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AN INEQUALITY FOR THE POWERS OF TRIANGLE SIDES

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Dedicated to the memory of Professor Gisbert Stoyan

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Abstract. An inequality will be presented for areas of triangles with side lengths given by the powers of a given triple. For small powers we provide separate proofs, for some further cases computer aided proofs are proposed. Finally we prove our statement in full.

1. Introduction

Inequalities for triangle areas which contain some powers of the sides are not uncommon, see e.g. the Area section in [1], especially Ono's and Weitzenböck's inequality. On further investigation we found an inequality which compares the area of a triangle with sides (a, b, c) with the area of the triangle having sides (a^k, b^k, c^k), $k \geq 2$. Hence Heron's formula

$$\Delta(a, b, c) = \tfrac{1}{4}\sqrt{(a+b-c)(b+c-a)(c+a-b)(a+b+c)}$$

for the area of a triangle will play a central role in our treatment. In what follows we will employ the square of this quantity.

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We introduce the notations

$$A_k = b^k + c^k - a^k, \quad B_k = a^k + c^k - b^k, \quad C_k = a^k + b^k - c^k,$$

and their sum

$$S_k = A_k + B_k + C_k = a^k + b^k + c^k.$$

Here a, b, c are positive numbers, and k is a natural number. With these quantities we define

$$G_k = \tfrac{1}{3} A_k B_k C_k S_k,$$

a scalar multiple of the squared area of a (possibly existing) triangle.

Remark 1.1. Although the degree of G_k is $4k$, this number can be halved by observing that by virtue of the identity

$$A_k B_k C_k S_k = A_{2k} B_{2k} + B_{2k} C_{2k} + C_{2k} A_{2k} \tag{1.1}$$

we can replace (a^2, b^2, c^2) by (a, b, c). More concretely, define g_k via

$$g_k(a, b, c) = G_k(\sqrt{a}, \sqrt{b}, \sqrt{c}), \tag{1.2}$$

then g_k is a polynomial of degree $2k$ in three variables.

Now we can formulate a preparative lemma and the main result.

Lemma 1.1. *If for some positive numbers a, b, c and for $n \in \mathbb{N}$ the triangle (a^n, b^n, c^n) exists, then the triangles (a^k, b^k, c^k), $1 \le k \le n$ also exist.*

Theorem 1.1. *Assume the triangle (a^k, b^k, c^k) exists for some k. Then*

$$g_k \le g_1^k.$$

In fact, $g_k \le g_1 g_{k-1}$ is also true. Equality holds if and only if all variables are equal: $a = b = c$. As a consequence, for the areas we have

$$\Delta(a^k, b^k, c^k) \le \left(\tfrac{4}{\sqrt{3}}\right)^{k-1} \Delta^k(a, b, c).$$

The theorem will be proven in the next sections. In order to emphasize the complexity and significance of the nonnegativity problem for polynomials, we give four proofs in case of $k = 2$, two for $k = 3$, and one for $k = 4$. If the degree is larger, we have a method applicable for all concrete k, using a growing amount of calculations. Finally, a computer-free, concise proof will be given for arbitrary k.

To begin with, we prove Lemma 1.1.

Proof of Lemma 1.1. Without loss of generality assume $a \ge b \ge c$, which means $a^k + b^k \ge c^k$ and $a^k + c^k \ge b^k$ holds for any k, thus it remains to show that $b^k + c^k \ge a^k$. By assumption $b^n + c^n \ge a^n$, and consequently $(b/a)^n + (c/a)^n \ge 1$, where both terms on the left are decreasing in n, i.e. $(b/a)^k + (c/a)^k \ge 1$ holds for $k \le n$. Rearranging gives $b^k + c^k \ge a^k$, proving our result. ∎

2. The proofs for $k = 2$

Proof 1. Elementary identities will be used. We have

$$\begin{aligned}
9(G_1^2 - G_2) &= (A_2B_2 + B_2C_2 + C_2A_2)^2 - 3A_2B_2C_2S_2 = \\
&= A_2^2B_2^2 + B_2^2C_2^2 + C_2^2A_2^2 + 2A_2B_2C_2S_2 - 3A_2B_2C_2S_2 = \\
&= A_2^2B_2^2 + B_2^2C_2^2 + C_2^2A_2^2 - A_2B_2C_2(A_2 + B_2 + C_2) = \\
&= \tfrac{1}{2}\left[A_2^2(B_2 - C_2)^2 + B_2^2(A_2 - C_2)^2 + C_2^2(A_2 - B_2)^2\right],
\end{aligned}$$

where we applied the identity $A_2 + B_2 + C_2 = S_2$. ■

Proof 2. Sum of squares representation. Let us denote

$$\alpha = a^2 + bc, \quad \beta = b^2 + ac, \quad \gamma = c^2 + ab,$$

then we have

$$g_1^2 - g_2 = \tfrac{1}{9}\left[(\alpha + \beta - 2\gamma)^2 + 3(\alpha - \beta)^2\right].$$

Note that our MATLAB program yields other solutions too, e.g.

$$g_1^2 - g_2 = \tfrac{1}{63}\left[3(\alpha + 2\beta - 3\gamma)^2 + (-5\alpha + 4\beta + \gamma)^2\right],$$

which, interestingly, also depends only on (α, β, γ). ■

Proof 3. Application of the Schur inequality. Observe that the polynomial $\frac{9}{4}(g_1^2 - g_2)$ assumes

$$a^4 + b^4 + c^4 - a^3b - a^3c - ab^3 - b^3c - ac^3 - bc^3 + a^2bc + ab^2c + abc^2. \tag{2.1}$$

Since this is a homogeneous symmetric polynomial, it can be written as a linear combination of the basis elements $m_4, m_{31}, m_{22}, m_{211}$, the monomial symmetric polynomials (see [3]) of degree 4. For convenience, we specify them here.

For nonnegative variables (a, b, c) and naturals (p, q, r), the corresponding monomial symmetric polynomial is the sum of all monomials with exponents ranging over all permutations of the triple (p, q, r) :

$$m_{p,q,r} = a^pb^qc^r + a^pb^rc^q + a^qb^pc^r + a^qb^rc^p + a^rb^pc^q + a^rb^qc^p.$$

Note the outcome of this term-normalization: for $a = b = c$ we always have $m_{p,q,r} = 6\,a^{p+q+r}$. We order the exponents to be monotone decreasing and omit possible zeroes, hence we write for instance $m_{3,1,0} = m_{3,1}$, except for the (likewise frequently used) bracket notation

$$[p, q, r] \equiv m_{p,q,r},$$

where normally all exponents are indicated like e.g. $[3,1,0]$.

In our case (three variables, fourth degree) they are

$$\begin{aligned} m_4 =& [4,0,0] = 2(a^4+b^4+c^4), \\ m_{3,1} =& [3,1,0] = a^3b+a^3c+b^3a+b^3c+c^3a+c^3b, \\ m_{2,2} =& [2,2,0] = 2(a^2b^2+b^2c^2+c^2a^2), \\ m_{2,1,1} =& [2,1,1] = 2(a^2bc+b^2ac+c^2ab). \end{aligned}$$

Schur's inequality ([5], Theorem 3) on monomials asserts that

$$\mathrm{Sch}_2(p,q) \equiv [p+2q,0,0] - 2[p+q,q,0] + [p,q,q] \geq 0 \tag{2.2}$$

is valid for all positive p and q. Choosing $p=2$ and $q=1$ just gives the desired result,

$$m_4 - 2m_{3,1} + m_{2,1,1} \geq 0. \qquad \blacksquare$$

Proof 4. Difference Substitution (DS). This means substituting (as in [4])

$$a = x, \quad b = x+y, \quad c = x+y+z, \tag{2.3}$$

which is equivalent to

$$x = a, \quad y = b-a, \quad z = c-b,$$

hence it holds

$$x \geq 0,\ y \geq 0,\ z \geq 0 \iff 0 \leq a \leq b \leq c.$$

Since the polynomial at issue is symmetric, it suffices to perform only one substitution of this kind instead of checking all possible 3! permutations. If all the coefficients are positive, we are done. In our case the quartic (2.1) becomes

$$x^2y^2 + x^2yz + x^2z^2 + 4xyz^2 + 2xz^3 + 3y^2z^2 + 3yz^3 + z^4,$$

proving that the original polynomial is indeed nonnegative for all $a,b,c \geq 0$. $\blacksquare$

3. The proofs for $k=3$ and $k=4$

To get on the case $k=3$ we could work with the polynomial $g_1^3 - g_3$, however it turns out that $g_1g_2 - g_3$ is easier to handle, and the nonnegativity of the latter obviously implies the nonnegativity of the former.

Theorem 3.1. $g_3 \leq g_1g_2$.

Proof 1. We make use of the following theorem of V. Cirtoaje [2, Theorem 1.1]: *Let $f_6(x, y, z)$ be a symmetric homogeneous polynomial of degree six which has the highest coefficient $A \leq 0$. The inequality $f_6(x; y; z) \geq 0$ holds for all nonnegative real numbers $x; y; z$ if and only if*

$$f_6(x, 1, 1) \geq 0 \text{ and } f_6(0, y, z) \geq 0 \text{ for all } x, y, z \geq 0.$$

Note that the highest coefficient A above is the coefficient of r^2 in

$$f_6(x, y, z) = Ar^2 + h_1(p, q) + h_2(p, q),$$

obtained after rewriting the original form using the substitution

$$p = x + y + z,\ q = xy + yz + zx,\ r = xyz. \tag{3.1}$$

Let us check the assumption of this theorem! For

$$\begin{aligned}
f_6 &= 9(g_1 g_2 - g_3) = \\
&= 4a^6 - 2a^5b - 2a^5c - a^4b^2 - 2a^4bc - a^4c^2 - 2a^3b^3 + 4a^3b^2c + 4a^3bc^2 - \\
&\quad - 2a^3c^3 - a^2b^4 + 4a^2b^3c - 6a^2b^2c^2 + 4a^2bc^3 - a^2c^4 - 2ab^5 - 2ab^4c + \\
&\quad + 4ab^3c^2 + 4ab^2c^3 - 2ab^4 - 2ac^5 + 4b^6 - 2b^5c - b^4c^2 - 2b^3c^3 - \\
&\quad - b^2c^4 - 2bc^5 + 4c^6
\end{aligned}$$

the polynomial transformed through (3.1) is

$$-9r^2 + (26p^3 - 50pq)r + (4p^6 - 26p^4q + 43p^2q^2 - 12q^3),$$

thus $A = -9 < 0$. Hence it is enough to examine the nonnegativity of our polynomial at the values $(a, 1, 1)$ and $(0, b, c)$. For these we obtain

$$4a^3(a+1)(a-1)^2 \quad \text{and} \quad 4b^4 + 6b^3c + 7b^2c^2 + 6bc^3 + 4c^4.$$

Both are nonnegative for all nonnegative a, b, c, thus the proof is complete. ■

Poof 2. We use again the monomial symmetric polynomials. In three variables, for degree 6 these are

$$\begin{aligned}
m_6 &= 2(a^6 + b^6 + c^6), \\
m_{5,1} &= a^5(b+c) + b^5(a+c) + c^5(a+b), \\
m_{4,2} &= a^4(b^2+c^2) + b^4(a^2+c^2) + c^4(a^2+b^2), \\
m_{4,1,1} &= 2abc(a^3 + b^3 + c^3), \\
m_{3,3} &= 2(a^3b^3 + b^3c^3 + c^3a^3), \\
m_{3,2,1} &= abc\big(a^2(b+c) + b^2(a+c) + c^2(a+b)\big), \\
m_{2,2,2} &= 6a^2b^2c^2.
\end{aligned}$$

In this basis our polynomial has the form

$$9(g_1g_2 - g_3) = 2m_6 - 2m_{5,1} - m_{4,2} - m_{4,1,1} - m_{3,3} + 4m_{3,2,1} - m_{2,2,2}.$$

We succeeded in decomposing this into three smaller components

$$\begin{aligned} s_1 &= m_6 - 2m_{5,1} + m_{4,1,1}, \\ s_2 &= m_6 - 2m_{4,2} - m_{4,1,1} + 2m_{3,2,1}, \\ s_3 &= m_{4,2} - m_{4,1,1} - m_{3,3} + 2m_{3,2,1} - m_{2,2,2}, \end{aligned}$$

all three of which are nonnegative on $\mathbb{R}^3_+$. As for the first form s_1, one can apply Schur's (two-parametric) inequality with $p = 4, q = 1$, i.e. $s_1 = \mathrm{Sch}_2(4,1)$.

The second form, s_2 can be handled by means of Schur's one-parametric inequality ([5], Corollary 4):

$$\text{(3.2)} \quad \mathrm{Sch}(t) = x^t(x-y)(x-z) + y^t(y-x)(y-z) + z^t(z-x)(z-y) \geq 0,$$

valid for all

$$x \geq 0, \; y \geq 0, \; z \geq 0, \; \text{and } t \geq 0.$$

Namely, we take $t = 3$ and multiply the left hand side by $(x+y+z)$. After expanding, we get s_2, i.e. $s_2 = \mathrm{Sch}(3)(a+b+c)$.

Finally, the third form is the expansion of the Vandermonde-like product:

$$(a-b)^2(b-c)^2(c-a)^2 = s_3.$$

In consequence, equality $9(g_1g_2 - g_3) = s_1 + s_2 + s_3$ proves the theorem. ■

Now we turn to the case $k = 4$. We prove the following statement.

Theorem 3.2. $g_4 \leq g_1g_3$.

Proof. Since all three terms in the decomposition

$$\begin{aligned} 9(g_1g_3 - g_4) = {} & 2\,\mathrm{Sch}(5)\,(a+b+c) + 2\,\mathrm{Sch}(4)\,(a^2+b^2+c^2) + \\ & + \big((a-b)(b-c)(c-a)(a+b+c)\big)^2 \end{aligned}$$

are nonnegative, the theorem follows. (Note that Schur's function (3.2) is applied here with the arguments (a,b,c) instead of (x,y,z) as above). ■

Corollary 3.1. *Putting together Theorem 1.1, Theorem 3.1 and Theorem 3.2 (i.e. the inequalities valid for $k = 2,3,4$) we obtain $g_4 \leq g_1^4$.*

4. The (computer aided) proof for some further values of k

Here we exhibit a possible proof using the method of difference substitution (2.3), capable for *all* moderate sizes (this restriction is necessary owing to the growing amount of calculations). Since these calculations are performed quite easily, using e.g. the Maple command `subs(a=x,b=x+y,c=x+y+z,sg);` we give only the number of terms in the expansions for polynomials $g_1 g_{k-1} - g_k$. Observe that, by induction, if $g_{k-1} \le g_1^{k-1}$, and $g_1 g_{k-1} - g_k \ge 0$, then

$$g_k \le g_1 g_{k-1} \le g_1 g_1^{k-1} = g_1^k.$$

The following table gives the number of terms for the (original) polynomials $g_1 g_{k-1} - g_k$ and that for the polynomials transformed by DS - the difference substitution.

k	2	3	4	5	6	7
original	12	28	36	36	36	36
transformed	8	19	34	53	76	103

It is seen that apart from the first two cases the number of terms in the polynomials $g_1 g_{k-1} - g_k$ is constant! As for the transformed polynomials - which all have positive coefficients - the sequence comes from the formula $2k^2 + k - 2$.

We also display the less fortunate results for $g_1^k - g_k$, demonstrating why the above choice is preferred. The 'lengths' of the transformed polynomials are the same, however, the size of the form $g_1^k - g_k$ grows with k :

k	2	3	4	5	6	7
original	12	28	45	66	91	120
transformed	8	19	34	53	76	103

Finally we describe the proof for the general case.

5. The proof of Theorem 1.1 for arbitrary k

First we observe that the polynomial $g_1 g_{k-1} - g_k$, where g_k is defined via (1.2), can be represented as

$$f = 2m_{2k} - 2m_{2k-1,1} + m_{2k-2,2} - m_{2k-2,1,1} - \\ - 2m_{k+1,k-1} - m_{k,k} + 4m_{k,k-1,1} - m_{k-1,k-1,2}. \tag{5.1}$$

Owing to the fixed number of terms (always 8, independently of k), this can be proved immediately. Next we split f into four terms

$$\begin{aligned}
f_1 =& m_{2k-2,2} - m_{2k-2,1,1} - m_{k,k} + 2m_{k,k-1,1} - m_{k-1,k-1,2},\\
f_2 =& m_{2k} - 2m_{2k-2,2} - m_{2k-2,1,1} + 2m_{2k-3,2,1},\\
f_3 =& m_{2k} - 2m_{2k-1,1} + m_{2k-2,1,1},\\
f_4 =& 2m_{2k-2,2} - 2m_{2k-3,2,1} - 2m_{k+1,k-1} + 2m_{k,k-1,1}.
\end{aligned}$$

The equality $f = f_1 + f_2 + f_3 + f_4$ can be checked by means of the table

	1	2	3	4	5	6	7	8	9
f	2	-2	1	-1	**0**	-2	-1	4	-1
f_1	0	0	1	-1	0	0	-1	2	-1
f_2	1	0	-2	-1	2	0	0	0	0
f_3	1	-2	0	1	0	0	0	0	0
f_4	0	0	2	0	-2	-2	0	2	0

where the coefficients correspond to the basis elements

$$\begin{aligned}
&1 : m_{2k},\ 2 : m_{2k-1,1},\ 3 : m_{2k-2,2},\ 4 : m_{2k-2,1,1},\ \mathbf{5} : m_{2k-3,2,1},\\
&6 : m_{k+1,k-1},\ 7 : m_{k,k},\ 8 : m_{k,k-1,1},\ 9 : m_{k-1,k-1,2}.
\end{aligned}$$

Notice the (bold) zero in the row of f, belonging to the monomial symmetric polynomial $m_{2k-3,2,1}$. Although a corresponding term is not present in the original f, it is contained in both f_2 and f_4.

We prove that all four components are nonnegative over $\mathbb{R}^3_+$. It holds that

$\mathbf{f_1}$: The first summand is a square of a k-th degree polynomial:

$$f_1 = \big(h_{k-3}(a-b)(b-c)(c-a)\big)^2,$$

where $h_i = h_i(a,b,c)$ denotes the i-th complete homogeneous symmetric polynomial in three variables, see 4.4 in [3].

$\mathbf{f_2}$: For the second polynomial we have

$$f_2 = 2\,(a+b+c)\,\mathrm{Sch}(2k-3),$$

where Sch is defined in (3.2).

$\mathbf{f_3}$: The third summand is easy to handle by the help of (2.2). We have

$$f_3 = \mathrm{Sch}_2(2k-2,1).$$

f$_4$: After a little calculation we get

$$\begin{aligned}\tfrac{1}{2}f_4 = (a-b)c^2\Big(a^k(a^{k-3}-c^{k-3}) + b^k(c^{k-3}-b^{k-3})\Big)+\\ +(b-c)a^2\Big(b^k(b^{k-3}-a^{k-3}) + c^k(a^{k-3}-c^{k-3})\Big)+\\ +(c-a)b^2\Big(c^k(c^{k-3}-b^{k-3}) + a^k(b^{k-3}-a^{k-3})\Big).\end{aligned}$$

To be able to apply the nonnegativity of (3.2), more precisely, its generalization, we factorize the differences above. Denoting $j = k-3$, we have e.g.

$$\frac{a^j - c^j}{a-c} = \left(\sum_{i=0}^{j-1} a^i c^{j-1-i}\right) \equiv \Sigma_{ac}^{(j)} = \Sigma_{ac}\,.$$

Using a similar modification for the remaining five differences we arrive at

$$\begin{aligned}\tfrac{1}{2}f_4 = (a-b)(a-c)\,a^k\Big(c^2\Sigma_{ac} + b^2\Sigma_{ab}\Big)+\\ +(b-c)(b-a)\,b^k\Big(a^2\Sigma_{ab} + c^2\Sigma_{bc}\Big)+\\ +(c-a)(c-b)\,c^k\Big(b^2\Sigma_{bc} + a^2\Sigma_{ac}\Big).\end{aligned}$$

At this point we utilize the Vornicu–Schur, or generalized Schur inequality [6, Theorem 1]: *Let $a; b; c$ be three reals, and let $x; y; z$ be three nonnegative reals. Then, the inequality*

$$x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b) \geq 0$$

holds if one of the following (sufficient) conditions is fulfilled:

(a) *We have $a \geq b \geq c$ and $x \geq y$,*

(b) *etc.*

There are ten further conditions in that theorem, however, we need only (a). Note that due to symmetry, $a \geq b \geq c$ can be assumed. We have to show that

$$a^k\Big(c^2\Sigma_{ac} + b^2\Sigma_{ab}\Big) \geq b^k\Big(a^2\Sigma_{ab} + c^2\Sigma_{bc}\Big),$$

which follows from

$$a^k c^2\Sigma_{ac} \geq b^k c^2 \Sigma_{bc} \quad \text{and} \quad a^k b^2 \Sigma_{ab} \geq b^k a^2 \Sigma_{ab}.$$

The first is true due to $a \geq b$, while the second can be divided by a^2b^2, giving

$$a^{k-2}\Sigma_{ab} \geq b^{k-2}\Sigma_{ab}, \tag{5.2}$$

which also holds true for $k \geq 2$.

With this the proof for $f_4 \geq 0$ and hence the proof of the inequality in Theorem 1.1 is complete.

As for equality, the "if" part immediately follows from the fact that in case of $a = b = c$ we have $g_k = a^{2k}$. (Note the additional benefit of the term-normalization: to check if (5.1) vanishes for these values, it is enough to show that the coefficients sum up to zero!)

As regards the "only if" part, it suffices to consider f_1, keeping in mind that the complete homogeneous symmetric polynomials are positive definite, cf. [9], or [10]. The proof is herewith complete. ■

Corollary 5.1. *The proof for $f_4 \geq 0$ makes possible a slight generalization. Replacing in f_4 the squares (a^2, b^2, c^2) by (a^s, b^s, c^s) yields a three-parameter homogeneous symmetric polynomial*

$$\begin{aligned} F_4(k,j,s) = (a-b)(a-c)\,a^k\left(c^s\Sigma_{ac}^{(j)} + b^s\Sigma_{ab}^{(j)}\right) + \\ + (b-c)(b-a)\,b^k\left(a^s\Sigma_{ab}^{(j)} + c^s\Sigma_{bc}^{(j)}\right) + \\ + (c-a)(c-b)\,c^k\left(b^s\Sigma_{bc}^{(j)} + a^s\Sigma_{ac}^{(j)}\right), \end{aligned}$$

which form coincides with

$$m_{k+j+1,s} - m_{k+1,j+s} + m_{k,j+s,1} - m_{k+j,s,1}. \tag{5.3}$$

(*The special case above corresponds to the choice $j = k-3$, $s = 2$.*) *The relevant part of proving $F_4(k,j,s) \geq 0$ is the analogue of* (5.2) *with s instead of 2, i.e.*

$$a^{k-s}\Sigma_{ab} \geq b^{k-s}\Sigma_{ab},$$

which obviously holds for $s \leq k$.

Remark 5.1. More is true: $F_4(k,j,s) \geq 0$ for $s \leq k+1$. To see this, observe that for $s = k+1$ (5.3) reduces to $m_{k+j+1,k,1} - m_{k+j,k+1,1}$, in which case Muirhead's inequality (see [7] or [8]) can be applied. Accordingly, for the monotone decreasing sequences $(p_i)_1^3$ and $(q_i)_1^3$ the inequality

$$m_{p_1,p_2,p_3} \geq m_{q_1,q_2,q_3}$$

holds, if the majorization inequalities

$$p_1 \geq q_1, \quad p_1 + p_2 \geq q_1 + q_2, \quad p_1 + p_2 + p_3 = q_1 + q_2 + q_3$$

are fulfilled - which is now the case, for we have

$$k+j+1 > k+j,\ 2k+j+1 = 2k+j+1,\ 2k+j+2 = 2k+j+2.$$

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