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ON UNIFORMLY SUMMABLE FUNCTIONS AND A PROBLEM OF HALMOS IN ERGODIC THEORY

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Dedicated to the memory of Professor János Galambos

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Abstract. Let $\mathcal{L}^*$ be the $\|.\|_1$-closure of l^∞, where

$$\|f\|_1 := \limsup_{N\to\infty} N^{-1} \sum_{n\le N} |f(n)|.$$

The linear functional Λ of the vector space $\{s \in l^\infty : s(n) = const\}$, defined by $\Lambda(s) := \lim\limits_{N\to\infty} N^{-1} \sum\limits_{n\le N} s(n)$, can be extended to a linear functional Λ^* on $\mathcal{L}^*$. We show that, for $f \in \mathcal{L}^*, \Lambda^*(f)$ can be written as an integral, and apply this to a problem formulated by Halmos in [5].

1. Introduction

Let $(X, \mathfrak{F}, \mu)$ be a measure space together with a measure-preserving transformation T, i.e. $T : X \to X$ and

$$\int\limits_X f(T^n x)d\mu = \int\limits_X f(x)d\mu$$

for all $n = 0, 1, 2, ...$ and all $f \in L_1 := L_1(X, \mathfrak{F}, \mu)$.

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Then P. Halmos [5] formulates Birkhoff's ergodic theorem as follows.

Individual Ergodic Theorem. ([5], p. 18) *If T is a measure preserving transformation and if $f \in L_1$, then $\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$ converges almost everywhere. The limit function f^* is integrable and invariant (i.e. $f^*(Tx) = f^*(x)$ almost everywhere). If $\mu(X) < \infty$, then $\int_X f^*(x)d\mu = \int_X f(x)d\mu$.*

In his "Comments on the Ergodic Theorem" (see [5], pp. 22–24) Halmos writes

> *I cannot resist the temptation of concluding these comments with an alternative "proof" of the ergodic theorem. If f is a complex-valued function on the non-negative integers, write $\int f(n)dn =$ $= \lim \frac{1}{n}\sum_{j=0}^{n-1} f(j)$ whenever the limit exists, and call such functions integrable. If T is a measure-preserving transformation on a space X and f is an integrable function on X, then*
>
> $$\iint |f(T^n X)|dndx = \iint |f(T^n x)|dxdn = \iint |f(x)|dxdn = \\ = \int |f(x)|dx < \infty.$$
>
> *Hence, by "Fubini's theorem" (!), $f(T^n x)$ is an integrable function of its two arguments and therefore, for almost every fixed x, it is an integrable function of n. Can any of this nonsense be made meaningful?*

We assume $\mu(X) < \infty$ and define, for $f \in L_1$, the arithmetic function f_x by

$$f_x(n) = f(T^{n-1}x).$$

Then we show, that f_x lies in the space $\mathcal{L}^*$ of *uniformly summable* functions for almost all $x \in X$.

Next, the linear functional Λ on the vector space

$$\{s \in l^\infty : s(n) = const. \text{ for all } n \in \mathbb{N}\},$$

defined by

$$\Lambda(s) = \lim_{N\to\infty} N^{-1} \sum_{n \le N} s(n)$$

can be extended to a linear functional Λ^* on $\mathcal{L}^*$. We show, that, for all $f \in$ $\in \mathcal{L}^*$, $\Lambda^*(f)$ can be written as an integral and solve the problem formulated by Halmos.

2. Uniformly summable functions

Let l^∞ be the linear space of all bounded functions on the positive integers with norm $\|f\| = \sup_{n\in\mathbb{N}} |f(n)|$. For $1 \le \alpha < \infty$ let

$$\mathcal{L}^\alpha := \{f : \mathbb{N} \to \mathbb{C}; \|f\|_\alpha < \infty\}$$

denote the linear space of arithmetic functions with bounded seminorm

$$\|f\|_\alpha := \left\{ \limsup_{x\to\infty} x^{-1} \sum_{n\le x} |f(n)|^\alpha \right\}^{\frac{1}{\alpha}} .$$

In the frame of investigations of the mean behaviour of *multiplicative functions* (see, for example, K.-H Indlekofer [6]) I introduced in 1980 the following

Definition 2.1. A function $f \in \mathcal{L}^1$ is said to be *uniformly summable* if

$$\lim_{K\to\infty} \sup_{N\ge1} N^{-1} \sum_{\substack{n\le N\\ |f(n)|\ge K}} |f(n)| = 0, \tag{2.1}$$

and the space of all uniformly summable functions is denoted by $\mathcal{L}^*$. It is obvious that, if $\alpha > 1$,

$$\mathcal{L}^\alpha \subsetneqq \mathcal{L}^* \subsetneqq \mathcal{L}^1.$$

Remark 2.1 The main underlying motivation for introducing this concept can be described as follows.

Let f be real-valued. For each number $N \ge 1$ we define the frequency function

$$F_{N,f}(y) := N^{-1} \sum_{\substack{n\le N\\ f(n)\le y}} 1.$$

If, as $N \to \infty$, the frequencies converge to a limiting distribution F_f in the usual probabilistic sense we say that f has a *limiting distribution* F_f and write $F_{N,f} \xrightarrow{\mathcal{D}} F_f$. Here we call the distribution function F_f *degenerate* if $F_f(y) = 0$ for $y < 0$ and $F_f(y) = 1$ for $y \ge 0$, and *nondegenerate* otherwise.

Now, look at the existence of the limiting distribution F_f of f and the existence of the *mean-value*

$$M(f) := \lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} f(n) \tag{2.2}$$

of f.

Because of

$$\frac{1}{N}\sum_{n\le N} f(n) = \int\limits_{-\infty}^{\infty} y dF_{N,f}(y)$$

one could expect that $M(f)$ exists if the limit law F_f of f exists. Unfortunately, this is not so! For example, define the multiplicative function f such that, for every prime p,

$$f(p^k) = \begin{cases} 1, & k = 1; \\ p^k, & k \ge 2. \end{cases}$$

Then (see, for example, Indlekofer [8], pp.117-119) f has a non-degenerate limiting distribution, but

$$\sum_{n\le N} f(n) > \sum_{n^2\le N} n^2 \gg N^{\frac{3}{2}}.$$

Thefore, $M(f)$ does not exist.

Now, let f be (real-valued and) uniformly summable. Then, if $\pm K$ are contininuity points of F_f one easily obtains

$$\frac{1}{N}\sum_{\substack{n\le N\\ |f(n)|\le K}} f(n) \underset{(N\to\infty)}{\to} \int\limits_{-K}^{K} y dF_f(y)$$

and

$$\frac{1}{N}\sum_{\substack{n\le N\\ |f(n)|\le K}} |f(n)| \underset{(N\to\infty)}{\to} \int\limits_{-K}^{K} |y| dF_f(y)$$

so that (2.1) implies

$$\left(\overline{\lim}_{N\to\infty} - \underline{\lim}_{N\to\infty}\right)\frac{1}{N}\sum_{v\le N} f(n) < 2\varepsilon$$

and $M(f)$ exists. Similarly for $|f|$.

This ends Remark 2.1.

The idea of uniform summability turned out to provide the appropriate tools for describing the mean behaviour of multiplicative functions and gave insight into exactly which additive functions belong to $\mathcal{L}^1$. (See [6] – [10] for generalizations of results of Delange [2], Wirsing [18], Halász [4], Elliott [3], Daboussi [1].)

A typical example of such a generalization is given in the following. In 1943 Wintner [17], in his book on *Erathostenian Averages*, asserted that if a multiplicative function g assumes only values ± 1, then the mean-value $M(g)$ always existed. But the sketch of his proof could not be substantiated, and the problem remained open as the *Erdős–Wintner conjecture.* In 1967 Wirsing [18] solved this problem, and in [6] we could show

Proposition 2.1. *Let $g \in \mathcal{L}^*$ be a real-valued multiplicative function. Then the existence of $M(|g|)$ implies the existence of $M(g)$.*

In Wintner's case $|g| = 1$ and $M(|g|) = \lim_{N\to\infty} N^{-1} \sum_{n\le N} 1 = 1.$

In this paper we note that $\mathcal{L}^*$ is the $\|.\|_1$-closure of l^∞. The mean-value $M(.)$ defines a linear functional on the (vector-) space $\{f \in l^\infty, M(f) \ \ exists\}$ and can be extended to a linear functional Λ^* on $\mathcal{L}^*$ that satisfies

$$|\Lambda^* f| \le \|f\|_1 \ \ for \ \ f \in \mathcal{L}^*$$

(see Rudin [15], p. 57). If τ is the translation operator defined on l^∞ by the equation

$$(\tau f)(n) = f(n+1) \ \ (n \in \mathbb{N})$$

then

$$\Lambda^*(\tau f) = \Lambda^*(f).$$

As an example we consider a measure space $(X, \mathfrak{F}, \mu)$ together with a measure preserving transformation T.

For $f \in L_1$ define f_x by $(T^0 x = x)$

$$f_x(n) := f(T^{n-1}x) \ \ (n \in \mathbb{N}). \tag{2.3}$$

Defining $\int_X f_x d\mu$ by

$$n \longmapsto \int_X f_x(n) d\mu = \int_X f(T^{n-1}x) d\mu \tag{2.4}$$

for $n \in \mathbb{N}$ the problem we consider is whether

$$\Lambda^*(\int_X f_x d\mu) = \int_X \Lambda^*(f_x) d\mu \ \ for \ \ almost \ \ all \ \ x \in X \tag{2.5}$$

holds for $f \in L_1$.

For the use of Fubini's Theorem we observe that, because of the definition of the integral, the general result follows easily from the special case of simple functions.

Now, for a given measurable $A \in \mathfrak{F}$, let $f := 1_A$ be the indicator function of A. Then

$$f_x^* := \Lambda^*(f_x) = M(f_x)$$

exists for almost all x and, by (2.4),

$$\int_X f_x(n) d\mu = \int_X f d\mu = \mu(A)$$

and

$$\Lambda^* \left(\int_X f_x d\mu \right) = \int_X f d\mu = \mu(A).$$

If $\mu(X) < \infty$ then

$$\int_X \Lambda^*(f_x) d\mu = \int_X f_x^* d\mu = \int_X f d\mu = \mu(A)$$

by the *Individual Ergodic Theorem*, and (2.5) holds for simple functions from L_1.

On the other hand, suppose X is the real line (i.e. $\mu(X) = \infty$), T is the translation $Tx = x + 1$, f is the indicator function of $[a, b)$, where $-\infty < a < < b < \infty$. Then obviously $\Lambda^* f_x = 0$ for all $x \in \mathbb{R}$ and

$$\int_X f_x^* d\mu = 0 \neq \int_X f d\mu = b - a.$$

Therefore we shall assume in this paper that $\mu(X) < \infty$.

3. Integration theory on $\mathbb{N}$

Suppose that $\mathcal{A}$ is an algebra of subsets of $\mathbb{N}$. Then, if $\mathcal{E}$ denotes the family of simple functions on $\mathbb{N}$, the set

$$\mathcal{E}(\mathcal{A}) = \{s \in \mathcal{E} : s = \sum_{j=1}^{m} \alpha_j 1_{A_j}; \alpha_j \in \mathbb{C}, A_j \in \mathcal{A}, j = 1, ..., m\}$$

of *simple functions on* $\mathcal{A}$ is a vector space.

Now, $\mathbb{N}$, endowed with the discrete topology, will be embedded in a compact space $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, and then any algebra $\mathcal{A}$ in $\mathbb{N}$ with an arbitrary finitely additive set function (a content or pseudo measure on $\mathbb{N}$) can be extended to an algebra $\bar{\mathcal{A}}$ in $\beta\mathbb{N}$ together with an extension of the pseudomeasure, which turns out to be a premeasure on $\bar{\mathcal{A}}$.

Proposition 3.1. *Let $\mathcal{A}$ be an algebra in $\mathbb{N}$ and $\delta : \mathcal{A} \to [0,\infty)$. Then the following assertions hold.*

(i) *The family*

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\}$$

is an algebra in $\beta\mathbb{N}$, where $\bar{A} = cl_{\beta\mathbb{N}}A$ is the closure of A in $\beta\mathbb{N}$.

(ii) *The map*

$$\bar{\delta} : \bar{\mathcal{A}} \to [0,\infty)$$

$$\bar{\delta}(\bar{A}) = \delta(A)$$

is σ-additive on $\bar{\mathcal{A}}$ and can uniquely be extended to a measure on the minimal σ-algebra $\sigma(\bar{\mathcal{A}})$ over $\bar{\mathcal{A}}$.

This was the starting point for an "Integrations theory on $\mathbb{N}$". For details see Indlekofer [11], [12], [13].

In [14] K.-H. Indlekofer and R. Wagner considered algebras $\mathcal{F}$ of real-valued functions from l^∞ such that

(i) $\mathcal{F}$ separates the points,

(ii) $\mathcal{F}$ contains the constants,

(iii) $\mathcal{F}$ is complete in the sup-norm,

and proved

Proposition 3.2. (See [14], Theorem 1.) *Let $\mathcal{F}$ be an algebra of real-valued bounded functions on $\mathbb{N}$ satisfying* (i), (ii) *and* (iii). *Let Λ be a positive linear functional on $\mathcal{F}$ with $\Lambda(1_{\mathbb{N}})$=1. Then there exist an algebra $\mathcal{A}$ of subsets on $\mathbb{N}$ and a content δ on $\mathcal{A}$ such that*

(i) *each $f \in \mathcal{F}$ belongs to the $\|.\|_u$-closure of $\mathcal{E}(\mathcal{A})$ and*

(ii) *for each $f \in \mathcal{F}$ the relation*

$$\Lambda(f) = \int\limits_{\beta\mathbb{N}} \bar{f} d\bar{s}$$

holds.

Let us apply Proposition 2.2 to Λ^* where $\mathcal{F} = \{s \in l^\infty : s \;\; real - valued\}$.

Observe that $s = u + iv \in l^\infty$, where u and v are real-valued, holds if and only if $u, v \in l^\infty$. Then there exists an algebra $\mathcal{A}^*$ and a content δ such that

$$\Lambda^*(s) = \int\limits_{\beta\mathbb{N}} \bar{s} d\bar{\delta} \;\; for \;\; all \;\; s \in l^\infty. \tag{3.1}$$

If $f \in \mathcal{L}^*$ there exists a sequence $\{s_n\}$, $s_n \in l^\infty$ such that

$$\lim_{n\to\infty} \|f - s_n\|_1 = 0 \;\; \text{and} \;\; \lim_{m,n\to\infty} \int\limits_{\beta\mathbb{N}} |s_m - s_n| d\bar{s} = 0.$$

Therefore we have

Theorem 3.1. *Let $f \in \mathcal{L}^*$. Then there exist an algebra $\mathcal{A}^*$ and a content δ such that (3.1) holds. Furthermore, there exist $s_n \in l^\infty$ such that*

$$\Lambda^*(f) = \lim_{n\to\infty} \Lambda^*(s_n) = \int\limits_{\beta\mathbb{N}} \bar{f} d\bar{\delta}$$

where $\bar{f} : \beta\mathbb{N} \to \mathbb{C}$ is unique modulo $\bar{\delta}$-null functions.

4. On the problem of Halmos

Let $(X, \mathfrak{F}, \mu)$ be a measure space with $\mu(X) < \infty$ and $f \in L_1 = L_1(\mu)$. For a given measure preserving mapping $T : X \to X$ and $x \in X$ define $f_x : \mathbb{N} \to \mathbb{C}$ by (2.3).

Then

$$f_x^* := \|f_x\|_1 = M(f_x) \;\; for \;\; almost \;\; all \;\; x \in X$$

is integrable and invariant under T. Choose a sequence $\{s_m\}$ of simple functions on X such that

$$\int\limits_X |f - s_m| d\mu \to 0 \;\; as \;\; m \to \infty.$$

Then, for almost all $x \in X$,

$$\|f_x - s_{m,x}\|_1 =: |f_x - s_{m,x}|^* \tag{4.1}$$

and

$$\int\limits_X |f_x - s_{m,x}|^* d\mu \leq \int\limits_X |f - s_m| d\mu \to 0 \;\; as \;\; m \to \infty.$$

There exists a subsequence m_k such that

$$|f_x - s_{m_k,x}|^* \to 0 \ \ as \ \ k \to \infty$$

for almost $x \in X$, i.e.

$$\|f_x - s_{m_k,x}\|_1 \to 0 \ \ as \ \ k \to \infty.$$

Thus $f_x \in \mathcal{L}^*$ for almost all $x \in X$.

Integrating f_x over $\mathbb{N}$ (or $\beta\mathbb{N}$) and X and using Fubini's Theorem we obtain by (2.4)

$$\begin{aligned} \int\limits_X f_x^* d\mu &= \int\limits_X \Lambda^*(f_x) d\mu = \int\limits_X \int\limits_{\beta\mathbb{N}} \bar{f}_x d\bar{\delta} d\mu = \int\limits_{\beta\mathbb{N}} \int\limits_X \bar{f}_x d\mu d\bar{\delta} = \\ &= \Lambda^* \left(\int\limits_X f_x d\mu \right) = \\ &= \int\limits_X f d\mu. \end{aligned}$$

References

[1] **Daboussi, H.,** Sur les fonctions multplicatives ayant une valeur moyenne non nulle., *Bull. Soc. Math. France*, **109** (1981), 183–205.

[2] **Delange, H.,** Sur les fonctions arithmetiques multiplicatives, *Ann. Sci. cole Norm. Sup (3)*, **78** (1961), 273–304.

[3] **Elliott, P. D. T. A.,** Mean value theorems for multiplicative functions bounded in mean α-power, $\alpha > 1$, *J. Australian Math. Soc.* (*Ser. A*) **29** (1980), 177–205.

[4] **Halász G.,** Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta. Math. Acad. Sci. Hung.*, **19** (1968), 365–403.

[5] **Halmos, Paul R.,** *Lectures on Ergodic Theory*, Chelsea Publ. Comp, New York, (1956).

[6] **Indlekofer, K.-H.,** A mean-value theorem for multiplicative functions, *Math. Z.*, **172** (1980), 255–271.

[7] **Indlekofer, K.-H.,** On multiplicative arithmetical functions, *Topics in Classical Number Theory*, Vol. I-II., Colloq. Math. Soc. Janos Bolyai 34, North-Holland, Amsterdam, 1981, 731–748.

[8] **Indlekofer, K.-H.,** Limiting distributions and mean-values of multiplicative arithmetical functions, *J. für Reine Angew. Math.*, **328** (1981), 116–127.

[9] **Indlekofer, K.-H.,** Properties of uniformly summable multiplicative functions, *Period. Math. Hungar.*, **17(2)** (1986), 143–161.

[10] **Indlekofer, K.-H.,** Gleichgradige Summierbarkeit bei verallgemeinerten Momenten additiver Funktionen, *Arch. Math.*, **49** (1987), 508–512.

[11] **Indlekofer, K.-H.,** A new method in probabilistic number theory, *Probability Theory and Applications, Math. Appl.*, **80** (1992), 299–308.

[12] **Indlekofer, K.-H.,** Number theory - probabilistic, heuristic, and computational approaches, *Comput. Math. Appl.*, **43** (2002), 1035–1061.

[13] **Indlekofer, K.-H.,** New approach to probabilistic number theory-compactifications and integration, *Advenced Studies in Pure Mathematics*, **49** (2005), 133–170.

[14] **Indlekofer, K.-H. and R. Wagner,** About positive linear functionals on spaces of arithmetical functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **40** (2013), 295–306.

[15] **Rudin, W.,** *Functional Analysis*, MacGwaw-Hill Book Company, New York, (1973).

[16] **Taylor, S. J.,** *Introduction to Measure and Integration*, Cambridge University Press, (1966).

[17] **Wintner, A.,** *Eratosthenian Averages*, Waverly Press, Baltimore, (1943).

[18] **Wirsing, E.,** Das asymptotische Verhalten von Summen multiplikativer Funktionen, II., *Acta Math. Acad. Sci. Hungar.*, **18** (1967), 411–467.

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