

DISCRETE RATIONAL BIORTHOGONAL SYSTEMS ON THE DISC

Sándor Fridli and Ferenc Schipp

(Budapest, Hungary)

Dedicated to the memory of Professor Gisbert Stoyan

Communicated by László Szili

(Received June 16, 2020; accepted September 12, 2020)

Abstract. In recent years we have considered various problems related to Malmquist–Takenaka (MT) functions, which form orthogonal systems on the torus. We have introduced their discrete versions and applied them successfully for compression and representation of human ECG signals [6, 7]. Also we have shown electrostatic interpretation of the discretization points. In these investigations we have taken the MT systems on the torus. In this paper we construct discrete MT type systems by generalizing our former discretization method applied on the torus to the unit disc.

1. Introduction

It is known that the roots of orthogonal polynomials play special role in numerical mathematics. They are frequently taken as nodal points of interpolation algorithms and quadrature formulas [4, 11]. Discrete polynomial systems

Key words and phrases: rational functions, Malmquist–Takenaka systems, unit disc, orthogonalization.

2010 Mathematics Subject Classification: Primary 26C15, Secondary 41A20, 42C05, 65T99.
EFOP-3.6.3-VEKOP-16-2017-00001: Talent Management in Autonomous Vehicle Control Technologies - The Project is supported by the Hungarian Government and co-financed by the European Social Fund.

can be constructed by means of these roots along with the Christoffel–Darboux formula. Our paper is related to this classical topic.

The Blaschke–functions

$$B_a(z) := \frac{z - a}{1 - \bar{a}z} \quad (a \in \mathbb{D}, z \in \overline{\mathbb{D}}),$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ stands for the open, and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ for the closed unit disc, play fundamental role in our investigations.

Let T denote the one–dimensional torus. It is known that the restrictions $B_a : \mathbb{D} \rightarrow \mathbb{D}$, and $B_a : \mathbb{T} \rightarrow \mathbb{T}$ are both bijections. We note that $B_a^{-1} = B_{-a}$ is the inverse of B_a . Moreover, B_a can be expressed in the following explicit form on the torus

$$B_a(e^{it}) = e^{i(\alpha + \gamma_r(t - \alpha))},$$

$$\gamma_r(t) := \int_0^t \frac{1 - r^2}{1 - 2r \cos \tau + r^2} d\tau \quad (t \in \mathbb{R}, a = re^{i\alpha} \in \mathbb{D}).$$

$\gamma_r : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function for which $\gamma_r(t + 2\pi) = \gamma_r(t) + 2\pi$ ($t \in \mathbb{R}$) holds, and

$$\gamma_r(t) = 2 \arctan(s(r) \tan(t/2)), \quad s(r) := \frac{1 + r}{1 - r}.$$

We will be concerned with finite products of Blaschke–functions of the form

$$B_N^{\mathbf{a}}(z) := B_N(z) := c \prod_{k=0}^{N-1} B_{a_k}(z)$$

($z \in \overline{\mathbb{D}}, \mathbf{a} = (a_0, \dots, a_{N-1}) \in \mathbb{D}^N$), where the factor $c \in \mathbb{T}$ will be fixed according to our need later. Since the numbers $a_k \in \mathbb{D}$ are the zeroes of B_N we have that the numbers $a_k^* := 1/\bar{a}_k$, the mirror image of a_k with respect to the unit circle, are the poles of B_N . Therefore, the a_k parameters will be called inverse poles. $B_N^{\mathbf{a}} : \mathbb{D} \rightarrow \mathbb{D}$ is an N -fold map on \mathbb{T} and can be expressed as

$$B_N^{\mathbf{a}}(e^{it}) = c \cdot e^{iN\theta_N(t)}, \quad \theta_N(t) := \frac{1}{N} \sum_{k=0}^{N-1} (\alpha_k + \gamma_{r_k}(t - \alpha_k))$$

$$(a_k = r_k e^{i\alpha_k}, \theta_N(t + 2\pi) = \theta_N(t) + 2\pi \ (t \in \mathbb{R})).$$

Here the parameter c was taken to make the $B_N^{\mathbf{a}}(1) = 1$ equation hold.

In our previous works the sets

$$\mathbb{T}_{N,u}^{\mathbf{a}} := \{z \in \mathbb{T} : u = B_N^{\mathbf{a}}(z)\} = \{z_k : 0 \leq k < N\}$$

with N elements were used for discretization, in which the parameter $u \in \mathbb{T}$ was a free parameter. It is easy to show that the discretization points can be expressed as $z_k = e^{i\tau_k}$, where $\tau_k = \theta_N^{-1}(t_k)$, $t_k := t_0 + 2k\pi/N$ ($0 \leq k < N$), $u = e^{it_0}$.

In this paper we consider the solutions of equations

$$\text{i) } B_N^{\mathbf{a}}(z) = u \quad (0 \leq |u| \leq 1), \quad \text{ii) } (B_N^{\mathbf{a}})'(w) = 0.$$

In the next sections we will show that they can be used in the constructions of discrete orthogonal and biorthogonal systems. For our previous work see [8].

2. Discrete orthogonal systems

Orthogonal and biorthogonal systems have been very effective in the theory of approximation, harmonic analysis and in many areas of applied mathematics. In numerical computations the discrete versions are of particular importance [3, 5]. In this section we consider discretization processes in which the discrete system is generated by restricting the original continuous system onto proper finite sets. There are well-known examples for this type of discretization. Namely, the taking equidistant subdivisions and the trigonometric system we obtain the discrete trigonometric system. Similarly, orthogonal polynomial systems and the set of their roots generate discrete polynomial systems. We note that this we construct interpolation methods as well.

In the rest of this section we are concerned with discretization of rational orthogonal systems. The elements of the set

$$(2.1) \quad \mathcal{Z}_{N,u}^{\mathbf{a}} := \{z \in \overline{\mathbb{D}} : B_N^{\mathbf{a}}(z) = u\} \quad (0 \leq |u| \leq 1)$$

will be chosen as the nodes of discretization. It is easy too see that the equation $B_N(z) = u$ has exactly N solutions counting with multiplicities. In particular, if $u \in \mathbb{T}$ then all of the roots are of multiplicity one, i.e. the equation has N distinct roots. In what follows we will always take such $u \in \mathbb{D}$ for which this condition holds, i.e. the set $\mathcal{Z}_{N,u}$ has N elements.

Rational orthogonal systems are generated from basic rational functions by means of orthogonalization. Let us take a sequence

$$\mathbf{a} = (a_n, n \in \mathbb{N}) \in \mathbb{D}^{\infty}$$

of inverse poles. The sequence of multiplicities in \mathbf{a} is defined as follows

$$m^{\mathbf{a}} := (m_n, n \in \mathbb{N}), \quad \text{where} \quad m_n := \sum_{k \leq n, a_k = a_n} 1 \quad (n \in \mathbb{N}).$$

Let us introduce the following subspaces

$$\mathfrak{R}_N^{\mathbf{a}} := \text{span} \{R_k^{\mathbf{a}} : 0 \leq k < N\}, \quad \mathfrak{R}^{\mathbf{a}} := \bigcup_{N=0}^{\infty} \mathfrak{R}_N^{\mathbf{a}}$$

generated by the basic rational functions

$$R_k^{\mathbf{a}}(z) := \frac{z^{m_k-1}}{(1 - \bar{a}_k z)^{m_k}} \quad (k \in \mathbb{N}, z \in \mathbb{D}).$$

There have several Euclidean spaces been studied that contain $\mathfrak{R}^{\mathbf{a}}$ as a proper subspace. They include the Hardy- and [12, 13, 14, 15] Bergman-spaces [1, 9] and their variant with weight functions. Here we take the Hardy-space $H^2(\mathbb{T})$ with the scalar product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} dt \quad (f, g \in H^2\mathbb{T}).$$

Applying Gram-Schmidt orthogonalization on the system $\{R_n^{\mathbf{a}} : n \in \mathbb{N}\}$ we receive the MT orthonormed system $\phi_n = \phi_n^{\mathbf{a}}$ ($n \in \mathbb{N}$).

Let us fix \mathbf{a} . Then we may simplify our notations above by omitting \mathbf{a} from them, i.e. we will use \mathfrak{R}_N instead of $\mathfrak{R}_N^{\mathbf{a}}$, B_n instead of $B_n^{\mathbf{a}}$ etc.

Below we give a list of some of the most important properties of the MT-systems [10]:

$$(2.2) \quad \begin{aligned} & \text{i) } \langle \phi_n, \phi_m \rangle = \delta_{mn} \quad (m, n \in \mathbb{N}), \\ & \text{ii) } \mathfrak{R}_N = \text{span} \{ \phi_n : 0 \leq n < N \} \quad (N \in \mathbb{N}), \\ & \text{iii) } \phi_n(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - \bar{a}_n z} B_n(z) \quad (z \in \mathbb{C}, n \in \mathbb{N}), \\ & \text{iv) } K_N(z, \zeta) := \sum_{k=0}^{N-1} \phi_k(z) \overline{\phi_k(\zeta)} = \frac{1 - B_N(z) \overline{B_N(\zeta)}}{1 - z \bar{\zeta}}, \\ & \quad (z, \zeta \in \overline{\mathbb{D}}, z \bar{\zeta} \neq 1), \\ & \text{v) } K_N(z, z) := \sum_{k=0}^{N-1} \frac{1 - |a_k|^2}{|1 - \bar{a}_k z|^2} \quad (z \in \overline{\mathbb{D}}), \\ & \text{vi) } \phi_n^{\mathbf{0}}(z) = z^n \quad (n \in \mathbb{N}, \mathbf{0} := (0, 0, \dots)). \end{aligned}$$

The relation in iv) can be viewed as the MT-analogues of the Christoffel-Darboux formula, and can be utilized in the discretization of MT-systems. Indeed, taking the nodal points

$$\mathcal{Z}_{N,u} := \{z \in \overline{\mathbb{D}} : B_N(z) = u\} \quad (0 \leq |u| \leq 1),$$

the weight function

$$\rho_N(z) := \frac{1}{K_N(z, z)} \quad (z \in \mathbb{T}),$$

and the discrete scalar product

$$\left[f, g \right]_N := \sum_{z \in \mathcal{Z}_{N,u}} f(z) \overline{g(z)} \rho_N(z)$$

the orthogonality relation

$$\left[\phi_n, \phi_m \right]_N = \delta_{mn} \quad (0 \leq m, n < N)$$

holds for $u \in \mathbb{T}$.

Here we will generalize the above results for biorthogonal systems taking parameters $u \in \mathbb{D}$. Recall (see (2.1)) that according to assumption $B_N^{\mathbf{a}}(z) = u$ has N distinct solutions. Let Ω denote the set of rational functions. For any $f \in \Omega$ the domain will be extended to $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by: $f(a) = \infty$, if a is a pole of f , and $f(\infty) := \lim_{z \rightarrow \infty} f(z)$. The following two types inversions will be defined on the set of rational functions by taking inversions of values and arguments of the functions:

$$f^*(z) := (f(z))^*, \quad f^*(z) := f(z^*) \quad (z \in \overline{\mathbb{C}}, f \in \Omega).$$

It is obvious that

$$z = z^*, \quad f^*(z) = f^*(z) = f(z) \quad (f \in \Omega)$$

hold for any $z \in \mathbb{T}$. Moreover in case of Blaschke–functions the two operations coincide

$$B_N^*(z) = B_N^*(z) = B_N(z^*) \quad (z \in \overline{\mathbb{C}}).$$

The system $\Phi^* := ((\phi_n)^*, n \in \mathbb{N})$ is called the dual of the MT–system $\Phi = (\phi_n^{\mathbf{a}}, n \in \mathbb{N})$.

Let us apply the Christoffel–Darboux formula in (2.2) iv) and v) for ζ instead of ζ^* to obtain

$$K_N(z, \zeta^*) = \sum_{k=0}^{N-1} \phi_k(z) \overline{\phi_k(\zeta^*)} = \frac{1 - B_N(z)/B_N(\zeta)}{1 - z/\zeta} \quad (z \neq \zeta),$$

and

$$K_N(z, z^*) = \sum_{k=0}^{N-1} \phi_k(z) \overline{\phi_k(z^*)} = \sum_{k=0}^{N-1} \frac{z(1 - |a_k|^2)}{(z - a_k)(1 - \overline{a_k}z)}.$$

The following surprising relation deserves special attention

$$\frac{d}{dz}(\log B_N(z)) = zK_N(z, z^*) \quad (z \in \overline{\mathbb{D}}).$$

Let us recall that the parameter u satisfies the conditions, that the set $\mathcal{Z}_{N,u}$ has exactly N elements $\mathcal{Z}_{N,u} = \{z_k : 0 \leq k < N\} \subset \overline{\mathbb{D}}$, i.e. $B_N(z) = u$ has N solutions. This implies that $B'_N(z) \neq 0$ at these points. Then by the relation above we have that $K_N(z, z^*) \neq 0$ for $z \in \mathcal{Z}_{N,u}$. Consequently,

$$(2.3) \quad \sum_{k=0}^{N-1} \frac{\phi_k(z)\overline{\phi_k(\zeta^*)}}{K_N(z, z^*)} = \delta_{z,\zeta} \quad (z, \zeta \in Z_{N,u}^{\mathbf{a}}).$$

Then introducing the matrices

$$A = \left[a_{ik} \right]_{i,k=0}^{N-1}, \quad a_{ik} = \phi_k(z_i)/K_N(z_i, z_i^*)$$

$$B = \left[b_{jk} \right]_{j,k=0}^{N-1}, \quad b_{kj} = \phi_k(z_j^*) = \phi_k^*(z_j)$$

(2.3) can be written in the form

$$AB^* = E \quad \iff \quad A = (B^*)^{-1} \quad \iff \quad B^*A = E.$$

Here B^* stands for the adjoint of B , and E denotes the identity matrix of \mathbb{C}^N .

An equivalent form is

$$\delta_{ij} = \sum_{k=0}^{N-1} \bar{b}_{kj} a_{ki} = \sum_{k=0}^{N-1} \overline{\phi_j^*(z_k)} \phi_i(z_k)/K_N(z_k, z_k^*) \quad (0 \leq i, j < N).$$

As a conclusion we have the following theorem which is the generalization of the result on discrete orthogonality of MT-systems.

Theorem 2.1. *Let u be a parameter for which the condition $\mathcal{Z}_u \cap \mathcal{K} = \emptyset$ holds. Then the ϕ_n, ϕ_n^* ($0 \leq n < N$) systems are biorthogonal*

$$\left[\phi_n, \phi_m^* \right]_{\mathbf{a},u} := \sum_{z \in Z_{N,u}^{\mathbf{a}}} \phi_n(z) \overline{\phi_m^*(z)} / K_N^{\mathbf{a}}(z, z^*) = \delta_{mn} \quad (0 \leq m, n < N).$$

References

- [1] **Duren, P. and A. Schuster**, *Bergman Spaces*, American Mathematical Society, Providence, RI (2004).
- [2] **Eisner, T. and M. Pap**, Discrete orthogonality of the MT system of the upper half plane and rational interpolation, *J. Fourier Anal. Appl.*, **20** (2014), 1–16.
- [3] **Fazekas, Z., F. Schipp and A. Soumelidis**, Utilizing the discrete orthogonality of Zernike functions in corneal measurements, in: S.I. Ao et al. (Eds.) *WCE 2009, London*, Lecture Notes in Engineering and Computer Science, 2009, 795–800.
- [4] **Fejér, L.**, Über Interpolation, *Göttinger Nachrichten*, (1916), 66–91.
- [5] **Fridli, S., Z. Gilián and F. Schipp**, Rational orthogonal systems on the plane, *Annales Univ. Sci. Budapest., Sect. Comp.*, **39** (2013) 63–77.
- [6] **Fridli, S., P. Kovács, L. Lócsi and F. Schipp**, Rational modeling of multi-lead QRS complexes in ECG signals, *Annales Univ. Sci. Budapest., Sect. Comp.*, **36** (2012), 145–155.
- [7] **Fridli, S., L. Lócsi and F. Schipp**, Rational function systems in ECG processing, in: R. Moreno-Díaz et al. (Eds.) *EUROCAST 2011, Part I*, Lecture Notes in Computer Science **6927**, Springer-Verlag, Berlin, Heidelberg, 2011, 88–95.
- [8] **Fridli, S. and F. Schipp**, Biorthogonal systems to rational functions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **35** (2011), 95–105.
- [9] **Hedenmalm, H.B. and K. Zhu Korenblum**, *Theory of Bergman Spaces*, Graduate Text in Mathematics, **199**, Springer Verlag, New York, 2000.
- [10] **Heuberger, P.S.C., P.M.J. Van den Hof and Bo Wahlberg**, *Modelling and Identification with Rational Orthogonal Basis Function*, Springer Verlag, London, 2005.
- [11] **Natanson, I.B.**, *Constructive Function Theory*, Frederick Ungar Publ., New York, 1965
- [12] **Pap, M.**, Hyperbolic wavelets and multiresolution in $H^2(\mathbb{T})$, *J. Fourier Anal. Appl.*, **17**(5) (2011), 755–766.
- [13] **Pap, M.**, Multiresolution in the Bergman space, *Annales Univ. Sci. Budapest., Sect. Comp.*, **39** (2013), 333–353.
- [14] **Pap, M. and F. Schipp**, The Voice transform on the Blaschke group II., *Annales Univ. Sci. Budapest., Sect. Comp.* **29** (2008), 157–173.

- [15] **Zhu, K.**, Interpolating and recapturing in reproducing Hilbert Spaces, *Bull. Hong Kong Math. Soc.*, **1** (1997), 21–33.

S. Fridli and F. Schipp

Department of Numerical Analysis

Eötvös Loránd University

H-1117 Budapest

Pázmány Péter sétány 1/C

Hungary

fridli@inf.elte.hu

schipp@numanal.inf.elte.hu