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LAUDATION TO Professor Antal Járai on his seventieth birthday

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Antal Járai was born on 25 August, 1950 in Biharkeresztes, Hungary. He attended secondary school in Debrecen, and then continued his studies at Kossuth Lajos University in Debrecen (today it is called Debrecen University). I met him during those years when I was a student there as well. Our friendship goes back to those days when we started to read and discuss excellent math textbooks like Dieudonné's treatise, Federer's Geometric Measure Theory and many others. We visited together the unforgettable seminars of our master, Jenő Erdős on mathematical logic, abstract harmonic analysis, Pontryagin's duality, C^*-algebras, universal algebra and completely reducible modules: shortly, on mathematics. Later, when we became colleagues, we visited our first international conferences together. We'll never forget the amazing journeys to the ISFE meetings by the tiny Trabant, owned and driven by our other mentor Zoltán Daróczy, through the Alps, on the highways of Austria, Germany and Switzerland, France and Spain.

After graduation Antal Járai started his professional career at the Department of Analysis in the Institute of Mathematics at Kossuth Lajos University. In 1976, he wrote his thesis "*On Measurable Solutions of Functional Equations*" and received his doctoral degree. After that he held various positions as a researcher at the University of Debrecen. In the period 1992–1997, he had a research position at the University of Paderborn, Germany. Since 1997, he has been a professor of Eötvös Loránd University, Budapest. He earned his PhD degree (formerly called "candidate degree") in 1990, and the Doctor of the Hungarian Academy of Sciences degree in 2001.

Antal Járai's scientific activities cover a wide range of various fields. He himself considers the following areas as his fields of interest: functional equations, measure theory, system programming, computational number theory, computer algebra and generalized number systems. This list demonstrates that Antal Járai is both a modern mathematician and a computer scientist as well. About his life and interests let me quote his own words: ,,*In my childhood, I was mostly interested in Jules Verne's stories. I was a second year pupil when I read "The Mysterious Island". I was extremely interested in the way how to produce nitro-glycerin. Verne's description was rather precise. Although my parents weren't really white collar workers, still we had a very good home library, and also they subscribed to the "Life and Science" periodical. Then there were other informative books by Perelman, Öveges, and Sztrókay – I read everything that was available in the public library in Biharkeresztes. The most serious book I read was the Hungarian translation of the Nobel prize holder Lawrence Bragg's "Electricity". There were a couple of chemical engineers in the family, and some of them gave me an experiment toolkit, so I could reconstruct a number of experiments from Sztrókay's book. My uncle told me about the periodic system. I figured out how to create exploding balls from red phosphorous and potassium permanganate – so it was quite obvious that I submitted my application to the Secondary School of Chemical Industry in Debrecen.*

I was 14 when – to my greatest surprise – I won the mathematical competition "Arany Dániel". Earlier I used mathematics only when I, say, designed a u-boat. My teacher inspired me to solve mathematical problems and to learn mathematics, but still my favourite subject remained chemistry. I was 15 when I got the second place at the national competition for secondary school students "What is your talent?" in chemistry. From all my excellent chemistry teachers I considereded József Forgács as my coach in the preparation for this contest. In the next two years I achieved the sixth and ninth place in the national physics contests for secondary schools – although that time we did not even learn physics anymore. Obviously, I had to choose physics, chemistry or maths – and I chose the last one. During my one year military service I learnt some set theory, but still I was interested in computers and electronics.

I had several excellent teachers at the university. The most I learnt – like other fellow students – from Jenő Erdős."

Antal Járai's work on pure mathematics is closely related to the field of functional equations – even his research in measure theory, related to invariant extension of the Haar measure in [5] and Steinhaus-type theorems in [6], may have connections to problems arising from functional equations.

He is a prominent representative of the Debrecen school of functional equations. His fundamental research in this field can be characterized by the word *regularization*, which, in fact, is mainly Járai's achievement. By regularization we mean the following process: we assume weak regularity conditions, say measurability, of the unknown function. It turns out that for a whole class of functional equations, the equation itself, which is an algebraic restriction, together with the weak regularity forces stronger regularity properties, like differentiability of several orders, possibly analyticity. Differentiability of the unknown function(s) in the functional equation often leads to differential equations that are easier to solve due to the extensively developed theory of ordinary and partial differential equations.

A typical result of this type is related to a problem J. Aczél and J. K. Chung [1] about the functional equation

$$\sum_{i=1}^{n} f_i(x + \lambda_i y) = \sum_{k=1}^{m} p_k(x) q_k(y),$$

which includes as a special case various classical functional equations: the functional equations of Cauchy, Pexider and d'Alembert, the square norm equation, the basic trigonometric functional equations characterizing the cosine and the sine, etc. These special cases were studied by several authors, and were solved under different conditions. Here all $n + 2m$ functions are supposed to be unknown, and Aczél and Chung consider the case, where the equation holds for x, y taken from some open intervals, moreover the functions satisfy appropriate independence conditions. They show that if all functions are locally Lebesgue integrable, then they are infinitely many times differentiable. It follows that all integrable solutions can be obtained by solving induced differential equations. Járai showed in [4] that the condition of integrability can be weakened to Lebesgue measurability, and ordinary linear independence of the functions is sufficient to get differentiability.

Járai's regularization is related to Hilbert's fifth problem (see Aczél's paper on some unsolved problems in the theory of functional equations [2], and also in [3]). He formulated the problem in his monograph [7] in the following way:

Problem. *Let T and Z be open sets in $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and let D be an open subset of $T \times T$. Let $f : T \to Z$, $g_i : D \to T$ $(i = 1, 2, \ldots, n)$, and $h : D \times Z^n \to Z$ be functions. Suppose that*

1) $f(t) = h(t, y, f(g_1(t, y), f(g_2(t, y), \ldots, f(g_n(t, y)))$ *for all* (t, y) *in* D*;*
2) $h, g_1, g_2, \ldots, g_n$ *is analytic;*
3) *for each* t *in* T *there is a* y *such that* (t, y) *is in* D*, and the rank of* $\frac{\partial g_i}{\partial y}$ *is* s *for* $i = 1, 2, \ldots, n$*.*

Is it true, that if f is measurable, or it has the Baire property, then f is also analytic?

This problem can be formulated for the case where $\mathbb{R}^s$ and $\mathbb{R}^m$ are replaced by $\mathcal{C}^\infty$ manifolds X, Y, and the analyticity is replaced by $\mathcal{C}^\infty$ – then the answer is affirmative for compact X. He also proposed a couple of steps to be taken in order to achieve a positive answer:

(I) Measurability implies continuity.

(II) Almost open solutions are continuous.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All p-times continuously differentiable solutions are $p+1$-times differentiable.

(VI) Infinitely many times differentiable solutions are analytic.

In our joint paper in 1996, we gave a survey about the progress in this program. Járai published a number of papers ([9, 10, 11, 12]) containing partial results on the above steps. One of his strongest results is the following (see [12]):

Theorem. *Suppose that the conditions of the Problem are satisfied and f has locally essentially bounded variation. Then f is infinitely many times differentiable.*

In his book [7], he also presents a summary account on the problem. I quote here the following results.

(1) If h is continuous and the functions g_i are continuously differentiable then every solution f, which is Lebesgue measurable or has Baire property, is continuous.

(2) If h and g_i are p-times continuously differentiable, then every almost everywhere differentiable solution f is p-times continuously differentiable.

(3) If h and g_i are $\max\{2,p\}$-times differentiable, and there exists a compact subset C of T such that for each t in T there is a y in T such that $g_i(t,y)$ is in C, besides the above stated rank condition on g_i, then every solution f, which is Lebesgue measurable or has the Baire property, is p-times continuously differentiable $(1 \leq p \leq \infty; i = 1, 2, \ldots, n)$.

In our personal communication Antal told me that he has never seen an equation satisfying the assumptions of the $\mathcal{C}^\infty$ problem and, using his results, still is impossible to prove that the solutions are $\mathcal{C}^\infty$ – this shows that these results are quite satisfactory from the point of view of applications.

Another area of Járai's interest is related to the interval filling sequences. The basic setting is the following: let a strictly decreasing sequence $\{\lambda_n\}$ of positive numbers, which is summable to L, be given with the property that

$\lambda_n \le \sum_{i=n+1}^{\infty} \lambda_i$. It is easy to see that, exactly in this case, every nonnegative number x, not greater than L, can be represented in the form

$$x = \sum_{n=1}^{\infty} \epsilon_n \lambda_n,$$

where each ϵ is 0 or 1. Such a sequence will then be called an *interval filling sequence*. Joint papers of Z. Daróczy, I. Kátai and Antal Járai (see [14, 15, 16]) studied the properties of interval filling sequences. There are several ways to construct the digits ϵ_n. One of them is the so-called *eager*, or *regular algorithm*, which is defined recursively by

$$\epsilon_n(x) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n-1} \epsilon_i(x)\lambda_i + \lambda_n \le x \\ 0, & \text{if } \sum_{i=1}^{n-1} \epsilon_i(x)\lambda_i + \lambda_n > x, \end{cases}$$

and results in

$$x = \sum_{n=1}^{\infty} \epsilon_n(x)\lambda_n.$$

The following concept is based on this algorithm. A function $F : [0, L] \to \mathbb{R}$ is called *additive* with respect to a sequence $\{\lambda_n\}$ if

$$\sum_{n=1}^{\infty} |F(\lambda_n)| < \infty\,,$$

and we have for each x in $[0, L]$ that

$$F(x) = F\big(\sum_{n=1}^{\infty} \epsilon_n(x)\lambda_n\big) = \sum_{n=1}^{\infty} \epsilon_n(x)F(\lambda_n).$$

The study of additive functions with respect to interval filling sequences attracted great interest in the family of functional equationists. Related problems concerning functions defined by digits of real numbers have been studied by Járai in cooperation with Daróczy and Kátai (see [17, 18]).

So far I have tried to give some insight into the mathematics of Antal Járai from the "pure math" side. However, as Zoltán Daróczy said in his dedication to his former student's 60th birthday: *"Antal Járai is a renaissance figure of our age. He is interested in physics, chemistry and electronics as well as in certain fields of geology and biology. Most of all he is a prominent developer of mathematics and computer science."* And now the main emphasis is on computer science. He has outstanding contributions to this applied part of mathematics. During his stay in Paderborn, Germany he became a real expert of that field, and his significant role ripened by now and so earned worldwide reputation. I remember the time when I was a young colleague of him in the analysis department and he devoted a lot of time and effort to improve my knowledge in the freshly discovered useful tool, the TeX editor. He was and is a real expert of TeX, that time he was the one who ”Hungarianized” TeX for us in the department, he sank into the deep layers of TeX to teach it how to understand the magic world of Hungarian accents.

Of course, his skills in TeX was not his only admirable talent in computer science: he was a professional programmer. I have just a rough estimate by himself about how many thousands of program lines he wrote in his life: it is approximately 200-300 thousand lines. There was a time when he was living on programming, beside his university affiliation – which definitely was not his main income source. But this is still not the top of his expertize in the world of bits and bytes. After he left Debrecen University for ELTE, Budapest in 1997, he started to make research in computer algebra and to teach professional courses on related subjects. Some of the courses he was teaching: Number Theory with Computers, Effective Computations with RISC Processors, Partial Differential Equations in Physics, Advanced Mathematics for Electric Engineers, Chaotic Dynamic Systems and Fractal Geometry, Computer Algebra Algorithms, Computational Models. This list clearly supports Zoltán Daróczy's words: *”His brain like a piece of sponge, as it absorbs everything; on the other hand, it is sharp like a knife as he is fast and creative in addressing any problem”.*

Although I am not the right person to analyze and exhibit Antal Járai's scientific achievement in computer science due to the lack of my expertize in that field, but I can rely on his own description of his professional experience. He is the author of over 20 system programs in assembly, including translation programs, database management systems, floating point arithmetic algorithms and time sharing systems. His programs, some of which proved to be the fastest on the given hardware all over the world, have been installed at about hundred sites. He is the manager and co-author in the development of 8 application systems. He was the project manager in the group of Karl-Heinz Indlekofer in 3 projects in computer science resulting in more than 10 ”world records”. He was the leader of 2 projects, and former fellow in 12 other ones in mathematics in Germany and Hungary. Here is the review of one of his joint papers [19] with his fellows T. Csajbók, G. Farkas, Z. Járai and J. Kasza:

Summary: ”The largest known Sophie Germain prime is

$$137211941292195 \cdot 2^{171960} - 1$$

and the largest known twin primes are

$$100314512544015 \cdot 2^{171690} \pm 1.”$$

Let me recall here his most important awards: Pro Universitate (Kossuth Lajos University, Debrecen, 1974), "Grünwald Géza Award" (Bolyai Mathematical Society, 1979), Ministry award of Ministry of Culture (1990), "For outstanding contribution to the conference" (ISFE, 1994), Award of Hungarian Academy of Science (2000), "Kalmár Award" (2008), "Knight Cross", the Order of Merit of the Hungarian Republic (2010).

I conclude this laudation with my personal short comment: I consider him as my best "math-friend". I am grateful that my fate made it possible that I have become a mathematician together with him and I have become also his colleague and his friend. On this wonderful occasion we all wish you, Tóni, a happy birthday, blessed with joy and love together with good health –from all of your students, colleagues and friends around you.

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