

ON SOME UNIFORMLY DISTRIBUTED FUNCTIONS ON THE SET OF SHIFTED PRIMES

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Abstract. The uniformly distributed functions on the set of shifted primes is defined and a theorem is proved.

1. Introduction

Let, as usual, \mathbb{N} , \mathbb{Z} , \mathbb{R} be the set of positive integers, integers and real numbers, respectively.

A positive arithmetical function h is said to be uniformly distributed, if

$$\frac{1}{x} \sum_{h(n) \leq x} 1 \rightarrow A, \quad \text{where } A > 0.$$

(see P. Erdős [4])

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There are several papers on this topics [1]–[8]. It is quite natural to extend the notation of uniformity for subsets of integers. Let \mathcal{B} be an infinite sequence of integers,

$$\mathcal{B}(x) := \#\{b \leq x \mid b \in \mathcal{B}\}.$$

We say that a function $h : \mathcal{B} \rightarrow (0, \infty)$ is uniformly distributed on \mathcal{B} , if

$$\lim_{x \rightarrow \infty} \frac{1}{\mathcal{B}(x)} \sum_{\substack{h(m) \leq x \\ m \in \mathcal{B}}} 1 = A, \quad \text{where } A > 0.$$

In this short paper we shall consider the case when \mathcal{B} is the set of shifted primes.

2. Formulation of the theorem

Let \mathcal{P} be the set of primes, g be a multiplicative function, $g(n) > 0$ for every $n \in \mathbb{N}$. Let $f(n) = \log g(n)$.

Conditions:

(C1): There exists a constant $B > 0$ for which

$$(2.1) \quad \frac{1}{(\log \log n)^B} \leq g(n) \leq (\log \log n)^B \quad \text{if } n > n_0, \quad n_0 \in \mathbb{N}.$$

(C2):

$$(2.2) \quad f(q^r)(\log q^r)^C \rightarrow 0 \quad (q^r \rightarrow \infty, \quad q \in \mathcal{P})$$

holds for every fixed C .

Let

$$S(x) := \#\{p \in \mathcal{P} \mid (p+1)g(p+1) \leq x\}.$$

Theorem 1. *Under the conditions (C1) and (C2), we have*

$$\lim_{x \rightarrow \infty} \frac{S(x)}{\pi(x)} = C_g,$$

where

$$C_g = \prod_{q \in \mathcal{P}} \xi_q,$$

$$\xi_2 = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha g(2^\alpha)}, \quad \xi_q = \left(1 - \frac{1}{q-1}\right) \left(1 + \sum_{l=1}^{\infty} \frac{q-1}{(q-2)q^l g(q^l)}\right) \quad \text{if } q > 2.$$

Here q runs over the set of primes.

Remarks. 1) The conditions (C1) and (C2) hold in particular for $g(n) = \left(\frac{\varphi(n)}{n}\right)^\lambda, \left(\frac{\sigma(n)}{n}\right)^\lambda$, where φ is Euler's totient function and σ the sum of divisors functions, with $\lambda \in \mathbb{R}$, and for many other functions.

2) We shall use the Bombieri–Vinogradov inequality. A weaker version is enough for our purpose. With the same method we could prove a more general theorem.

Theorem 2. *Let a_1, \dots, a_k be distinct non zero integers, g_1, \dots, g_k be multiplicative functions for which the conditions (C1) and (C2) hold. Let*

$$s(n) = g_1(n + a_1) \cdots g_k(n + a_k).$$

Then

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{ps(p) \leq x} 1 = C_s, \quad C_s \neq 0.$$

We shall not prove this assertion.

3) We are unable to prove that

$$(2.3) \quad \lim_{x \rightarrow \infty} \frac{1}{\pi\left(\frac{x}{\log x}\right)} \sum_{p\tau(p+1) \leq x} 1 = C, \quad C \neq 0,$$

where $\tau(n)$ is the number of divisors of n .

3. Lemma

$$\text{Let } \text{Li}(x) := \int_2^x \frac{dt}{\log t} \text{ and } \pi(u, m, l) := \#\{p \leq u \mid p \equiv l \pmod{m}\}.$$

Lemma 1. (Bombieri–Vinogradov) *Let $\delta > 0$ be fixed, A be an arbitrary positive constant. Then*

$$(3.1) \quad \sum_{m \leq X^{1/2-\delta}} \max_{(l,m)=1} \max_{u \leq X} \left| \pi(u, m, l) - \frac{\text{Li}(u)}{\varphi(m)} \right| \leq C(\delta) \frac{X}{(\log X)^A}.$$

(See Theorem 17.6 in H. Iwaniec and E. Kowalski [6]).

4. Proof of Theorem 1

I. Let

$$y = \frac{1}{16} \log x \quad \text{and} \quad Q = \prod_{\substack{2 < \pi \leq y \\ \pi \in \mathcal{P}}} \pi.$$

Let $\mathcal{P}_{\alpha, D}$ be the set of those primes p for which $p + 1 \equiv 0 \pmod{2^\alpha D}$, and $\left(\frac{p+1}{2^\alpha D}, 2Q\right) = 1$.

We shall consider only those D 's all prime factors of which divide Q .

II. Let $w = 2(\log \log x)^B$. From (C1) we have

$$\begin{aligned} \text{if } (p+1)g(p+1) \leq x, & \text{ then } p \leq 2xw, \\ \text{if } (p+1)g(p+1) > x, & \text{ then } p > \frac{x}{2w}. \end{aligned}$$

III. Let

$$\Omega_Q(p+1) = \sum_{\substack{p^r | p+1 \\ q | Q}} 1.$$

Then, with a suitable constant c ,

$$\begin{aligned} \sum_{p \leq 2xw} \Omega_Q(p+1) &\leq \sum_{q | Q} \sum_{r \geq 1} \pi(xw, q^r, -1) \leq \\ (4.1) \quad &\leq \frac{cxw}{\log x} \sum_{\substack{r \geq 1 \\ q | Q}} \frac{1}{\varphi(q^r)} \leq \frac{3Ccxw \log \log y}{\log x}. \end{aligned}$$

Here we used the Brun–Titchmarsh inequality. Thus

$$(4.2) \quad \#\{p \leq xw \mid \Omega_Q(p+1) > w(\log \log y)^2\} = o(\pi(x)).$$

Let $\mathcal{R}^{(1)}$ be the set of primes listed in (4.2). For the other primes p ,

$$\begin{aligned} \prod_{\substack{p^r | p+1 \\ q | Q}} q^r &\leq y^{w(\log \log y)^2} = \exp(w(\log y)(\log \log y)^2) \leq \\ (4.3) \quad &\leq \exp(c_2(\log \log y)^{B+1}) =: T_x. \end{aligned}$$

Similarly,

$$(4.4) \quad \pi(xw, 2^\nu, -1) \leq \frac{cxw}{2^\nu \log x} \quad \text{if } 2^\nu \leq \sqrt{x}.$$

Let ν_x be such an integer for which $2^{\nu_x} \leq w^2 < 2^{\nu_x+1}$. Let

$$\mathcal{R}^{(2)} = \{p \mid p \leq xw, 2^{\nu_x} \mid p+1\}.$$

Then

$$\mathcal{R}^{(2)} = o(\pi(x)).$$

IV. Let

$$f_y(p+1) = \sum_{\substack{q^r \geq y \\ q^r \mid p+1}} f(q^r).$$

By using the Brun–Titchmarsh inequality, we have

$$\sum_{p \leq xw} |f_y(p+1)| \leq \frac{cxw}{\log x} \sum_{y < q^r \leq \sqrt{x}} \frac{|f(q^r)|}{q} + O(x^{\frac{4}{5}}) \leq \frac{cxw}{\log x} \frac{1}{(\log \log x)^C},$$

where C is an arbitrary constant and c depends on C . Consequently

$$(4.5) \quad \frac{1}{\pi(x)} \#\{p \leq xw \mid |f_y(p+1)| > (\log \log x)^{-C'}\} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

C' is an arbitrary large constant.

Let $\mathcal{R}^{(3)}$ be the set of prime counted in (4.5). Then $\#\mathcal{R}^{(3)} = o(\pi(x))$.

V. Let

$$(4.6) \quad \Pi(X, 2^\alpha D) = \#\{p \leq X \mid p \in \mathcal{P}_{\alpha, D}\}.$$

It is clear that

$$(4.7) \quad \Pi(X, 2^\alpha D) = \sum_{\delta \mid 2^Q} \mu(\delta) \pi(X, 2^\alpha D\delta, -1)$$

We are interested only on those D which are smaller than T_x and those α which are less than ν_x .

From Lemma 1 we can deduce that

$$\Pi(X, 2^\alpha D) = (\text{Li}(X)) \sum_{\delta \mid 2^Q} \frac{\mu(\delta)}{\varphi(2^\alpha D\delta)} + O\left(\frac{X}{(\log X)^A}\right),$$

where A is arbitrary large. Thus

$$\Pi(X, 2^\alpha D) = (\text{Li}(X)) \frac{\varrho(D)}{2^\alpha D} S_Q,$$

where ϱ is the multiplicative function defined on prime powers π^l by

$$(4.8) \quad \varrho(\pi^\ell) = \varrho(\pi) = \begin{cases} \frac{\pi-1}{\pi-2} & \text{if } \pi \mid Q \\ 1 & \text{if } \pi \nmid Q, \end{cases}$$

$$(4.9) \quad S_Q := \prod_{\pi \mid Q} \frac{1}{\varrho(\pi)}.$$

For every $p \in \mathcal{P}_{\alpha,D}$, let

$$\kappa_y(p+1) := \frac{g(p+1)}{g(2^\alpha D)}.$$

If $p \notin \mathcal{R}^{(3)}$, then $|\kappa_y(p+1) - 1| \leq \frac{2}{(\log \log x)^{C'}}$, (C' is arbitrary large), therefore the number of primes $p \in \mathcal{P}_{\alpha,D}$ for which $p \in \mathcal{P}_{\alpha,D}$ is in between

$$\Pi \left(\frac{x \pm \frac{2x}{(\log \log x)^{C'}}}{g(2^\alpha)g(D)} \mid 2^\alpha D \right).$$

VI. Let

$$(4.10) \quad R = \sum_{\substack{1 \leq \alpha \leq \nu_x \\ D \leq T_x}} \max_{u \leq xw} \left| \Pi(u \mid 2^\alpha D) - \frac{\varrho(D)}{2^\alpha D} S_Q \text{Li}(u) \right|.$$

Starting from (4.7) and Lemma 1, and letting τ stand for the number of divisors function, we have

$$\begin{aligned} R \leq & \sum_{1 \leq \alpha \leq \nu_x} \sum_{k \leq T_x Q} \tau(k) \left\{ \max_{u \leq xw} \left| \Pi(u, 2^\alpha k, -1) - \frac{\text{Li}(u)}{\varphi(2^\alpha k)} \right| + \right. \\ & \left. + \max_{u \leq xw} \left| \Pi(u, 2^{\alpha+1} k, -1) - \frac{\text{Li}(u)}{\varphi(2^{\alpha+1} k)} \right| \right\} = \Sigma_1 + \Sigma_2. \end{aligned}$$

Here we observed that for odd k the number of those D, δ for which $D\delta = k$ is at most $\tau(k)$. Since δ is either odd, or $2 \mid \delta$, therefore $D\delta = 2k$ holds for at most $\tau(k)$ distinct cases. In Σ_1 we sum over those k for which $\tau(k) \leq (\log x)^E$, and in Σ_2 over those k for which $\tau(k) > (\log x)^E$. Here E is an appropriate large constant. From Lemma 1,

$$\Sigma_1 = O \left(\frac{x}{(\log x)^{C_1}} \right),$$

C_1 is arbitrary large.

Since

$$\Sigma_2 \leq \frac{c}{(\log x)^E} \sum_{\alpha \leq \nu_x} \sum_{k \leq T_x Q} \frac{\tau^2(k) \text{Li}(x)}{\varphi(2^\alpha k)} \leq \frac{cx}{(\log x)^{E+1}} \sum_{k \leq T_x Q} \frac{\tau^2(k)}{\varphi(k)},$$

we have

$$\sum_{k \leq T_x Q} \frac{\tau^2(k)}{\varphi(k)} \leq \prod_{\pi|Q} \left(1 + \frac{2^2}{\pi - 1} + \frac{3^2}{\pi(\pi - 1)} + \dots \right) \leq c \exp(4 \log \log y) \leq c_1 (\log y)^4,$$

therefore

$$\Sigma_2 = o(\pi(x)).$$

Consequently

$$R = o(\pi(x)).$$

It remains to estimate

$$U(x) := \sum_{\substack{\alpha \leq \nu_x \\ D \leq T_x}} \Pi \left(\frac{x}{g(2^\alpha D)} \middle| 2^\alpha D \right).$$

We have

$$U(x) = \sum_{\substack{1 \leq \alpha \leq \nu_x \\ D \leq T_x}} \text{Li} \left(\frac{x}{g(2^\alpha D)} \right) \frac{\varrho(D)}{2^\alpha D} S_Q + o(\pi(x)).$$

Since

$$\text{Li} \left(\frac{x}{g(2^\alpha D)} \right) = (1 + o_x(1)) \frac{\text{Li}(x)}{g(2^\alpha D)},$$

we easily obtain that

$$\lim \frac{U(x)}{\text{Li}(x)} = \left(\sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha g(2^\alpha)} \right) \prod_{\substack{q \in \mathcal{P} \\ q \neq 2}} \left(1 - \frac{1}{q-1} \right) \left(1 + \sum_{l=1}^{\infty} \frac{q-1}{q-2} \frac{1}{q^l g(q^l)} \right).$$

The product on the right hand side is clearly convergent, since

$$1 - g(q) = 1 - e^{f(q)} = -f(q) + O(f^2(q)) = O\left(\frac{1}{(\log q)^2}\right).$$

Thus

$$\lim \frac{U(x)}{\text{Li}(x)} = C_g.$$

Since

$$|S(x) - U(x)| \leq \left| U\left(x + \frac{2x}{(\log \log x)^C}\right) - U\left(x - \frac{2x}{(\log \log x)^C}\right) \right| + o(\pi(x)),$$

the theorem follows. ■

5. Final remark

Theorem 3. *Under the conditions of Theorem 1,*

$$\frac{1}{\text{Li}(x)} \sum_{p \leq x} \frac{1}{g(p+1)} \rightarrow C_g.$$

This can be proved easily, applying the method of proof of Theorem 1.

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