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LAUDATION TO Professor Antal Járai on his seventieth birthday

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1. Introduction

Antal Járai is truly a polymath: a mathematician by profession who has made major contributions to the theory of functional equations, but he also has publications in computer science and even geology, he designed a novel kind of A/D converter, he is able to follow the latest results in physics and writes chemistry books as a "hobby". He has broken several world records in computational mathematics, has influenced several generations as a mentor, supervisor and research group leader in Debrecen, Paderborn and Budapest. It is an impossible task to summarize his scientific achievements. Below, we present highlights of his fruitful career.

2. Computer science and programming

He has been interested in electronic devices since his early years, reading electronics books and building his own electronic projects. Computers fit into this line and sparked his enthusiasm already in his first year at the university. His first attempt at programming was to implement the simplex method – characteristically of the era, the program did not work not because of his lack of skills but because of a bad algorithm description in an economy textbook. He continued to experiment with programming, learning ALGOL in 5 days for example. When they learnt about the simplex method it at the university, he also tested its practical running times by solving zero-sum two-player matrix games.

He got involved in professional programming in the 80s. Due to the changes at the end of the socialist era, one was allowed to found so-called Economic Productive Communities (Gazdasági Munkaközösség – GMK), a kind of private enterprise. This was a way to gain some extra salary for an underpaid young researcher. The started to develop programs for enterprises in BASIC, but since they worked with floats and conversion on the Z80 machines was expensive, they decided to move to assembly soon. One of the first pieces of commercial software he developed was an arithmetic package started with help from his wife during their honeymoon. He made an interpreter, resulting in a speedup of the original arithmetic package by a factor of 20 to 30. He became more involved in system programming, his wife continued to work on user programs. During these years he developed a relational database software; a file manager capable of handling virtual files for handling query records; a Forth compiler; sorting and cryptographic algorithms; later also modifying operating systems – altogether over 20 system programs.

His attitude towards programming meant he would always dig deep, understanding the machine to the last logic gate and looking for better algorithms, optimizing every bit. This mindset would later be beneficial for his research in computational mathematics. Several stories illustrate how thorough he was. Due to commercial restrictions between capitalist and socialist countries, software for socialist-produced hardware was scarce or not available, opening many opportunities to develop missing software. When one of their customers started using the Forth language, he modified his Forth compiler to support timesharing, thus 5 instances could be run at the same time (on only 64 KBs of memory, this might be one of the tiniest timesharing systems ever!), enabling data recording, reading, searching, deleting and modifying on 20 terminals in parallel, boosting the customer's efficiency. On another occasion he wrote a disassembler, disassembled the operating system and modified the poorly written disc handler to enable sorting an almost full disk in space as fast as a disc copy had worked earlier.

After the birth of his daughter in 1986, he focused on mathematical research. In the early 90s, he moved to Paderborn, Germany to work with Karl-Heinz Indlekofer. He was the project manager of a group aiming to break records in computational mathematics, e.g. the largest known pair of twin primes, Sophie Germain primes etc. He thoroughly analyzed the available methods, made projections on the expected progress made by competing research groups and estimated the effort and time needed to implement and execute the algorithms. This enabled him to select problems based on a well-established probability of success. The success rate was around 50 percent, they held over 10 records at the time. The effect of these projects are more than just a "sport achievement": the group implemented a Fast Fourier Transform and multiprecision arithmetic functions, which have uses well beyond prime number search. The programs were coded largely by him. The Paderborn years also saw him complete his habilitation thesis.

He brought his insights from computer science and programming to ELTE University too. He continued to lead projects in computational number theory. He gave courses on "Compilers" and "High performance computing and computer architecture". In a special course with his students they planned a microprocessor based on Knuth's RISC circuit but capable of supporting a true time-sharing system as Minix. They wrote a simulator, an assembler and a C compiler for it.

3. Functional equations

Antal Járai has made contributions in several areas of mathematics but his results in the theory of functional equations stand out.

Much but not all of his work is related to the second part of Hilbert's fifth problem. In his celebrated address to the 1900 International Congress of Mathematicians, in his fifth problem Hilbert asked (in the language of present day mathematics) whether is it true that every locally Euclidean group is a Lie group? In the second part of his fifth problem Hilbert goes on as follows:

"Moreover, we are thus led to the wide and interesting field of functional equations which have been heretofore investigated usually only under the assumption of the differentiability of the functions involved. In particular the functional equations treated by Abel (Oeuvres, vol. 1, pp. 1, 61, 389) with so much ingenuity ... and other equations occurring in the literature of mathematics, do not directly involve anything which necessitates the requirements of the differentiability of the accompanying functions. ... In all these cases, then, the problem arises: *In how far are the assertions which we can make in the case of differentiable functions true under proper modifications without this assumption?" (Hilbert's emphasis).*

After this, Hilbert quotes a result of Minkowski, stating that under certain conditions, the solutions of the functional inequality

$$f(x+y) \le f(x) + f(y) \qquad x, y \in \mathbb{R}$$

are partially differentiable, and remarks that certain functional equations, for example the system of functional equations

$$\begin{aligned} f(x+\alpha) - f(x) &= g(x) \\ f(x+\beta) - f(x) &= 0, \end{aligned}$$

where α, β are given real numbers, may have solutions f which are continuous but not differentiable, even if the given function g is analytic. This special case is one of many examples showing that for one-variable equations continuity does not usually imply differentiability.

The origins of the quest for regularity in functional equations thus go back to Hilbert. Many of Antal Járai's major contributions to the area of functional equations are concerned with proving stronger regularity properties assuming only weak regularity. We summarize some of his results below.

3.1. A general strategy for solving functional equations

A general strategy for solving functional equations can be described as follows:

(A) Show how "weak" regularity implies "strong" regularity;

(B) Obtain a differential equation (integral equation, etc.) from the functional equation;

(C) Solve this differential equation using the rich theory and algorithmic tools for differential equations.

We note that today all three of the above steps are vastly supported by computer algorithms, developed in part by his co-authors, colleagues and students, e.g. Zs. Páles, A. Gilányi, A. Házy and S. Czirbusz.

The above three-step heuristic does not always work: for equations with one variable we have Hilbert's above mentioned counterexample. For iterated equations (i.e. when the unknown function appears as an input as well), one has the Aczél–Benz equation:

$$f\big(x+f(y)\big) = f(x) + f\big(x+y-f(x)\big).$$

Continuous real solutions for this equations were analyzed by Z. Daróczy. The function $x \mapsto \frac{1}{2}\big(x \pm |x|\big)$ is a solution.

Also, for what could be considered the "most general" equation without iteration:

$$H\Big(x, y, f\big(G(x,y)\big), f_0\big(G_0(x,y)\big), \ldots, f_n\big(G_n(x,y)\big)\Big) = 0$$

where the f's are the unknown functions, there is no regularity phenomenon unless we make some additional assumptions. Thus, one of the main challenges was to find the most general case of functional equations for which regularity results can be stated. Antal Járai managed to prove theorems for such a large class of equations.

3.2. The main problem

Antal Járai generalized earlier results by Andrade and Kac, summarized in the book of Aczél. These classical results showed for some one-variable equations that all continuous solutions are also $\mathcal{C}^\infty$. He managed to generalize these results in the following four senses.

- I. His theorems assume weak regularity, starting from $\mathcal{C}^{-1}$, i.e., from measurability or from Baire property (sometimes called Baire-measurability in the literature);

- II. The unknown function can have several (real) variables;
- III. The equations to which the theorems apply can be strongly nonlinear;
- IV. It is sufficient for the equations to be satisfied only almost everywhere.

Based on this work, his student S. Czirbusz wrote software that can prove regularity automatically.

To formulate his achievements on regularity, we define the setting for the problem he investigated.

Problem 1. *Let X, Y, and Z be open subsets of $\mathbb{R}^r$, $\mathbb{R}^s$, and $\mathbb{R}^t$, respectively, and let D be an open subset of $X \times Y$ and let W be an open subset of $D \times Z^n$. Let $f : X \to Z$, $g_i : D \to X$ $(i = 1, 2, \ldots, n)$, and $h : W \to Z$ be functions. Suppose that*

(FE) *if $(x, y) \in D$ then*

$$\Big(x, y, f\big(g_1(x, y)\big), \ldots, f\big(g_n(x, y)\big)\Big) \in W$$

and

$$f(x) = h\Big(x, y, f\big(g_1(x, y)\big), \ldots, f\big(g_n(x, y)\big)\Big);$$

(S) *h and g_i $(i = 1, 2, \ldots, n)$ are in $\mathcal{C}^\infty$;*

(RC) *for each $x \in X$ there exists a y for which $(x, y) \in D$ and $\frac{\partial g_i}{\partial y}(x, y)$ has rank $r = \dim(X)$ $(i = 1, 2, \ldots, n)$.*

Is it true that every f, which is in $\mathcal{C}^{-1}$ (i.e., measurable or has the Baire property) is in $\mathcal{C}^\infty$?

Note the "rank condition" (RC) that rank $\frac{\partial g_i}{\partial y} = r$, which implies $\dim(Y) = $ $= s \geq r$.

The transfer principle. One might ask whether the use of only one unknown function is too restrictive? It turns out that if we have a general non-composite functional equation with several variables and several unknown functions, and we can express *each* unknown function from it, then (after writing different variables in each equation) we may consider a vector-valued function having the different unknown functions as coordinates and use results concerning equations with only one, but vector-valued unknown function. This shows how important it is to discuss vector-valued unknown functions with vector variables.

The above problem is well-formed in the following sense. None of the conditions of the problem above can be omitted without introducing new conditions. This justifies that the above problem can be called "the main regularity problem of non-composite functional equations with several variables".

We note that Sincov's equation (for example from introduction of thermodynamical temperature scale) shows that the rank condition cannot be simply omitted:

$$f(x_1,x_2)=f(x_1,y)+f(y,x_2), \qquad x_1,x_2,y\in\mathbb{R}.$$

No regularity, one has $f(x_1,x_2)=g(x_1)-g(x_2)$.

3.3. The main results

In several papers, he published many results on the main regularity problem. Put together, these partial answers for the regularity problem basically solve it. For practically all problems one encounters, for which the conditions of the main problem hold, the partial results can be used to prove regularity – sometimes with a bit of additional manual manipulation of the equations. Not even the compactness conditions are restrictive. Perhaps the following three theorems stand out of the many published results by him.

Partial answer. *If X is a compact manifold then every $\mathcal{C}^{-1}$ solution f is in $\mathcal{C}^{\infty}$.*

Partial answer. *If equation* (FE) *is quasilinear:*

$$f(x)=\sum_{i=1}^{n}h_i\Big(x,y,f\big(g_i(x,y)\big)\Big),$$

where the functions g_i and $h_i : D\times Z\to\mathbb{R}^t$ are in $\mathcal{C}^{\infty}$, then every solution $f\in\mathcal{C}^{-1}$ is in $\mathcal{C}^{\infty}$.

Partial answer. *If there exists a compact subset C of X such that for each $x\in X$ there exists a $y\in Y$ satisfying $g_i(x,y)\in C$ besides other conditions, then every $\mathcal{C}^{-1}$ solution f is in $\mathcal{C}^{\infty}$. (This also hold on manifolds.)*

The additional compactness condition in this theorem is retained by the transfer principle.

We note that some refinements of the problem by M. Laczkovich, Zs. Páles and A. Gilányi have implications for regularity of composite equations.

3.4. Almost solutions

A typical situation where measurable almost solutions of functional equations arise naturally are characterization problems of probability distributions. A prototype of such results is the well-known theorem stating that if ξ and η are independent random variables and $\xi+\eta$ and $\xi-\eta$ are independent, too, then all have normal distributions. Supposing that ξ, η, $\xi+\eta$, and $\xi-\eta$ have density functions f_ξ, f_η, $f_{\xi+\eta}$, and $f_{\xi-\eta}$, respectively, we obtain that

$$f_{\xi+\eta}(u)f_{\xi-\eta}(v) = 2f_\xi(u+v)f_\eta(u-v)$$

for almost all $(u,v) \in \mathbb{R}^2$. Other typical situations where measurable almost solutions of functional equation appear are harmonic analysis, distribution methods for solving functional equations and functional equations arising by differentiation of functional equations almost everywhere. The following deep theorem was proved by Antal Járai for almost solutions of functional equations.

Theorem. *Let Z be a regular space, Z_i $(i = 1, 2, \dots, n)$ topological spaces, and T a first countable topological space. Let Y be an open subset of $\mathbb{R}^k$, X_i an open subset of $\mathbb{R}^{r_i}$, D an open subset of $T \times Y$, and $W \subset D \times Z_1 \times \cdots \times Z_n$. Let $T' \subset T$ be a dense subset, $f : T' \to Z$, $g_i : D \to X_i$, and $h : W \to Z$. Suppose that the function f_i is almost everywhere defined on X_i with values in Z_i and the following conditions are satisfied:*

(1) *for all $t \in T'$ for almost all $y \in D_t$ we have*

$$\Big(t, y, f_1\big(g_1(t,y)\big), \dots, f_n\big(g_n(t,y)\big)\Big) \in W$$

and

$$f(t) = h\Big(t, y, f_1\big(g_1(t,y)\big), \dots, f_n\big(g_n(t,y)\big)\Big);$$

(2) *for each fixed y in Y, the function h is continuous in the other variables;*

(3) *f_i is Lusin λ^{r_i} measurable on its domain;*

(4) *g_i and the partial derivative $\frac{\partial g_i}{\partial y}$ is continuous on D;*

(5) *for each $t \in T$ there exists a y such that $(t, y) \in D$ and the partial derivative $\frac{\partial g_i}{\partial y}$ has rank r_i at $(t, y) \in D$.*

Then f has a unique continuous extension to T.

3.5. Steinhaus type-theorems and zero-one laws for functional equations

A famous theorem of Steinhaus from 1920 asserts that, for any measurable set $A_{\mathbb{R}}$ with positive Lebesgue measure the set $A \subset \mathbb{R}$ contains an interval.

This theorem allows various generalizations and modifications. Such results are of independent interest but also have implications for functional equations. In the generalizations the following problem is treated: if we replace the subtraction by a binary operation $F(x, y)$, under what conditions on F can we prove that $F(A,B)$ contains a nonvoid open set? The first step was done by Erdős and Oxtoby proving in the case $x, y \in \mathbb{R}$ that, if F is a continuously differentiable function with non-vanishing partial derivatives, then $F(A,B)$ contains a nonvoid open set.

We will treat a generalization for function F with more than two variables. Of course, if F maps $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ we obtain a problem already solved by the theorem of Erdős and Oxtoby. To obtain a really interesting new problem, we have to consider a function with values in $\mathbb{R}^2$. The condition about the nonvanishing partial derivatives will be substituted with the condition that the null space of the derivative (as a linear map) is in general position. Antal Járai obtained the following general theorem.

Theorem. *Let X be an r-dimensional Euclidean space, and let $X_1, \dots, X_n$ be orthogonal subspaces of X with dimensions $r_1, \dots, r_n$. Suppose, that $r_i \geq 1$ $(1 \leq i \leq n)$ and $\sum_{i=1}^{n} r_i = r$. Let U be an open subset of X and $F : U \to \mathbb{R}^m$ be a continuously differentiable function. For each $x \in U$ let N_x denote the null space of $F'(x)$. Let A_i be a subset of X_i $(i = 1, \dots, n)$. If N_x is in general position, i.e. $\dim N_x = r - m$ and $p_i(N_x) = X_i$ for all $x \in U$ $(i = 1, 2, \dots, n)$, moreover $A_1 \times A_2 \times \ldots \times A_n \subset U$, $\lambda^{r_i}(A_i) > 0$ $(i = 1, 2, \dots, n)$, and the set A_i is λ^{r_i} measurable for $2 \leq i \leq n$, then $F(A_1 \times \ldots \times A_n)$ contains a nonvoid open set.*

One implication of this Steinhaus-type theorem is on functional equations in multiplicatice form. Several equations arising from probability theory has multiplicative form. For these the following theorem is important.

Theorem. *Suppose that the 0–1 valued functions f_i, $i = 1, \dots, m$ and g_j, $j = 1, \dots, n$ are Lebesgue measurable and the functional equation*

$$\text{(MFE)} \qquad \prod_{i=1}^{m} f_i(x_i) = \prod_{j=1}^{n} g_j(y_j)$$

is satisfied almost everywhere on $X_1 \times \cdots \times X_m$, where

(1) *the domains X_i and Y_j of the unknown functions f_i, $i = 1, \dots, m$ and g_j, $j = 1, \dots, n$, respectively, are nonvoid connected open sets of finite dimensional Euclidean spaces;*

(2) *the inner functions $y_1, \dots, y_n$ of $x_1, \dots, x_m$ are such that the mapping $F : (x_1, \dots, x_m) \mapsto (y_1, \dots, y_n)$ is a one-to-one $\mathcal{C}^1$-mapping of the Cartesian product of the X_i's to the Cartesian product of the Y_j's and its inverse mapping $G : (y_1, \dots, y_n) \mapsto (x_1, \dots, x_m)$ is also $\mathcal{C}^1$-mapping, hence both are $\mathcal{C}^1$-diffeomorphisms;*

(3) *fixing any of the vector variables* x_i $(i = 1, \ldots, m)$ *and choosing any of the dependent vectors* y_j $j = 1, \ldots, n$*, the corresponding mapping is a submersion, i. e., its derivative has rank equal to the dimension of the chosen dependent vector, and the same remains true if we change the roles of* x_i, $i = 1, \ldots, m$ *and* y_j $j = 1, \ldots, n$.

Then either on both sides one of the functions is zero almost everywhere or all of the functions f_i, $i = 1, \ldots, m$ *and* g_j, $j = 1, \ldots, n$ *are equal to 1 almost everywhere.*

The proof depends an "almost" version of the previous Steinhaus type theorem.

Corollary. *Let us suppose that the real valued functions* f_i *and* g_j *satisfies* (MFE) *almost everywhere and* (1)–(3) *are also satisfied. Then either on both sides one of the functions is zero almost everywhere, or the functions* f_i *and* g_j *are almost equal to real valued nonzero* $\mathcal{C}^\infty$*-functions satisfying* (MFE) *everywhere.*

3.6. Equations with few variables

The first general regularity results which overcome the difficulty when the "rank condition" is not satisfied were given by Światak. She applied her distribution method to generalizations of the mean value equation. She investigated the equation

$$\sum_{i=1}^{n} h_i(x, y) f\big(g_i(x, y)\big) = h_0(x, y), \tag{1}$$

$$x \in \mathbb{R}^r,\ y \in Y \subset \mathbb{R}^s. \tag{2}$$

with unknown function f and proved that $\mathcal{C} \implies \mathcal{C}^\infty$

Regularity results for functional equations with "few" variables, i. e. with r place unknown function but with less than $2r$ (but more than r) variables have been proved. These have the structure $f \in \mathcal{C}^{q-1} \implies f \in \mathcal{C}^q$ for $q = 0, 1, \ldots$ where $\mathcal{C}^{-1}$ is understood as the class of measurable functions or as the class of functions having the property of Baire. Most of the results can be applied for equation (1) except $f \in \mathcal{C}^0 \implies f \in \mathcal{C}^1$ which is proved only for special h linear in the f-terms.

The proofs use special function spaces, which — roughly speaking — interpolate between measurability and continuity, between Baire property and continuity and between continuity and continuous differentiability. Although the general theory can be applied for systems of functional equations with several unknown functions, here only less general but easy to apply corollaries for the case of one unknown function will be given.

Antal Járai has made recent contributions to the area of functional equations where the rank condition (RC) is lacked. Instead of stating his theorems in the most general abstract form, we give a few corollaries. The key difficulties lie in proving continuous differentiability from continuity. One important tool is the investigation of "critical subspaces".

Corollary 1. *Let $X \subset \mathbb{R}^r$ be an open set and $f : X \to \mathbb{R}^m$ a function. Suppose that*

(LFES) *the linear functional equations*

$$f(x) = h_{i,0}(x,y) + \sum_{j=1}^{n_i} h_{i,j}(x,y) f\big(g_{i,j}(x,y)\big)$$

are satisfied, whenever $i \in I$, $(x,y) \in D_i$ (here I is an index set), moreover

(S2) *$D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, the functions $h_{i,0} : D_i \to \mathbb{R}^m$ and $h_{i,j} : D_i \to \mathbb{R}$ are in $\mathcal{C}^1$, the functions $g_{i,j} : D_i \to X$ are in $\mathcal{C}^2$, moreover*

(D) *for each $x \in X$ and for each proper linear subspace V of $\mathbb{R}^r$ there exists an $i \in I$ and a y such that $(x,y) \in D_i$ and*

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) > \dim(V)$$

whenever $1 \le j \le n_i$.

Then $f \in \mathcal{C}^0$ implies $f \in \mathcal{C}^1$.

Observe, that if $\dim(Y_i) > 0$, then the dimension condition (D) is satisfied "in general", because "in general"

$$\det\left(\frac{\partial g_{i,j}}{\partial x}(x,y)\right) \ne 0, \quad \dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V)\right) = \dim(V),$$

$$\operatorname{rank}\left(\frac{\partial g_{i,j}}{\partial y}(x,y)\right) = \min\{r, \dim(Y_i)\} > 0.$$

Corollary 2. *Let $X_{\mathbb{R}}^r$ be an open set and $f : X \to \mathbb{R}^m$ a function. Suppose that*

(FES) *we have*

$$\Big(x, y, f\big(g_{i,1}(x,y)\big), \dots, f\big(g_{i,n_i}(x,y)\big)\Big) \in W_i$$

and the functional equation

$$f(x) = h_i\Big(x, y, f\big(g_{i,1}(x,y)\big), \dots, f\big(g_{i,n_i}(x,y)\big)\Big)$$

is satisfied, whenever $i \in I$, $(x,y) \in D_i$ (here I is an index set), moreover

(SI) *$D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, W_i is an open subset of $D_i \times (\mathbb{R}^m)^{n_i}$, all the partial derivatives $\partial_t^{\alpha_0}\partial_{z_1}^{\alpha_1}\dots\partial_{z_{n_i}}^{\alpha_{n_i}} h_i$ of the functions $h_i : W_i \to \mathbb{R}^m$ are continuously differentiable, the functions $g_{i,j} : D_i \to X$ are in $\mathcal{C}^\infty$, moreover*

(D) *for each $x \in X$ and for each proper linear subspace V of $\mathbb{R}^r$ there exists an $i \in I$ and a y such that $(x,y) \in D_i$ and*

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) > \dim(V)$$

whenever $1 \le j \le n_i$.

Then $f \in \mathcal{C}^1$ implies $f \in \mathcal{C}\infty$.

Critical subspaces. Let $X \subset \mathbb{R}^r$ be an open set and for each $i \in I$ let $D_i \subset X \times Y_i$ be an open set, where Y_i is a Euclidean space and let the functions $g_{i,j} : D_i \to X$, $1 \le j \le n_i$ be in $\mathcal{C}^1$. Suppose, that for each $x \in X$ there is a $i \in I$ and a y such that $(x,y) \in D$. For a proper linear subspace V of $\mathbb{R}^r$ we will say that it is a critical subspace at x if for each $i \in I$ and for each y for which $(x,y) \in D_i$ we have

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x,y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x,y)\right) \le \dim(V)$$

for some $1 \le j \le n_i$.

It is clear, that the dimension condition (D) can be formulated on the way that there is no critical subspace for any $x \in X$. It is also clear that if a linear subspace V is critical then any proper linear subspace of $\mathbb{R}^r$ containing V is critical too. Hence it is enough to consider minimal critical linear subspaces.

Sincov equation. For the Sincov equation

$$f(x_1, x_2) = f(x_1, y) + f(y, x_2), \qquad x_1, x_2, y \in \mathbb{R} \tag{3}$$

if we try to apply our general results, we find the coordinate axis as critical subspaces. In this case we cannot remove these with substitutions, as in several other cases. Let us observe the connection of the critical directions with the general solution $f(x_1, x_2) = g(x_1) - g(x_2)$.

Corollary 3. *Let $X \subset \mathbb{R}^r$ be an open set, $f : X \to \mathbb{R}^m$ a function and let $K \subset \{0, 1, \ldots, r\}$ containing 0 and r. Suppose that*

(FES) *we have*

$$\Big(x, y, f\big(g_{i,1}(x, y)\big), \ldots, f\big(g_{i,n_i}(x, y)\big)\Big) \in W_i$$

and the functional equation

$$f(x) = h_i\Big(x, y, f\big(g_{i,1}(x, y)\big), \ldots, f\big(g_{i,n_i}(x, y)\big)\Big)$$

is satisfied, whenever $i \in I$, $(x, y) \in D_i$ (here I is an index set), moreover

(S1) *$D_i \subset X \times Y_i$ is an open set, Y_i is a Euclidean space, W_i is an open subset of $D_i \times (\mathbb{R}^m)^{n_i}$, for all $y \in Y_i$, h_i is continuous in the other variables and the functions $g_{i,j} : D_i \to X$ are in $\mathcal{C}^1$, moreover*

(CD) *for each $x_0 \in X$ and for each proper linear subspace V_0 of $\mathbb{R}^r$ with $k_0 = \dim(V_0) \in K$ there exists an $i \in I$ and a y_0 such that $(x_0, y_0) \in D_i$ and*

$$\dim\left(\frac{\partial g_{i,j}}{\partial x}(x, y)(V) + \operatorname{rng}\frac{\partial g_{i,j}}{\partial y}(x, y)\right)$$

is the same constant k in K and greater than k_0 for $1 \le j \le n_i$, whenever x is close enough to x_0, y is close enough to y_0 and V is close enough to V_0 in the Grassmann manifold $\mathbb{G}(r, k)$.

Then $f \in \mathcal{C}^{-1}$ implies $f \in \mathcal{C}^0$.

Here again if $\dim(Y_i) > 0$, the "constant dimension" condition (CD) is satisfied "in general", but not if there is a critical subspace for some $x \in X$.

To this day he remains active and he is opening new attack lines on regularity and many more interesting challenges in functional equations and other math topics.

4. Closing thoughts

Working as a student and colleague with Antal has been a privilege for many of us. However intimidatingly bright his achievements and ideas seemed to us, he has always been approachable and open to discussions on and beyond mathematics and computer science. In the name of all colleagues at ELTE who have been lucky to work with him and learn from him, I wish him many more happy productive years to come.