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ON THE CONSTRUCTION OF p-EIGENVECTORS

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Dedicated to the memory of Professor Gisbert Stoyan

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Abstract. Recent investigations led to the definition of p-eigenvectors: such vectors for a matrix, that the fraction of the vector norms of the matrix-vector product and the nonzero vector (as in defining a natural matrix norm) is independent of the applied p-norm. This paper elaborates this concept further, presenting general results, proof of existence for arbitrary matrices, and constructing new solutions in the case of 3×3 diagonal matrices.

1. Introduction

We elaborate on the topic of p-eigenvectors, introduced in [3]. The mentioned paper laid down the foundations for this area, solved some basic problems (such as the case of 2×2 diagonal and rotation matrices), and formulated many related questions. Solutions for many of the raised questions will be presented, e.g.

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- The existence and construction problem of non-trivial p-eigenvectors for general matrices of $\mathbb{R}^{n\times n}$ $(n \geq 2)$.
- Further, previously unknown p-eigenvectors of diagonal matrices are described numerically, analytically and visually.

1.1. Preliminaries

Recall the power norms or p-norms for vectors of $\mathbb{R}^n$ with $2 \leq n \in \mathbb{N}$:

$$\|.\|_p : \mathbb{R}^n \to \mathbb{R}, \qquad \|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \qquad (p \in [1,\infty)),$$

and

$$\|x\|_\infty = \max_{k=1}^{n} |x_k|.$$

It is well known that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ $(x \in \mathbb{R}^n)$. Let us now consider a matrix $A \in \mathbb{R}^{n\times n}$. The p-norm of A is defined as

$$\|.\|_p : \mathbb{R}^{n\times n} \to \mathbb{R}, \qquad \|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \qquad (p \in [1,\infty]).$$

Definition 1 (p-eigenvectors, cf. [3], Definition 2)**.** Given a matrix $A \in \mathbb{R}^{n\times n}$ $(2 \leq n \in \mathbb{N})$, a vector $x \in \mathbb{R}^n, x \neq 0$ shall be called a *p-eigenvector* of A, if there exists a constant $\gamma \in \mathbb{R}_0^+$, such that for all $p \in [1,\infty]$:

$$\frac{\|Ax\|_p}{\|x\|_p} = \gamma.$$

The value γ is the *p-eigenvalue* associated with the p-eigenvector x.

Remark that regular eigenvectors are *trivial* p-eigenvectors, that $p \in (0,1)$ may also be considered, and that one may also speak of p-eigendirections.

A basic finding of [3] was that in the most simple case with $a, b > 0$, for the diagonal matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathbb{R}^{2\times 2} \quad \text{the vector} \quad x = \begin{pmatrix} \pm\sqrt{b} \\ \pm\sqrt{a} \end{pmatrix},$$

is a (non-trivial) p-eigenvector. Real valued non-trivial p-eigenvectors of 2×2 rotation matrices have also been described in a similar manner.

A straightforward tool to visualize the behaviour of the fraction in Definition 1 and to spot the p-eigenvectors was introduced, the so-called induction set (induction curve, induction surface in low dimensions), this shall be recalled and used in Section 3.4.

2. General results

In this section, we provide a characterization for p-eigenvalues and eigenvectors of a matrix $A \in \mathbb{R}^{n\times n}$, based on regular eigenvalues and eigenvectors of some specific matrices obtained by transforming A. We also discuss some observations regarding the number of p-eigenvalues of a matrix.

2.1. Characterization using regular eigenpairs

First, let us present the following useful observation.

Lemma 1. *Suppose that $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is a p-eigenvector of $A \in \mathbb{R}^{n\times n}$ with the p-eigenvalue $\gamma \geq 0$. Then, there exists a permutation $\pi : \{1, \dots, n\} \to$ $\to \{1, \dots, n\}$ such that*

$$|(Ax)_i| = \gamma \left|x_{\pi(i)}\right|, \qquad (i = 1, \dots, n).$$

Proof. Let us notate the vector $Ax \in \mathbb{R}^n$ by $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. We prove our claim by constructing such a permutation π. By the definition of p-eigenvalues we have $\|Ax\|_\infty = \|y\|_\infty = \gamma\|x\|_\infty$, i.e.

$$\max_{k=1}^{n} |y_k| = \gamma \max_{k=1}^{n} |x_k|,$$

consequently there exists a pair of indices $i, j \in \{1, \dots, n\}$ such that $|y_i| = \gamma|x_j|$, so let $\pi(i) = j$.

Now, for any $1 \leq p < \infty$ we have

$$\left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p} = \gamma \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p},$$

consequently

$$\sum_{k=1}^{n} |y_k|^p = \sum_{k=1}^{n} \gamma^p |x_k|^p.$$

Since $|y_i|^p = \gamma^p|x_j|^p$, we conclude that

$$\sum_{\substack{k=1\\k\neq i}}^{n} |y_k|^p = \sum_{\substack{k=1\\k\neq j}}^{n} \gamma^p |x_k|^p,$$

and so

$$\left(\sum_{\substack{k=1\\k\neq i}}^{n} |y_k|^p\right)^{1/p} = \gamma \left(\sum_{\substack{k=1\\k\neq j}}^{n} |x_k|\right)^{1/p}.$$

Introducing two new the vectors $x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ and $y' = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) \in \mathbb{R}^{n-1}$, we find that $\|x'\|_p = \|y'\|_p$ holds for every $p \in [1, \infty)$.

By taking the limit as $p \to \infty$, we also get $\|x'\|_\infty = \|y'\|_\infty$, so now we can repeat the same process to find a new pair of corresponding indices, constructing a suitable permutation π in n steps. ■

We remark that the permutation π is not unique in general, as the infinity norm of x and y can be equal to the absolute values of multiple entries. However, if the absolute values of the entries of x are pairwise different, it is unique.

Interpreting this result, for all of the p-eigenvectors $x \in \mathbb{R}^n$ of a matrix $A \in \mathbb{R}^{n\times n}$, we must have that the vector $Ax \in \mathbb{R}^n$ contains a permutation of ± 1 times the entries of γx, i.e. $\gamma x_i = \pm(Ax)_{\pi(i)}$ for some $\pi \in S_n$ (where S_n denotes the symmetric group over n elements). Notate the matrix representation of π as the permutation matrix $P \in S_n$, and observe that the sign setting for each entry of $(Ax) \circ \pi$ can be identified with a multiplication by a diagonal matrix having ± 1 values on its main diagonal.

So we have that, for any p-eigenvector, the relation

$$I^{\pm} P A x = \gamma x, \qquad \big(P \in S_n, I^{\pm} = \operatorname{diag}(\pm 1, \pm 1, \dots, \pm 1) \in \mathbb{R}^n\big)$$

must hold with some $\gamma \geq 0$. In fact, the set of product matrices $I^{\pm} P$ is finite group of order $2^n n!$ (every pair of $I^{\pm}$ and P defines a unique element), which we notate by $S_n^{\pm}$ from now on, and refer to its elements as *signed permutations*.

This leads us to the vital conclusion that every p-eigenvector must also be a regular eigenvector, corresponding to a nonnegative real eigenvalue of at least one of the finitely many matrices $P^{\pm} A \in \mathbb{R}^{n\times n}$, where $P^{\pm} \in S_n^{\pm}$. Now it just remains to observe that if the real $\lambda \geq 0$ with $0 \neq v \in \mathbb{R}^n$ is an eigenvalue-eigenvector pair of a matrix $P^{\pm} A \in \mathbb{R}^{n\times n}$, then we have

$$\|Av\|_p = \|P^{\pm} A v\|_p = \|\lambda v\|_p = \lambda \|v\|_p, \qquad (p \in [1, \infty])\,,$$

a p-eigenvalue with a p-eigenvector. Notice that if a matrix $P^{\pm} A \in \mathbb{R}^{n\times n}$ has a negative real eigenvalue $\lambda < 0$, then $(-1) \cdot P^{\pm} A$ has the eigenvalue $(-1) \cdot \lambda > 0$ with the same eigensubspace.

Summarizing our train of thought, we have the following result.

Theorem 1. *The vector $0 \neq x \in \mathbb{R}^n$ is a p-eigenvector of $A \in \mathbb{R}^{n\times n}$ with p-eigenvalue $\gamma \geq 0$ if and only if there exists a transformation $P^{\pm} \in S_n^{\pm}$ such that γ and x is an eigenvalue-eigenvector pair of $P^{\pm} A \in \mathbb{R}^{n\times n}$.*

Provided we can efficiently find the regular eigenpairs of a matrix, this result also gives us a practical algorithm for finding p-eigenvectors and eigenvalues, as we can cycle through the $2^n n!$ elements of the group $S_n^{\pm}$, and solve the regular problem for each. Technically, it is enough to consider $2^{n-1}n!$ elements, because after checking $P^{\pm}$ we can skip $(-1)\cdot P^{\pm}$. However, this method is only efficient for smaller values of n.

2.2. Number of p-eigenvalues and eigenvectors

Now we may define the algebraic multiplicity of a p-eigenvalue as the sum of its algebraic multiplicities over all possible signed permutations of A, and say that the number of p-eigenvalues of a matrix is the sum of algebraic multiplicities of distinct p-eigenvalues.

Based on Theorem 1 and our corresponding remarks, it is imminent that the number of p-eigenvalues has the following upper bound.

Corollary 1. *The number of p-eigenvalues (with multiplicity) of a matrix $A \in \mathbb{R}^{n\times n}$ cannot be greater than $n \cdot 2^{n-1}n!$.*

However, it is unclear if this number or anything close to it may be reached for arbitrary values of n, i.e. if there exists a matrix $A \in \mathbb{R}^{n\times n}$ with this number of p-eigenvalues. The particularly interesting part would be to solve the open problem whether or not the following conjecture is true.

Conjecture 1. *For every $n = 2, 3, \ldots$ there exists a matrix $A \in \mathbb{R}^{n\times n}$ such that for every permutation $P \in S_n$, all the eigenvalues of PA are real.*

In our experiments, we have verified that this is true up until $n = 7$, but solving it in general could be a subject of future research. We provide two example matrices below.

$$M_3 = \begin{pmatrix} 1 & 4 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 1 & 2^{17} & 2^{32} & 2^{45} & 2^{56} & 2^{65} & 2^{72} \\ 0 & 1 & 2^{18} & 2^{32} & 2^{45} & 2^{56} & 2^{65} \\ 0 & 0 & 1 & 9\cdot 2^{14} & 171\cdot 2^{26} & 2^{45} & 2^{56} \\ 0 & 0 & 0 & 1 & 9\cdot 2^{14} & 2^{32} & 2^{45} \\ 0 & 0 & 0 & 0 & 1 & 2^{18} & 2^{32} \\ 0 & 0 & 0 & 0 & 0 & 1 & 2^{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the following, we study the problem of counting p-eigenvalues of diagonal matrices through constructing *directionally distinct* p-eigenvectors. Two vectors will be called directionally distinct, if they are not collinear.

The notation $\operatorname{diag}(a, b, c)$ will be used to denote a diagonal matrix with entries a, b and c on its main diagonal, respectively.

In the previous work [3], the case of 3×3 diagonal matrices was investigated in detail. It turns out that, based on Theorem 1, we may now investigate the number of p-eigenvectors of a $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}$ diagonal matrix. We may assume it has positive $d_i > 0$ entries.

In order to have a p-eigenvector $x \in \mathbb{R}^n$, the equation

$$Dx = \gamma P^{\pm} x$$

must hold with some signed permutation $P^{\pm} \in S_n^{\pm}$, i.e. with a permutation $\pi \in S_n$ we have

$$d_i x_i = \pm\gamma x_{\pi(i)}, \qquad (i = 1, \ldots, n).$$

The permutation π may be decomposed into disjoint cycles. Suppose that the orbit of 1 consists of k elements of $\{1, \ldots, n\}$, i.e.

$$\begin{aligned} d_1 x_1 &= \pm\gamma x_{\pi(1)}; \\ d_{\pi(1)} x_{\pi(1)} &= \pm\gamma x_{\pi^2(1)}; \\ &\vdots \\ d_{\pi^{k-1}(1)} x_{\pi^{k-1}(1)} &= \pm\gamma x_1. \end{aligned}$$

Combining these, we obtain the relation

$$d_1 d_{\pi(1)} d_{\pi^2(1)} \cdots d_{\pi^{k-1}(1)} x_1 = \pm\gamma^k x_1,$$

so the p-eigenvalue must be $\gamma = \sqrt[k]{|d_1 d_{\pi(1)} d_{\pi^2(1)} \cdots d_{\pi^{k-1}(1)}|}$. By choosing an arbitrary $x_1 \neq 0$ value, or, since any scalar multiple of x is also a corresponding p-eigenvector, by setting $x_1 = 1$, we can solve through the above equations and choose arbitrary $\pm$ signs for each entry $x_{\pi(1)}, x_{\pi^2(1)} \ldots, x_{\pi^k(1)}$.

Now, if we set $x_j = 0$ for all $j \in \{1, \ldots, n\} \setminus \{1, \pi(1), \ldots, \pi^{k-1}(1)\}$, it is easy to verify that x is a p-eigenvector for the above γ: the same equations for every other cycle in π become $0 = 0$ identities.

Based on this observation, we can immediately get a lower estimation for the number of p-eigenvectors.

Theorem 2. *Every $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}$ diagonal matrix has at least*

$$\frac{3^n - 1}{2}$$

directionally distinct p-eigenvectors and eigenvalues (*with multiplicity*).

Proof. Notice that for the described construction, we could choose any k-element subset of $\{1, \ldots, n\}$, (as we can identify any *derangement* [1] of its elements to a cycle in the decomposition of a permutation of the whole set), and we can choose the signs of $k-1$ vector entries, so we have constructed at least

$$\sum_{k=1}^{n}\binom{n}{k}2^{k-1}=\frac{1}{2}\left(-1+\sum_{k=0}^{n}\binom{n}{k}2^{k}\right)=\frac{3^n-1}{2}$$

different (i.e. directionally distinct) p-eigenvectors. The reason for this is that for different subsets of $\{1,\ldots,n\}$, the construction always gives directionally distinct eigenvectors, obviously. ■

Omitting the division by 2 can be interpreted as considering a unit norm p-eigenvector and its (-1) scalar multiple distinct, and as such gives a lower bound on the number of intersections of induction curves/surfaces.

Compare this result to Lemma 1 and Corollary 1 of [3]: we find that this estimation is sharp for $n=2$, and constructs all distinct p-eigenvectors. However, it is far off from the theoretical upper bound of our Corollary 1 as n increases, so we may try to improve it.

We should investigate if, for a given subset of diagonal entries, there exists different derangements yielding the same eigenvectors or not (because in this case, due to the normalization requiring the first nonzero entry to be 1 for the constructed vectors, two eigenvectors are not directionally distinct exactly if they are equal).

It turns out that with an additional (mild) assumption on the diagonal entries, we can compute the exact number of p-eigenvectors and eigenvalues.

Theorem 3. *Suppose that the diagonal matrix $D=\operatorname{diag}(d_1,\ldots,d_n)\in\mathbb{R}^{n\times n}$ has the following property: for every subset $I\subset\{1,2,\ldots,n\}$, the values*

$$\gamma_I:=\left(\prod_{i\in I}|d_i|\right)^{1/|I|}$$

are pairwise distinct. Then, the number of directionally distinct p-eigenvectors and eigenvalues (with multiplicity) of D is exactly

$$\sum_{k=1}^{n}\binom{n}{k}2^{k-1}\left[\frac{k!}{e}\right],$$

where $[x]$ is the nearest integer function.

Proof. With the described property at hand, we can see that for any permutation $\pi\in S_n$, exactly one of its cycles has to contain every nonzero entry of x: otherwise, if there are nonzero entries for at least two disjoint cycles of indices $I,J\subset\{1,2,\ldots,n\}$, $I\cap J=\emptyset$, then the p-eigenvalue for x must be γ_I and γ_J at the same time, but we have $\gamma_I\neq\gamma_J$.

Consequently, we can enumerate all possible p-eigenvectors by first choosing a subset $I \subset \{1, 2, \ldots, n\}$ for the indices of nonzero entries of x, and then choose a derangement of its elements to represent the cycle itself, finally set the first nonzero entry to 1 and solve the system of equations with every possible choice of signs for the entries.

We already know from the proof of Theorem 2, that for a given $k = |I| = 1, 2, \ldots, n$, there are $\binom{n}{k}$ possible subsets, and that there are 2^{k-1} options for choosing the signs. The number of derangements of k elements is known (see e.g. [2]) to be

$$!k = k! \sum_{i=0}^{k} \frac{(-1)^i}{i!} = \left[\frac{k!}{e}\right].$$

So we have the upper bound for the number of directionally distinct eigenvectors at

$$\sum_{k=1}^{n} \binom{n}{k} 2^{k-1} \cdot !k.$$

Finally, if we can show that all these constructed p-eigenvectors are directionally distinct, then our proof is complete. As we saw earlier, different choices of I always yield distinct eigenvectors corresponding to different eigenvalues, and different choices of signs also give directionally distinct vectors, so we only left to prove that there are no two different derangements of I that give us the same vector x.

Notate the smallest element of I by i, and suppose there exists two derangements $\pi, \widetilde{\pi} : I \to I$ generating the same x solution. We may assume that we have $\pi(i) \neq \widetilde{\pi}(i)$. Since both are derangements over I, there exists an integer $1 < l < |I|$ such that $\widetilde{\pi}(i) = \pi^l(i)$.

Now from the equations for $\widetilde{\pi}$, it follows that

$$x_{\widetilde{\pi}(i)} = \frac{\pm d_i}{\gamma_I},$$

and from the equations for π we also have

$$x_{\widetilde{\pi}(i)} = x_{\pi^l(i)} = \frac{\pm d_i d_{\pi(i)} d_{\pi^2(i)} \cdots d_{\pi^{l-1}(i)}}{\gamma_I^l}.$$

But now, simplifying this equation implies that the relation

$$\gamma_I = \sqrt[l-1]{\left|d_{\pi(i)} d_{\pi^2(i)} \cdots d_{\pi^{l-1}(i)}\right|}$$

must hold, i.e. for the index set $J := \left\{\pi(i), \pi^2(i), \ldots, \pi^{l-1}(i)\right\}$ we have $\gamma_J = \gamma_I$, a contradiction, so for different derangements we always get directionally distinct p-eigenvectors.

Notice that if any pair of the constructed vectors would originate from the same eigensubspace of a γ_I p-eigenvalue for some permutation, then all their infinitely many linear combinations should be p-eigenvectors as well, which clearly cannot happen, as we have constructed every possible solution under normalization. So the number of p-eigenvalues is the same as the number of constructed vectors. ■

Remarks. 1. By the proof, it is also clear that the γ_I values are the corresponding p-eigenvalues, and that the multiplicity of any γ_I is exactly

$$2^{|I|-1}\left[\frac{|I|!}{e}\right].$$

2. By saying that the assumption on the diagonal entries is mild, we mean that in fact it is true for *almost every* $\mathbb{R}^n$ vector of diagonal entries, we leave the details for our reader to work out. Nonetheless, a simple choice of integer diagonal entries would be a set of any n distinct primes $\{p_1,\ldots,p_n\}$: in this case, $\gamma_I=\gamma_J$ for two subsets of indices I,J would mean that we have

$$\left(\prod_{i\in I}|p_i|\right)^{|J|}=\left(\prod_{j\in J}|p_j|\right)^{|I|},$$

which cannot happen if $I\neq J$.

3. The number of p-eigenvalues (with multiplicity) in this case turns out to be comparable to the theoretical maximum. In fact, we have the following result.

Corollary 2. *For the class of restricted diagonal matrices of Theorem 3, the number of directionally distinct p-eigenvectors is asymptotically*

$$\frac{n!\cdot 2^{n-1}}{\sqrt{e}}.$$

Proof. We have that

$$\sum_{k=1}^{n}\binom{n}{k}2^{k-1}\left[\frac{k!}{e}\right]\approx\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}2^{k-1}\frac{k!}{e}=\frac{n!}{e}\sum_{k=1}^{n}\frac{2^{k-1}}{(n-k)!},$$

so with the change of index $k\to n-k$ this equals to

$$=\frac{n!\cdot 2^{n-1}}{e}\sum_{k=0}^{n-1}\frac{(1/2)^k}{k!}\approx\frac{n!\cdot 2^{n-1}}{e}\sqrt{e}.$$ ■

4. Finally, we note that if we omit our restriction for the diagonal entries, determining the exact number of p-eigenvalues becomes much more tricky.

- On one hand, there could be additional vectors which we didn't take into consideration. E.g. if we have disjoint I, J subsets for which $\gamma_I = \gamma_J$, then we have infinitely many directionally distinct x solutions originating from higher dimensional eigensubspace of γ_I for some permutation. To see this, may once again construct every x solution corresponding to the derangements of I, and also every x corresponding to derangements of J, but now we can see that for these two orthogonal vectors, every linear combination is also a solution, and so a p-eigenvector. We provide an example of this in our detailed discussion of the 3×3 case in the next section.

- Also, now choosing a subset I doesn't guarantee that all the derangements over I generate directionally distinct solutions, so there may be coinciding p-eigenvectors which the theorem's formula counts multiple times. We state without proof the fact that this can only happen if $n \geq 5$, and provide the following numerical example: Let $n = 5$, $I = \{1, 2, 3, 4, 5\}$, and
$$D = \operatorname{diag}\left(2, \frac{1}{2}, 1, 3, \frac{1}{3}\right) \in \mathbb{R}^{5\times 5}.$$
Now we can easily verify that the derangements (12345) and (12453) both generate the same solutions $x = (1, \pm 2, \pm 1, \pm 1, \pm 3)$.

3. Some constructive results for 3 × 3 diagonal matrices

It was already clear from Lemma 1, Theorem 1 and Example 4 of [3] that a 3×3 diagonal matrix with distinct non-zero elements in the diagonal has non-trivial p-eigenvectors, although still having a zero element. I.e. consider with $a, b, c > 0$ distinct real numbers
$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \text{ and } \begin{pmatrix} \pm\sqrt{b} \\ \pm\sqrt{a} \\ 0 \end{pmatrix}, \begin{pmatrix} \pm\sqrt{c} \\ 0 \\ \pm\sqrt{a} \end{pmatrix}, \begin{pmatrix} 0 \\ \pm\sqrt{c} \\ \pm\sqrt{b} \end{pmatrix}.$$
Then the listed vectors are p-eigenvectors of D with the associated p-eigenvalues $\sqrt{ab}$, $\sqrt{ac}$ and $\sqrt{bc}$ respectively.

However, the question was raised whether such p-eigenvectors exist, that do not have a zero element, i.e. do not lie on a coordinate plane, and as our results in the previous section verify, the answer is positive.

In this section, we investigate the possible ways of constructing p-eigenvectors based on those results.

3.1. Numerical investigation

Some numerical experiments were carried out in order to find such p-eigenvectors. The calculations and visualizations were done in the Octave numerical computing environment.*

Figure 1 shows two notable examples in the cases of $D = \mathrm{diag}(1,4,9)$ (first row) and $D = \mathrm{diag}(1,2,4)$ (second row). These images present the standard deviation of the set of fractions $\|Dv\|_p / \|v\|_p$ with some p values for different vectors (or directions) $v \in \mathbb{R}^3$ spanning an eight part of the (Euclidean) unit sphere, i.e. v having positive elements. The color assigned to a direction is darker when the standard deviation is larger; except for very small values, when the color black is applied. On the left-hand-side images, the parameters $p = 1, 2$ and 0.5 were used. This plot suggests that there exist p-eigenvectors without zeros elements, maybe even infinitely many, along a curve. But the right-hand-side image in the first case rather pinpoints only two separate directions with significantly small values, while in the second case we have infinitely many solutions indeed, as it will turn out later on. In this case the values for p were chosen as $0.5, 1, 2$ and ∞, and the function values were maximized by 0.1.

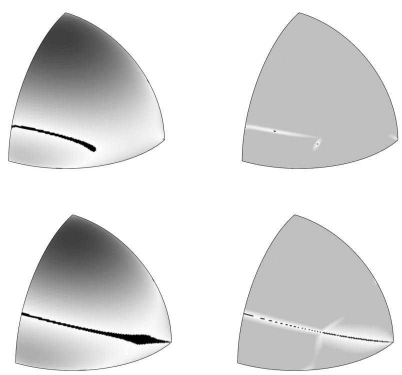

Figure 1. Results of numerical investigation about the existence of further p-eigenvectors in case of 3×3 diagonal matrices.

In the following, we will construct these p-eigenvectors in two different ways, and then conclude these observations.

*The GNU Octave is a free, open-source numerical computation and programming software, largely compatible with Mathworks' well-known Matlab environment. The author L. Lócsi had the honour to teach Matlab and numerical computing to bachelor students in applied mathematics together with *Prof. Gisbert Stoyan* in the spring semester of 2009/2010.

3.2. Construction by coordinates

Our first approach is based on the construction used to prove Theorem 3. In order to find a p-eigenvector of nonzero entries, we must choose the index set $\{1, 2, 3\}$ of all diagonal entries (as the entries left out would end up being zeroes), and choose any of its derangements, for now let's say $\pi = (123)$, i.e. we have $\pi(1) = 2$, $\pi(2) = 3$ and $\pi(3) = 1$.

Suppose we want to find the p-eigenvector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with all positive entries, so the system of equations $Dx = \gamma P_\pi x$ simplifies to

$$\begin{aligned} ax_1 &= \gamma x_2; \\ bx_2 &= \gamma x_3; \\ cx_3 &= \gamma x_1. \end{aligned}$$

Now we have $\gamma = \sqrt[3]{abc}$, and setting $x_1 = 1$ for this system of rank 2 gives us the solution

$$x = \frac{1}{\gamma^2}\begin{pmatrix}\gamma^2 \\ a\gamma \\ ab\end{pmatrix} = \frac{1}{\sqrt[3]{a^2b^2c^2}}\begin{pmatrix}\sqrt[3]{a^2b^2c^2} \\ \sqrt[3]{a^4bc} \\ \sqrt[3]{a^3b^3}\end{pmatrix}.$$

Since any vector of this direction is also a solution, we could multiply by $\sqrt[3]{bc^2}$ to obtain a more symmetric representative

$$x = \begin{pmatrix}\sqrt[3]{bc^2} \\ \sqrt[3]{ca^2} \\ \sqrt[3]{ab^2}\end{pmatrix}.$$

Remark that we could have started off with the other derangement (132) of the diagonal entries to obtain a directionally distinct solution. But instead, we will demonstrate a different construction to find the other solution.

3.3. Construction by eigenvectors

Based on Theorem 1, we shall now construct a p-eigenvector using regular eigenvectors. Again let $D = \operatorname{diag}(a, b, c)$ with $a, b, c \in \mathbb{R}^+$ distinct. Consider

$$P^\pm D \cdot v = \lambda \cdot v, \qquad (P^\pm \in S_n^\pm),$$

and find a real valued eigenvector of $P^\pm D$. Let us use

$$P^\pm = \begin{pmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0\end{pmatrix}, \quad \text{such that} \quad P^\pm D = \begin{pmatrix}0 & b & 0 \\ 0 & 0 & c \\ a & 0 & 0\end{pmatrix}.$$

Now

$$\begin{vmatrix} -\lambda & b & 0 \\ 0 & -\lambda & c \\ a & 0 & -\lambda \end{vmatrix} = -\lambda^3 + abc = 0,$$

having the only real solution $\lambda = \sqrt[3]{abc}$, and the corresponding homogeneous linear equation—calculation skipped—has the solutions as the scalar multiple of the particular representative vector

$$v = \begin{pmatrix} \sqrt[3]{b^2c} \\ \sqrt[3]{c^2a} \\ \sqrt[3]{a^2b} \end{pmatrix}$$

Remark that we could have started with a different permutation matrix $P^{\pm}$, which cycles the three elements in the other direction. This case we would arrive at the solution found in Section 3.2. In fact, these are the only two viable choices in order to get p-eigenvectors with all nonzero and positive entries.†

3.4. Conclusions for the 3 × 3 diagonal case

Based on the above calculations, we can make the following statement.

Corollary 3. *Let* $D = \operatorname{diag}(a, b, c)$ *with* $a, b, c \in \mathbb{R}$ *satisfying the condition of Theorem 3. There exist exactly* 8 *directionally distinct non-trivial* p*-eigenvectors of* D *having no zero elements. These may be grouped into* 2 *groups of* 4 *directions. Namely*

$$v_1 = \begin{pmatrix} \sqrt[3]{|bc^2|} \\ \pm\sqrt[3]{|ca^2|} \\ \pm\sqrt[3]{|ab^2|} \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} \sqrt[3]{|b^2c|} \\ \pm\sqrt[3]{|c^2a|} \\ \pm\sqrt[3]{|a^2b|} \end{pmatrix}.$$

The associated p*-eigenvalue is* $\sqrt[3]{abc}$*.*

Example 1. (Cf. [3], Example 4.) Consider the below matrix and vectors.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \text{ and } \begin{pmatrix} \pm 2 \\ \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 3 \\ 0 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 3 \\ \pm 2 \end{pmatrix}, \begin{pmatrix} \pm\sqrt[3]{324} \\ \pm\sqrt[3]{9} \\ \pm\sqrt[3]{16} \end{pmatrix}, \begin{pmatrix} \pm\sqrt[3]{144} \\ \pm\sqrt[3]{81} \\ \pm\sqrt[3]{4} \end{pmatrix}.$$

The listed vectors are p-eigenvectors of D with the associated p-eigenvalues $2, 3, 6$ and $\sqrt[3]{36} \approx 3.302$ (two times) respectively.

†A third, rather initial approach was to search for the p-eigenvector with its elements of the form $a^x b^y c^z$, permuting just the powers x, y, z for the coordinates. This is motivated by the fact, that in the 2×2 case, the p-eigenvectors also were expressed as products of some powers of the diagonal elements, with expression such as $\sqrt{ab} = a^{1/2}b^{1/2}$ involved [3]. A solution $x = 1/3$, $y = 2/3$, $z = 3/3$ could be even guessed. In general, we would arrive at a linear system of equations for the powers, which could be solved, and we got the same results.

To have a visual impression of these kind of p-eigenvectors, p-eigendirections, let us recall the definition of the $\mathcal{I}_p(A)$ induction set of a matrix A (see [3], Definition 1). Given a matrix $A \in \mathbb{R}^{n\times n}$ with $2 \le n \in \mathbb{N}$ and $p \in [1,\infty]$ or even $p \in (0,\infty]$, the set of points

$$\mathcal{I}_p(A) := \left\{ \frac{\|Ax\|_p}{\|x\|_p} \cdot \frac{x}{\|x\|_2} \in \mathbb{R}^n \,:\, 0 \neq x \in \mathbb{R}^n \right\} \subset \mathbb{R}^n$$

is called the *induction set* of A with parameter p. If $n = 3$, then this set is a surface symmetric with respect to the origin. These sets basically geometrically describe the effect of the multiplication with A on the norm of the vectors.

Figure 2 shows 6 induction surfaces of the diagonal matrix $\mathrm{diag}(1,2,3) \in$ $\in \mathbb{R}^{3\times 3}$. The applied values for p are 0.5, 1, 2, 4, 8 and ∞. One may observe the transition of the surface as the value of p increases (and imagine the smooth transition interpolated between the shown objects). Furthermore the p-eigenvectors, more precisely the intersection points of the p-eigendirections with these surfaces are also marked with small spheres. Dark gray spheres correspond to the trivial p-eigenvectors (on the axes), mid-gray color is used for the p-eigenvectors on the coordinate planes, and light gray is used to show the newly discovered p-eigenvectors with all non-zero coordinates. Observe that these points show the same constellation on all subfigures, these are common intersection points of all these surfaces.

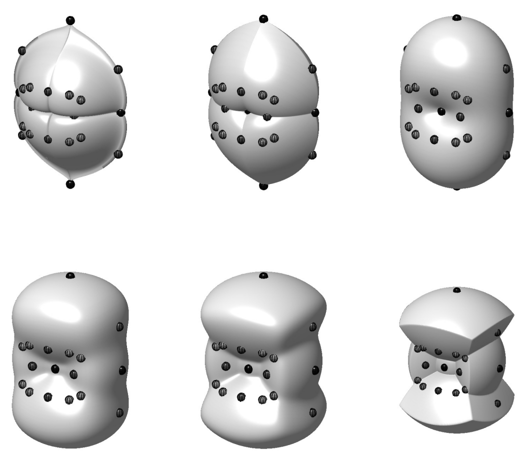

Figure 2. Some induction surfaces of the diagonal matrix $\mathrm{diag}(1,2,3) \in \mathbb{R}^{3\times 3}$ with different p-norms. The intersection points of the surfaces and the p-eigendirections are marked with small spheres.

Thus we may formulate the next Corollary.

Corollary 4. *In case of the restricted diagonal matrix $A \in \mathbb{R}^{3\times 3}$ of Corollary 3, the set*

$$S := \bigcap_{p\in[1,\infty]} \mathcal{I}_p(A) \subset \mathbb{R}^3$$

contains exactly 34 distinct elements (cf. Theorem 3), namely

$$S = \left\{ \begin{pmatrix} \pm a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pm c \end{pmatrix}, \begin{pmatrix} \pm b' \\ \pm a' \\ 0 \end{pmatrix}, \begin{pmatrix} \pm c' \\ 0 \\ \pm a' \end{pmatrix}, \begin{pmatrix} 0 \\ \pm c' \\ \pm b' \end{pmatrix}, \begin{pmatrix} \pm u_1 \\ \pm u_2 \\ \pm u_3 \end{pmatrix}, \begin{pmatrix} \pm v_1 \\ \pm v_2 \\ \pm v_3 \end{pmatrix} \right\},$$

where $a' = r_{ab} \cdot \sqrt{|a|}$, $b' = r_{ab} \cdot \sqrt{|b|}$, $r_{ab} = \sqrt{|ab| / (|a| + |b|)}$ (and similarly for the pairs a, c and b, c), and the u and v values in the vectors as presented in Corollary 3 scaled to have the length of their corresponding p-eigenvalue.

Example 2. As a final note, we present an example of a diagonal matrix not satisfying the condition of Theorem 3: for $D = \operatorname{diag}(1, 2, 4)$, we have that $2 = \sqrt{1 \cdot 4}$, i.e. we have $\gamma_I = \gamma_J = 2$ for the disjoint index sets $I = \{1, 3\}$ and $J = \{2\}$, and since both of them have a single derangement, we can construct the p-eigenvectors

$$x_I = \begin{pmatrix} 2 \\ 0 \\ \pm 1 \end{pmatrix}, \qquad \text{and} \qquad x_J = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since they satisfy $Dx_I = 2 \cdot Px_I$ and $Dx_J = 2 \cdot Px_J$ for the permutation $P = (13)(2)$, so does any linear combination of the two, yielding us infinitely many nontrivial p-eigenvectors $c_I x_I + c_J x_J$, $(c_I, c_J \in \mathbb{R})$. We visualize this phenomena on Figure 3. The applied values for p are 1, 2 and ∞.

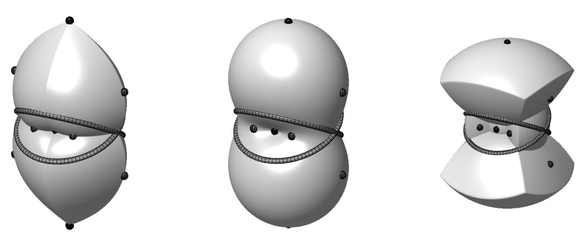

Figure 3. Some induction surfaces of the diagonal matrix $\operatorname{diag}(1, 2, 4) \in \mathbb{R}^{3\times 3}$ with different p-norms.

4. Conclusions and further research

In this paper, we managed to answer most of the questions raised in the original paper [3]. Nonetheless, some open problems regarding p-eigenvalues and eigenvectors would still be interesting to investigate. Here we collect a handful of possible research directions.

- Prove or disprove Conjecture 1. We suspect that the statement is true, but may be very difficult to prove either constructively or existentially. An easier problem could be to find a class of matrices for which the number of p-eigenvalues (with multiplicity) is closer to the theoretical maximum compared to the class of restricted diagonal matrices of Theorem 3.
- The algorithm for constructing p-eigenpairs based on regular ones becomes very inefficient as n increases. We may want to find a method of lower computational complexity.
- At this point, it seems the idea of p-eigenpairs is interesting for real vectors only. But still, the question of finding an appropriate complex analogue/generalization remains an open problem.

The scripts and other software created during the research are available to download at:

`http://numanal.inf.elte.hu/~locsi/indsets/`

The reader is encouraged to try these programs. E.g. on the computer screens it is possible to grab a plotted induction surface and rotate it in 3 dimensions, observing its structure (and the noted special points and curves) interactively.

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