_{i-k} l_{i-k}}{d_{i-k}}}{d_i - \dfrac{l_{i-k} r_{i-k}}{d_{i-k}}} & \text{if } i = k, \dots, n-k,\ j = i+k, \\ -\dfrac{R_i}{d_i} + \dfrac{\left(r_{i+k} - \dfrac{R_i l_i}{d_i}\right) r_i}{\left(d_{i+k} - \dfrac{l_i r_i}{d_i}\right) d_i} & \text{if } i = 0, \dots, n-2k,\ j = i+2k, \\ 0 & \text{otherwise} \end{cases}$$

$$t_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{l_j}{d_j} & \text{if } i = j+k,\ j = 0, \dots, k-1, \\ -\dfrac{l_j - \dfrac{L_{j-k} r_{j-k}}{d_{j-k}}}{d_j - \dfrac{l_{j-k} r_{j-k}}{d_{j-k}}} & \text{if } i = j+k,\ j = k, \dots, n-k, \\ -\dfrac{L_j}{d_j} + \dfrac{\left(l_{j+k} - \dfrac{L_j r_j}{d_j}\right) l_j}{\left(d_{j+k} - \dfrac{l_j r_j}{d_j}\right) d_j} & \text{if } i = j+2k,\ j = 0, \dots, n-2k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have proved

Theorem 6.1. *Assuming $(n+1)/3 \le k \le n/2$ and that all diagonal elements in (3.1) are different from zero the determinants S, T are well defined and*

$$A_{n+1,k,2k}(\mathbf{L}, \mathbf{l}, \mathbf{d}, \mathbf{r}, \mathbf{R})^{-1} = S D^{-1} T.$$

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