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REVISITING RATES OF CONVERGENCE AND PENULTIMATE APPROXIMATIONS FOR EXTREMES

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Dedicated to the memory of Professor János Galambos

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Abstract. In the broad area of *extreme value theory* (EVT), my main scientific connection with János Galambos is in the field of *rates of convergence* in EVT. During my stay in Sheffield, UK (1975–1978), my PH.D. supervisor, Clive Anderson, sequentially provided me with different topics of research, among which *rates of convergence and penultimate approximations.* And indeed I still think that there is some kind of magic in this topic, because this my first passion has been intermittently revisited until 1999, in the framework of EVT. More recently, and after 2012, I was able to find a possible strong link between this topic in the field of EVT and another field I am interested in, *statistical process control and reliability.* Indeed, in reliability theory any coherent system can be represented as a *series-parallel* or a *parallel-series* (PS) system. Its lifetime can thus be written as the minimum of maxima or the maximum of minima. For large-scale coherent systems it is sensible to assume that the number of system components goes to infinity and work with the possible non-degenerate EV distributions either for maxima or for minima to get adequate lower and upper bounds for the system reliability. But rates of convergence to these limiting laws are often slow and penultimate approximations can provide a faster rate of convergence. The identification of the possible limit laws for the system reliability of homogeneous PS systems is sketched and the gain in accuracy is assessed whenever penultimate approximations are used instead of the ultimate limiting one.

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1. Introduction and motivation

The first time I met János Galambos and his wife, Éva, was in 1983, at Vimeiro's meeting, a NATO *Advanced Study Institute* (ASI) on *Statistics of Extremes and Application* (SEA), which took place at Vimeiro in the summer of 1983, from 31st of August until September 14, and where János spoke about *rates of convergence in extreme value theory* (Galambos, 1984). János was then one of the invited speakers, among other prominent researchers in the field of *extreme value theory* (EVT), among whom I mention, Clive Anderson, Paul Deheuvels, Benjamin Epstein, Laurens de Haan, Ross Leadbetter, Georg Lindgren, Yashaswini Mittal, James Pickands III, Sid Resnick, Holger Rootzen, Masaaki Sibuya, J. Tiago de Oliveira, Jef Teugels and Ishay Weissman.

Repeating what I already mentioned several times (see for instance Fraga Alves and de Carvalho, 2015), it was indeed true that when Richard Davis, one of the young participants at Vimeiro 1983, at the biannual conference on *Extreme Value Analysis* (EVA), in 2009, spoke about Vimeiro's meeting as EVA–0, and when I read at EVA 2013 website: '*It has been 30 years since the so-called zero-th* EVA *conference took place in* 1983 *in Vimeiro, a small town near the beach in Portugal*' ... I indeed felt some "nostalgia" ... And among a few people I missed at the conference on *Extremes in Vimeiro Today* (EVT 2013), organized by my dear colleagues and friends, Antónia Amaral Turkman, Isabel Fraga Alves and Manuela Neves, to commemorate the thirty years of the NATO ASI on SEA, I refer János Galambos, who could not be present at EVT 2013.

During my talk at EVT 2013, I mentioned that it would possibly be necessary to renumber EVA meetings. Indeed, prior to EVA meetings, which began in 1998, at Gothenburg, and thinking only on events that occurred after my PhD degree, got in 1978, we cannot forget Oberwolfach 1987 *Conference on Extremewertheorie*, where János Galambos was present, as can be seen in Figure 1, the unique photo where I could find János.

I further mentioned Gaithersburg 1993 *Conference on Extreme Value Theory and its Applications*, a conference under the organization of János Galambos, where I was an invited speaker, and where, apart from a talk on J. Tiago de Oliveira, who had unfortunately died in 1992, I also talked on the *penultimate behaviour of extremes* (Gomes, 1994), the topic that is now under discussion, and my main scientific link with János Galambos, a pioneer in the topic of rates of convergence in EVT (Galambos, 1978).

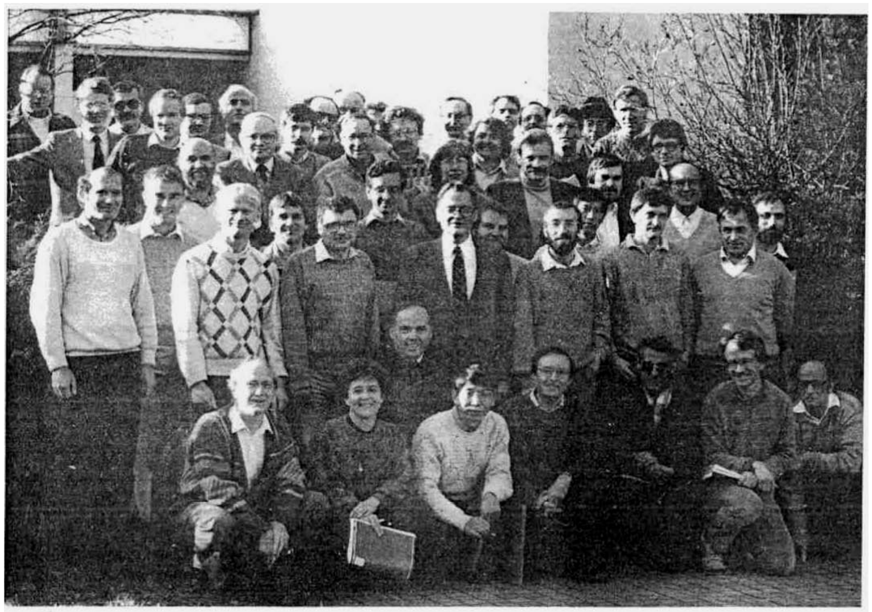


Figure 1. Oberwolfach 1987 *Conference on Extremewertheorie*

1.1. Scope of the article

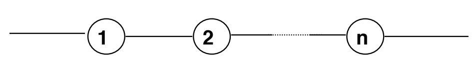
The study of the exact *right-tail function* (RTF) or *reliability function* (RF) of any complex and coherent technologic or biometric system, $\mathcal{S}$, with lifetime $T_{\mathcal{S}}$ and associated *cumulative distribution function* (CDF) $F_{T_{\mathcal{S}}}(\cdot)$, given by

$$R_{T_{\mathcal{S}}}(t) := \mathbb{P}(T_{\mathcal{S}} > t) = 1 - F_{T_{\mathcal{S}}}(t) =: \overline{F}_{T_{\mathcal{S}}}(t), \tag{1.1}$$

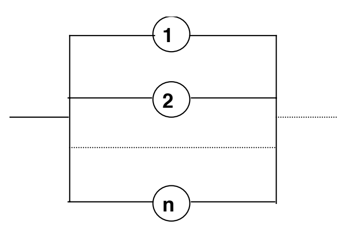
the most common notation for the RTF of $F_{T_{\mathcal{S}}}$, whenever working in the field of EVT, can be an intractable problem due to the large number, n, of system's components and/or to the way the operating process uses such components. Among others, we mention transport networks of oil, gas and water, telecommunication and electrical energy distribution networks, charge and discharge networks. Let us denote by $(T_1, \ldots, T_n)$ the lifetimes of those n components, and by $(T_{1:n} \leq \cdots \leq T_{n:n})$ the sample of associated ascending *order statistics* (OSs), with $T_{1:n} = \min_{1\leq i\leq n} T_i$ and $T_{n:n} = \max_{1\leq i\leq n} T_i$. As proved in Samaniego (1985), the lifetime $T_{\mathcal{S}}$ of any coherent system can always be written as an OS associated with $(T_1, \ldots, T_n)$. Further details on the role of OSs in reliability are thus given in Section 2, where we also refer possible and accurate reliability lower and upper bounds for the RF, in (1.1). On the basis of EVT, our strategy is here similar to the one in Reis and Canto e Castro (2009), Reis (2012), Reis *et al.* (2015) and Gomes *et al.* (2017), being essentially the following: Assuming that the number of components of a system $\mathcal{S}$ goes to infinity, asymptotic EV or ultimate models often provide a good interpretation and accurate estimation of the RF of $\mathcal{S}$, or at least of lower and upper bounds for such an RF. Considering that n is fixed despite of large, pre-asymptotic or penultimate models provide an improvement of the convergence rate and a better approximation to the RF of $\mathcal{S}$. A brief reference to the main asymptotic result in EVT is provided in Section 3, introducing the notions of rate of convergence and pre-asymptotic or penultimate behaviour of extreme OSs. In Section 4, ultimate and penultimate approximations for the RFs of regular and homogeneous *parallel series* (PS) systems, parallel structures with components connected in series, are discussed. In Section 5, alternative penultimate approximations are put forward and a few overall comments are drawn in Section 6.

2. Order statistics and reliability bounds

It is obvious that $T_{\mathcal{S}} = T_{1:n} = \min(T_1, \ldots, T_n)$ is the lifetime of a *series system*, the one that works if and only if all its n components work, represented by



For a *parallel system*, the one that works if and only if at least one of its n components work, represented by



we get $T_{\mathcal{S}} = T_{n:n} = \max(T_1, \ldots, T_n)$. More generally, we can always write the distributional identity, $T_{\mathcal{S}} = T_{I:n}$, where I is a discrete *random variable* (RV) with support $\{1, \ldots, n\}$. The vector $\underline{s} := (s_1, \ldots, s_n)$, with $s_i := \mathbb{P}(I = i)$,

$1 \le i \le n$ ($\sum_{i=1}^n s_i = 1$), is the so-called *signature* of the system (Samaniego, 1985). For a series system, the signature is $\underline{s} := (1, \ldots, 0)$, and for a parallel system, the signature is $\underline{s} := (0, \ldots, 1)$.

For any system $\mathcal{S}$, the most crude lower and upper RF bounds are respectively given by the RFs of the associated *series* (S) and *parallel* (P) systems, with all components working independently. Let us assume that the lifetimes T_i, $1 \le i \le n$, are *independent, identically distributed* (IID) from a model F, and associated RF, $R = 1 - F$. Then, with

$$M_n(t) = F^n(t) = (1 - R(t))^n \quad \text{and} \quad m_n(t) = 1 - (1 - F(t))^n = 1 - R^n(t),$$

the CDFs of $T_{n:n}$ and $T_{1:n}$, respectively, we have

$$L_{_S}(t) := 1 - m_n(t) \le R_{T_{\mathcal{S}}}(t) \le 1 - M_n(t) =: U_{_P}(t). \tag{2.1}$$

Moreover, without mentioning the definition of a coherent system (see Barlow and Proschan, 1975, for details on the topic), we further state the following relevant result in reliability: Any coherent structure can be represented either as a PS or a *series-parallel* (SP) structure, a series structure with components connected in parallel. To find the aforementioned PS and SP representations of a system, we need to identify the so-called *minimal paths*—paths without irrelevant components that enable the operation of the system, and the so-called *minimal cuts*—a set of relevant components that will imply the failure of the system whenever removed.

Example 2.1. As an illustration, consider the structure in Figure 2.

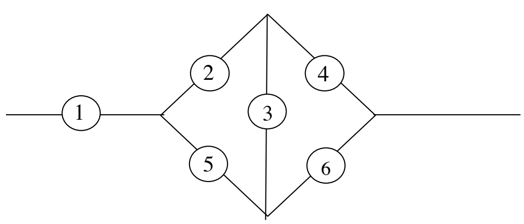


Figure 2. A series-bridge structure

For this structure, we have the *minimal paths*, {1, 2, 4}, {1, 5, 6}, {1, 2, 3, 6}, {1, 5, 3, 4}, and the *minimal cuts*, {1}, {2, 5}, {4, 6}, {2, 3, 6}, {5, 3, 4}. Consequently, we can write the distributional identities

$$T_{\mathcal{S}} = \max\big\{ \min(T_1, T_2, T_4),\ \min(T_1, T_5, T_6),\ \min(T_1, T_2, T_3, T_6),\\ \min(T_1, T_3, T_4, T_5)\big\}$$

and

$$T_{\mathcal{S}} = \min\big\{T_1, \max(T_2, T_5), \max(T_4, T_6), \max(T_2, T_3, T_6), \max(T_3, T_4, T_5)\big\}.$$

We obviously need to pay attention to the strong dependence of the different RVs under play either in the overall max or min operators. But we can easily build reliable upper and lower bounds for the reliability, respectively on the basis of the minimal paths (assuming they are disjoint) and the minimal cuts (assuming they are independent).

Generally speaking, let $P_j, 1 \le j \le p = p_n$, denote the minimal paths, and C_j, $1 \le j \le s = s_n$, the minimal cuts. Then, and for non-necessarily ID components, i.e. assuming that the lifetime T_i comes from a CDF F_i, $1 \le i \le n$, we have

$$\prod_{j=1}^{s}\Big(1 - \prod_{i\in C_j}(F_i(t))\Big) \le R_{T_{\mathcal{S}}}(t) \le 1 - \prod_{j=1}^{p}\Big(1 - \prod_{i\in P_j}(1 - F_i(t))\Big).$$

For sake of simplicity, we now assume that all minimal paths have the same size $l = l_n$ and that all minimal cuts have a size $r = r_n$ (the so-called *regular* system), and that $R_i(t) = R(t)$, $1 \le i \le n$ (the so-called *homogeneous* system). Then, using the notation,

$$L_{SP}(t) := \big(1 - (1 - R(t))^{r_n}\big)^{s_n} =: \big(1 - M_{r_n}(t)\big)^{s_n}$$

and

$$U_{PS}(t) := 1 - \big(1 - R^{l_n}(t)\big)^{p_n} =: 1 - \big(m_{l_n}(t)\big)^{p_n},$$

with $n = r_n s_n = l_n p_n$, we get

$$L_{SP}(t) \le R_{T_{\mathcal{S}}}(t) \le U_{PS}(t). \tag{2.2}$$

The lower and upper bounds in (2.2) are quite reliable, particularly when compared with the crude lower and upper bounds, in (2.1), respectively given by associated series and parallel systems, with all the n components of our system $\mathcal{S}$. The bounds L_{SP} and L_{PS} are much more accurate, as can be seen from Figure 3, where we consider the static counterpart of the RF, writing $p := R(t)$, and represent for $l_n = r_n = 2$, $s_n = p_n = 10$ ($n = 20$), as well as for $l_n = r_n = 4$, $s_n = p_n = 15$ ($n = 60$), the lower bounds $L_S \equiv L_S(p) = p^n$, $L_{SP} \equiv L_{SP}(p) = \big(1 - (1-p)^{r_n}\big)^{s_n}$, and the upper bounds $U_P \equiv U_P(p) = 1 - (1-p)^n$, $U_{PS} \equiv U_{PS}(p) = 1 - \big(1 - p^{l_n}\big)^{p_n}$, as functions of p.

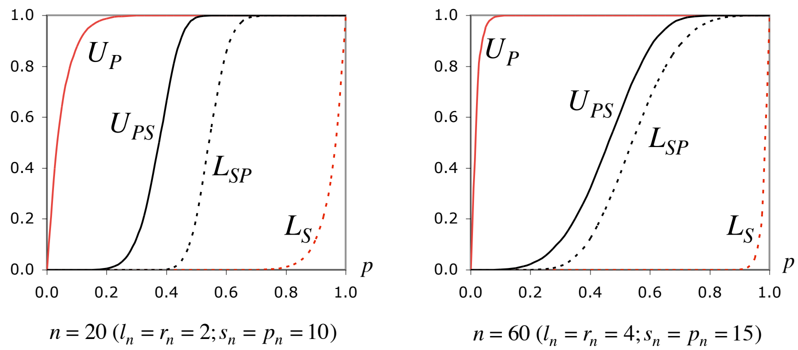


Figure 3. Lower and upper reliability bounds

3. The main limiting results in EVT

EVT provides a great variety of limiting results that enable us to deal with alternative approaches in the statistical analysis of extreme events. The main *limiting* result in EVT is due to Gnedenko (1943). In this seminal paper, Boris Gnedenko has fully characterized the possible non-degenerate limiting CDF of the linearly normalized maximum, $(T_{n:n} - b_n)/a_n,\ a_n > 0,\ b_n \in \mathbb{R}$. Such a limit is of the type of the general EV *distribution for maxima* (EV_MD), the unique max-stable law, given by

$$(3.1)\qquad G(t) \equiv G_\xi(t) := \begin{cases} \exp(-(1+\xi t)^{-1/\xi}),\ 1+\xi t > 0, & \text{if} \quad \xi \neq 0, \\ \exp(-\exp(-t)),\ t \in \mathbb{R}, & \text{if} \quad \xi = 0. \end{cases}$$

The shape parameter ξ, the so-called *extreme value index for maxima* (EVI_M), measures the heaviness of the RTF or RF $\overline{F}(t) \equiv R(t) = 1 - F(t)$, as $t \to +\infty$, and the heavier the right-tail, the larger ξ is.

The EV_MD is sometimes separated in the three following types,

$$\begin{array}{ll} \text{Type I (Gumbel)}: & \Lambda(t) = \exp(-\exp(-t)),\ t \in \mathbb{R}, \\ \text{Type II (Fréchet)}: & \Phi_\alpha(t) = \exp(-t^{-\alpha}),\ t \geq 0, \\ \text{Type III (max-Weibull)}: & \Psi_\alpha(t) = \exp(-(-t)^\alpha),\ t \leq 0, \end{array}$$

with $\alpha > 0$, the types considered in Gnedenko (1943).

As mentioned before, the parameter ξ, in (3.1), measures essentially the weight of the RTF, $\overline{F} = 1 - F$: If $\xi < 0$ (max-Weibull, with $\alpha = -1/\xi$), the right tail is light, and F has a finite *right endpoint* ($t^F := \sup\{t : F(t) < 1\} < +\infty$); If $\xi > 0$ (Fréchet, with $\alpha = 1/\xi$), the right tail is heavy, of a negative polynomial type, and F has an infinite *right endpoint*; If $\xi = 0$ (Gumbel), the right tail is of an exponential type. The *right endpoint* can then be either finite or infinite.

Remark 3.1. Any result for maxima has its counterpart for minima due to the fact that $\min_{1\le i\le n} T_i = -\max_{1\le i\le n}(-T_i)$. We thus have the min-stable laws or $\mathrm{EV_mD}$

$$G^*(t) \equiv G^*_\theta(t) := \begin{cases} 1 - \mathrm{e}^{-(1-\theta t)^{-1/\theta}},\ 1-\theta t > 0, & \text{if} \quad \theta \neq 0, \\ 1 - \mathrm{e}^{-\mathrm{e}^t},\ t \in \mathbb{R}, & \text{if} \quad \theta = 0. \end{cases} \tag{3.2}$$

We then say that the CDF F of an RV T is in the min-domain of attraction of G^*_θ, using the notation $F \in \mathcal{D}_\mathrm{m}(G^*_\theta)$, if the CDF of $-T$ is in the max-domain of attraction of G_θ, i.e. with the notation $H(t) = 1 - F(-t)$, $H \in \mathcal{D}_\mathcal{M}(G_\theta)$. The shape parameter θ, the so-called *extreme value index for minima* ($\mathrm{EVI_m}$), measures the heaviness of the *left-tail function* $F(t)$, as $t \to -\infty$ and the heavier the left-tail, the larger θ is.

Similarly to what happens in the max-scheme, the $\mathrm{EV_mD}$ is sometimes separated in the three following types:

$$\begin{array}{ll} \text{Type I (min-Gumbel)}: & \Lambda^*(t) = 1 - \exp(-\exp(t)),\ t \in \mathbb{R}, \\ \text{Type II (min-Fréchet)}: & \Phi^*_\alpha(t) = 1 - \exp(-(-t)^{-\alpha}),\ t \le 0, \\ \text{Type III (Weibull)}: & \Psi^*_\alpha(t) = 1 - \exp(-t^\alpha),\ t \ge 0. \end{array}$$

Remark 3.2. In most applications involving lifetimes the limit laws G^*_θ, in (3.2), are restricted to the case $\theta \le 0$. In fact, a lifetime T is always nonnegative. Thus $-T$ is an RV with a finite right endpoint and can only be in the max-domain of attraction of a Weibull or a Gumbel. However, since there are systems with large durability, we also often consider the case $\theta > 0$.

3.1. Rates of convergence and penultimate approximations

An associated important problem in EVT concerns the rate of convergence of $F^n(a_n t + b_n)$ towards $G_\xi(t)$, in (3.1), or, equivalently, the finding of estimates of the difference

$$d_n(F, G_\xi, t) := F^n(a_n t + b_n) - G_\xi(t). \tag{3.3}$$

Indeed, parametric inference on the right-tail of F, usually unknown, is done on the basis of the identification of $F^n(a_n t + b_n)$ and of $G_\xi(t)$, replacing $F^n(t)$ by $G_\xi((t - b_n)/a_n)$, with b_n and $a_n > 0$ being unknown parameters to be estimated from an adequate sample. The rate of convergence is thus important because it may validate or not the most usual models in *statistics of extremes*, and this was also already detected by Gnedenko. In EVT there exists no analogue of the Berry-Esséen theorem that, under broad conditions, gives a rate of convergence of the order of $1/\sqrt{n}$ in the Central Limit Theorem for sums. The rate of convergence depends here strongly on the right-tail of F, on the choice of the attraction coefficients (a_n, b_n), and can be rather slow, as first detected by Fisher and Tippet (1928). These authors were indeed the first ones to provide a so-called penultimate max-Weibull approximation for $\Phi^n(x)$, with Φ the normal CDF.

The modern theory of rates of convergence in EVT began with Anderson (1971, 1976), Galambos (1978) and Gomes (1978). Developments have followed different directions that can be found in Gomes (1994) and also in the more recent review papers by Beirlant *et al.* (2012) and Gomes and Guillou (2015). We refer here only the study of the structure of the remainder $d_n(F, G_\xi, t)$, in (3.3), with $F \in \mathcal{D}_{\mathcal{M}}(G_\xi)$, $\xi \in \mathbb{R}$, i.e. the finding of $d_n \to 0$, as $n \to \infty$, and $\varphi(t)$ such that, either uniformly in $t \in \mathbb{R}$ or at least uniformly in finite intervals of $t \in \mathbb{R}$,

$$F^n(a_n t + b_n) - G_\xi(t) = d_n \varphi(t) + o(d_n). \tag{3.4}$$

We then say that the rate of convergence of $F^n(a_n t + b_n)$ towards $G_\xi(t)$ is of the order of d_n.

In this same framework, the possible penultimate behaviour of $F^n(a_n t + b_n)$ has been studied, i.e. the possibility of finding $H(t) = H_n(t)$, perhaps a max-stable CDF, such that

$$F^n(a_n t + b_n) - H_n(t) = O(r_n), \quad r_n = o(d_n), \tag{3.5}$$

with d_n given in (3.4). We refer Gomes (1978, 1984b, 1986), Gomes and Pestana (1987), and Gomes and de Haan (1999), who derived, for all $\xi \in \mathbb{R}$, exact penultimate approximation rates, under von Mises-type conditions and some extra differentiability assumptions. Kaufmann (2000) proved a similar result, but under weaker conditions. This penultimate or pre-asymptotic behaviour has further been studied by Raoult and Worms (2003), and Diebolt and Guillou (2005), among others.

Despite of crucial, we shall not go into detail on first and second-order conditions in the field of extremes. We just mention that: The first-order conditions are just necessary and sufficient conditions (or sufficient conditions) to have $F \in \mathcal{D}_{\mathrm{M}}(G_\xi)$. The second-order conditions essentially measure the rate of convergence in the first-order conditions and depend upon a second-order parameter ρ (≤ 0) (see, de Haan and Ferreira, 2006, and Fraga Alves *et al.*, 2007, among others).

We next come to the question answered in Gomes and de Haan (1999): under which circumstances (i.e. for which combination of ξ and ρ) can the convergence rate be improved by the use of penultimate approximations? The answer is: To get any improvement we need to have $\rho = 0$ and to choose $\xi(t) \equiv \eta(t) = v''(t)/v'(t)$, with $v(t) := u^{\leftarrow}(t)$, $u(x) := -\ln(-\ln F(x))$. We further note that $\rho = 0$ is valid for a large variety of models, including the normal CDF.

4. PS systems: ultimate and penultimate models

We first state a theorem in Reis and Canto e Castro (2009). For the choice of attraction coefficients, see also the above mentioned article.

Theorem 4.1 (Reis and Canto e Castro, 2009)**.** *Any stable law for minima, i.e.* G^*_θ*, in* (3.2)*, belongs to* $\mathcal{D}_\mathcal{M}(G_0)$*, i.e. there exist sequences* $\{a_n > 0\}_{n\ge1}$ *and* $\{b_n \in \mathbb{R}\}_{n\ge1}$ *such that*

$$(G^*_\theta(a_n t + b_n))^n \underset{n\to\infty}{\longrightarrow} G_0(t),$$

uniformly in $t \in \mathbb{R}$*.*

We further state the following result (Reis *et al.*, 2015):

Theorem 4.2 (Reis *et al.*, 2015)**.** *Let* $F \in \mathcal{D}_\mathrm{m}(G^*_\theta)$*, the min-domain of attraction of* G^*_θ*, the* EV$_\mathrm{m}$D*, i.e. let us assume that there exist sequences* $\{a_n > 0\}_{n\ge1}$ *and* $\{b_n \in \mathbb{R}\}_{n\ge1}$ *such that*

$$1 - (1 - F(a_n t + b_n))^n \underset{n\to+\infty}{\longrightarrow} G^*_\theta(t) = 1 - G_\theta(-t),$$

$\forall t \in \mathbb{R}$ *and where* G_θ *and* G^*_θ *are the* EV$_\mathrm{M}$D *and the* EV$_\mathrm{m}$D*, given in* (3.1) *and* (3.2) *respectively. Then, for all* $\theta \in \mathbb{R}$*, and adequate* $(l_n, p_n) \to (\infty, \infty)$*, as* $n \to \infty$*, there exist sequences* $\{\alpha_n > 0\}_{n\ge1}$ *and* $\{\beta_n \in \mathbb{R}\}_{n\ge1}$ *such that for all* $t \in \mathbb{R}$*,*

$$F_{T_\mathcal{S}}(\alpha_n t + \beta_n) := \Big(1 - (1 - F(\alpha_n t + \beta_n))^{l_n}\Big)^{p_n} \underset{n\to\infty}{\longrightarrow} \Lambda(t) \equiv G_0(t).$$

Consequently, for a regular homogeneous PS *system, composed by* p_n *parallel subsystems with* l_n *components in series, the sequence of* RFs*, suitably normalized is such that*

$$R_{T_\mathcal{S}}(\alpha_n t + \beta_n) = 1 - F_{T_\mathcal{S}}(\alpha_n t + \beta_n) \underset{n\to\infty}{\longrightarrow} 1 - G_0(t),$$

for all $t \in \mathbb{R}$*.*

4.1. Right tail penultimate behaviour of min-stable laws

Also proved in Reis *et al.* (2015), we further state:

Theorem 4.3 (Reis *et al.*, 2015)**.** *For all $\theta \neq -1$, the min-stable law G^*_θ, in (3.2), is under the conditions of the main theorem in* Gomes and de Haan (1999)*. Consequently,*

$$\lim_{n\to\infty} \frac{(G^*_\theta(a_n t + b_n))^n - G_{\xi_n}(t)}{(\theta+1)/\ln^2 n} = \frac{t^3 G_0'(t)}{6},$$

uniformly for all $t \in \mathbb{R}$, with (a_n, b_n) the attraction coefficients in Theorem 4.1*, and where ξ_n is asymptotically given by*

$$\xi_n = -\frac{\theta+1}{\ln n} + O\left(\frac{1}{n}\right).$$

We further have

$$(G^*_\theta(a_n t + b_n))^n - G_0(t) = O(1/\ln n).$$

Remark 4.1. Note that if $\theta = -1$, von Mises first-order condition holds, but the *ultimate* aproximation $\left(G^*_{-1}\left(a_n t + b_n\right)\right)^n \approx G_0\left(x\right)$ cannot possibly be improved. If $\theta < -1$, $\xi_n > 0$, and G_{ξ_n} is a *penultimate sequence* of Fréchet distributions for $(G^*_\theta)^n$. If $\theta > -1$, $\xi_n < 0$, and G_{ξ_n} is a *penultimate sequence* of max-Weibull distributions for $(G^*_\theta)^n$.

5. Alternative penultimate approximations

In Reis *et al.* (2015) and Gomes *et al.* (2017), several $\mathrm{PS}_{(l_n,p_n)}$ systems have been simulated, with lifetime components from different models, including the $\mathrm{EV_m D}(\theta)$, for $\theta = -2(0.5)1$. The hypothesis $\mathcal{H}_0 : G^*_n = 1 - (1-F)^{l_n} \in \mathcal{D}_\mathcal{M}\left(G_\xi\right),$ for some $\xi \in \mathbb{R}$, was not rejected, and no typical behaviour was detected on the variation of l_n. The ultimate law G_0 was also tested, and the null hypothesis, $\mathcal{H}_0 :\ F_n\left(t\right) = G_0\left((t-\lambda)/\delta\right),$ with $F_n\left(t\right)$ the CDF of the lifetime of a PS system and $(\lambda, \delta) \in \mathbb{R} \times \mathbb{R}^+$ a vector of unknown (location and scale) parameters, was rejected except for $\theta = -1$ (showing consistency between simulated and theoretical results).

The main question is the following one: Are the estimates of ξ closer to a penultimate parameter $\xi_n = -(\theta+1)/\ln n$ rather than to the ultimate parameter zero? In the aforementioned articles, the parameter ξ in the EV model has been estimated through $\hat{\xi}^{(1)}$, the *maximum likelihood* (ML) estimate, and a positive answer to the aforementioned question was provided.

We now suggest also the consideration in (3.5) of the penultimate model considered in Smith (1987), which further depends on an unknown shape parameter r, being given by

$$\text{PEV}_\xi(t;r) = \exp\left(-(1+\xi t)^{-1/\xi}\left(1+r(1+\xi t)^{-1/\xi}\right)\right),\ 1+\xi t>0. \tag{5.1}$$

Let us denote by $\hat{\xi}^{(2)}$ any adequate estimate of ξ in (5.1). On the basis of R runs, it is thus sensible to simulate, for $j=1,2$, the *root mean square error* (RMSE) and BIAS-values,

$$\text{RMSE}_{P_j} = \sqrt{\tfrac{1}{R}\sum_{i=1}^{R}\left(\hat{\xi}_i^{(j)}-\xi_n\right)^2}, \quad \text{RMSE}_{U_j} = \sqrt{\tfrac{1}{R}\sum_{i=1}^{R}\left(\hat{\xi}_i^{(j)}\right)^2},$$

$$\text{BIAS}_{P_j}=\tfrac{1}{R}\sum_{i=1}^{R}\left(\hat{\xi}_i^{(j)}-\xi_n\right), \quad \text{BIAS}_{U_j}=\tfrac{1}{R}\sum_{i=1}^{R}\widehat{\xi}_i^{(j)}.$$

We then often have $\text{MSE}_{P_2} < \text{MSE}_{P_1} < \text{MSE}_{U_1} < \text{MSE}_{U_2}$, for $\xi \neq -1$, and a similar relation for |BIAS|, in most simulations. Only for $\xi=-1$ are we led to the adoption of a Gumbel model for F_n.

6. Overall conclusions

Due to the finiteness of n, the number of system's components, the assumption that n goes to infinity can be considered somewhat restrictive. Hence the reason for considering a fixed large number of components and pre-asymptotic approximations. We are conscious that the restriction that the RF of all components of the system is the *same* is quite strong, and such an assumption was used *only* as a simplification. More intricate but similar work can be done for non-homogeneous systems. Applications of the developed theory are feasible, but still at the beginning. And now, that we have access to highly sophisticated computational techniques, a great variety of *parametric models* can further be considered, like the penultimate EV parametric model in (5.1), which surely deserves further attention. The penultimate or pre-asymptotic behaviour can also be worked under a multivariate framework. The rate of convergence depends then not only on the marginals, but also on the dependence function (Omey and Rachev, 1991). The road from univariate to multivariate EVT is confronted from the beginning with a problem: there is not a single way to order multivariate observations. Barnett (1976) considers different categories of order relations for multivariate data, each being of potential use. We particularly like the one related to concomitants of OSs (see David and Galambos, 1974; Gomes, 1984a, 1985, among others). But the most useful order relation in multivariate EVT is a special case of what is called marginal ordering. The sample maximum needs not to be a sample point and the definition might thus seem artificial. Still, from its study a rich theory emanates that leads to a broad set of statistical tools for analysing extremes of multivariate data (see Beirlant *et al.*, 2004, and de Haan and Ferreira, 2006, among others), most of the times based on multivariate max-stable models. Penultimate models that are appropriate in the most flexible existing model for multivariate extremes, the one related to conditional extremes, introduced by Heffernan and Tawn (2004), have been recently discussed in Lugrin *et al.* (2019). And we believe that both aforementioned multivariate frameworks have an important role in the reliability of stress-strength models (Eryilmaz, 2008), where systems are subjected to a random stress over time, a research topic still in progress.

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