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DISCRETE RATIONAL BIORTHOGONAL SYSTEMS ON THE DISC

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Dedicated to the memory of Professor Gisbert Stoyan

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Abstract. In recent years we have considered various problems related to Malmquist–Takenaka (MT) functions, which form orthogonal systems on the torus. We have introduced their discrete versions and applied them successfully for compression and representation of human ECG signals [6, 7]. Also we have shown electrostatic interpretation of the discretization points. In these investigations we have taken the MT systems on the torus. In this paper we construct discrete MT type systems by generalizing our former discretization method applied on the torus to the unit disc.

1. Introduction

It is known that the roots of orthogonal polynomials play special role in numerical mathematics. They are frequently taken as nodal points of interpolation algorithms and quadrature formulas [4, 11]. Discrete polynomial systems

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can be constructed by means of these roots along with the Christoffel–Darboux formula. Our paper is related to this classical topic.

The Blaschke–functions

$$B_a(z) := \frac{z-a}{1-\overline{a}z} \qquad (a \in \mathbb{D}, z \in \overline{\mathbb{D}}),$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ stands for the open, and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ for the closed unit disc, play fundamental role in our investigations.

Let T denote the one–dimensional torus. It is known that the restrictions $B_a : \mathbb{D} \to \mathbb{D}$, and $B_a : \mathbb{T} \to \mathbb{T}$ are both bijections. We note that $B_a^{-1} = B_{-a}$ is the inverse of B_a. Moreover, B_a can be expressed in the following explicit form on the torus

$$\begin{aligned} &B_a(e^{it}) = e^{i(\alpha+\gamma_r(t-\alpha))}, \\ &\gamma_r(t) := \int_0^t \frac{1-r^2}{1-2r\cos\tau+r^2}\,d\tau \qquad (t \in \mathbb{R},\ a = re^{i\alpha} \in \mathbb{D}). \end{aligned}$$

$\gamma_r : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function for which $\gamma_r(t+2\pi) = \gamma_r(t) + 2\pi$ $(t \in \mathbb{R})$ holds, and

$$\gamma_r(t) = 2\,\arctan(s(r)\,\tan(t/2))\,, \quad s(r) := \frac{1+r}{1-r}.$$

We will be concerned with finite products of Blaschke–functions of the form

$$B_N^{\mathbf{a}}(z) : = B_N(z) := c\prod_{k=0}^{N-1} B_{a_k}(z)$$

$\big(z \in \overline{\mathbb{D}}, \mathbf{a} = (a_0, \ldots, a_{N-1}) \in \mathbb{D}^N\big)$, where the factor $c \in \mathbb{T}$ will be fixed according to our need later. Since the numbers $a_k \in \mathbb{D}$ are the zeroes of B_N we have that the numbers $a_k^* := 1/\overline{a}_k$, the mirror image of a_k with respect to the unit circle, are the poles of B_N. Therefore, the a_k parameters will be called inverse poles. $B_N^{\mathbf{a}} : \mathbb{D} \to \mathbb{D}$ is an N-fold map on $\mathbb{T}$ and can be expressed as

$$\begin{gathered} B_N^{\mathbf{a}}(e^{it}) = c \cdot e^{iN\theta_N(t)}\,, \quad \theta_N(t) := \frac{1}{N}\sum_{k=0}^{N-1}(\alpha_k + \gamma_{r_k}(t-\alpha_k)) \\ (a_k = r_k e^{i\alpha_k}, \theta_N(t+2\pi) = \theta_N(t) + 2\pi\ (t \in \mathbb{R}))\,. \end{gathered}$$

Here the parameter c was taken to make the $B_N^{\mathbf{a}}(1) = 1$ equation hold.

In our previous works the sets

$$\mathbb{T}_{N,u}^{\mathbf{a}} := \{z \in \mathbb{T} : u = B_N^{\mathbf{a}}(z)\} = \{z_k : 0 \leq k < N\}$$

with N elements were used for discretization, in which the parameter $u \in \mathbb{T}$ was a free parameter. It is easy to show that the discretization points can be expressed as $z_k = e^{i\tau_k}$, where $\tau_k = \theta_N^{-1}(t_k), t_k := t_0 + 2k\pi/N \;\; (0 \le k < N)$, $u = e^{it_0}$.

In this paper we consider the solutions of equations

$$\text{i)} \;\; B_N^{\mathbf{a}}(z) = u \quad (0 \le |u| \le 1)\,, \qquad \text{ii)} \;\; (B_N^{\mathbf{a}})'(w) = 0\,.$$

In the next sections we will show that they can be used in the constructions of discrete orthogonal and biorthogonal systems. For our previous work see [8].

2. Discrete orthogonal systems

Orthogonal and biorthogonal systems have been very effective in the theory of approximation, harmonic analysis and in many areas of applied mathematics. In numerical computations the discrete versions are of particular importance [3, 5]. In this section we consider discretization processes in which the discrete system is generated by restricting the original continuous system onto proper finite sets. There are well-known examples for this type of disretization. Namely, the taking equidistant subdivisions and the trigonometric system we obtain the discrete trigonometric system. Similarly, orthogonal polynomial systems and the set of their roots generate discrete polynomial systems. We note that this we construct interpolation methods as well.

In the rest of this section we are concerned with discretization of rational orthogonal systems. The elements of the set

$$\mathcal{Z}_{N,u}^{\mathbf{a}} := \{z \in \overline{\mathbb{D}} : B_N^{\mathbf{a}}(z) = u\} \;\; (0 \le |u| \le 1) \tag{2.1}$$

will be chosen as the nodes of discretization. It is easy too see that the equation $B_N(z) = u$ has exactly N solutions counting with multiplicities. In particular, if $u \in \mathbb{T}$ then all of the roots are of multiplicity one, i.e. the equation has N distinct roots. In what follows we will always take such $u \in \mathbb{D}$ for which this condition holds, i.e. the set $\mathcal{Z}_{N,u}$ has N elements.

Rational orthogonal systems are generated from basic rational functions by means of orthogonalization. Let us take a sequence

$$\mathbf{a} = (a_n, n \in \mathbb{N}) \in \mathbb{D}^\infty$$

of inverse poles. The sequence of multiplicities in $\mathbf{a}$ is defined as follows

$$m^{\mathbf{a}} := (m_n, n \in \mathbb{N})\,, \quad \text{where} \qquad m_n := \sum_{k \le n, a_k = a_n} 1 \qquad (n \in \mathbb{N})\,.$$

Let us introduce the following subspaces

$$\mathfrak{R}_N^{\mathbf{a}} := \operatorname{span}\{R_k^{\mathbf{a}} : 0 \le k < N\}\,, \quad \mathfrak{R}^{\mathbf{a}} := \bigcup_{N=0}^{\infty} \mathfrak{R}_N^{\mathbf{a}}$$

generated by the basic rational functions

$$R_k^{\mathbf{a}}(z) := \frac{z^{m_k-1}}{(1-\overline{a}_k z)^{m_k}} \qquad (k \in \mathbb{N}, z \in \mathbb{D})\,.$$

There have several Euclidean spaces been studied that contain $\mathfrak{R}^{\mathbf{a}}$ as a proper subspace. They include the Hardy– and [12, 13, 14, 15] Bergman–spaces [1, 9] and their variant with weight functions. Here we take the Hardy–space $H^2(\mathbb{T})$ with the scalar product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\,dt \qquad (f, g \in H^2\mathbb{T})\,.$$

Applying Gram–Schmidt orthogonalization on the system $\{R_n^{\mathbf{a}} : n \in \mathbb{N}\}$ we receive the MT orthonormed system $\phi_n = \phi_n^{\mathbf{a}}$ $(n \in \mathbb{N})$.

Let us fix **a**. Then we may simplify our notations above by omitting **a** from them, i.e. we will use $\mathfrak{R}_N$ instead of $\mathfrak{R}_N^{\mathbf{a}}$, B_n instead of $B_n^{\mathbf{a}}$ etc.

Below we give a list of some of the most important properties of the MT–systems [10]:

$$\text{i)}\quad \langle \phi_n, \phi_m\rangle = \delta_{mn} \qquad (m, n \in \mathbb{N})\,,$$

$$\text{ii)}\quad \mathfrak{R}_N = \operatorname{span}\{\phi_n : 0 \le n < N\} \qquad (N \in \mathbb{N})\,,$$

$$\text{iii)}\quad \phi_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a}_n z} B_n(z) \qquad (z \in \mathbb{C}, n \in \mathbb{N})\,,$$

$$\text{iv)}\quad K_N(z,\zeta) := \sum_{k=0}^{N-1} \phi_k(z)\overline{\phi_k(\zeta)} = \frac{1 - B_N(z)\overline{B_N(\zeta)}}{1 - z\overline{\zeta}}\,, \tag{2.2}$$

$$(z, \zeta \in \overline{\mathbb{D}}, z\overline{\zeta} \ne 1)\,,$$

$$\text{v)}\quad K_N(z,z) := \sum_{k=0}^{N-1} \frac{1-|a_k|^2}{|1-\overline{a}_k z|^2} \qquad (z \in \overline{\mathbb{D}})\,,$$

$$\text{vi)}\quad \phi_n^{\mathbf{0}}(z) = z^n \qquad (n \in \mathbb{N}, \mathbf{0} := (0, 0, \dots))\,.$$

The relation in iv) can be viewed as the MT–analogues of the Christoffel-Darboux formula, and can be utilized in the discretization of MT–systems. Indeed, taking the nodal points

$$\mathcal{Z}_{N,u} := \{z \in \overline{\mathbb{D}} : B_N(z) = u\} \qquad (0 \le |u| \le 1),$$

the weight function

$$\rho_N(z) := \frac{1}{K_N(z,z)} \qquad (z \in \mathbb{T}),$$

and the discrete scalar product

$$\Big[f, g\Big]_N := \sum_{z \in \mathcal{Z}_{N,u}} f(z)\overline{g(z)}\rho_N(z)$$

the orthogonality relation

$$\Big[\phi_n, \phi_m\Big]_N = \delta_{mn} \qquad (0 \leq m, n < N)$$

holds for $u \in \mathbb{T}$.

Here we will generalize the above results for biorthogonal systems taking parameters $u \in \mathbb{D}$. Recall (see (2.1)) that according to assumption $B_N^{\mathbf{a}}(z) = u$ has N distinct solutions. Let $\mathfrak{Q}$ denote the set of rational functions. For any $f \in \mathfrak{Q}$ the domain will be extended to $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by: $f(a) = \infty$, if a is a pole of f, and $f(\infty) := \lim_{z\to\infty} f(z)$. The following two types inversions will be defined on the set of rational functions by taking inversions of values and arguments of the functions:

$$f^*(z) := (f(z))^*\,, \qquad f^{\circledast}(z) := f(z^*) \qquad (z \in \overline{\mathbb{C}}, f \in \mathfrak{Q}).$$

It is obvious that

$$z = z^*\,, \qquad f^*(z) = f^{\circledast}(z) = f(z) \qquad (f \in \mathfrak{Q})$$

hold for any $z \in \mathbb{T}$. Moreover in case of Blaschke–functions the two operations coincide

$$B_N^*(z) = B_N^{\circledast}(z) = B_N(z^*) \qquad (z \in \overline{\mathbb{C}}).$$

The system $\Phi^{\circledast} := ((\phi_n)^{\circledast}, n \in \mathbb{N})$ is called the dual of the MT–system $\Phi = (\phi_n^{\mathbf{a}}, n \in \mathbb{N})$.

Let us apply the Christoffel–Darboux formula in (2.2) iv) and v) for ζ instead of ζ^* to obtain

$$K_N(z, \zeta^*) = \sum_{k=0}^{N-1} \phi_k(z)\overline{\phi_k(\zeta^*)} = \frac{1 - B_N(z)/B_N(\zeta)}{1 - z/\zeta} \qquad (z \neq \zeta),$$

and

$$K_N(z, z^*) = \sum_{k=0}^{N-1} \phi_k(z)\overline{\phi_k(z^*)} = \sum_{k=0}^{N-1} \frac{z(1 - |a_k|^2)}{(z - a_k)(1 - \overline{a}_k z)}.$$

The following surprising relation deserves special attention

$$\frac{d}{dz}(\log B_N(z)) = zK_N(z, z^*) \qquad (z \in \overline{\mathbb{D}}).$$

Let us recall that the parameter u satisfies the conditions, that the set $\mathcal{Z}_{N,u}$ has exactly N elements $\mathcal{Z}_{N,u} = \{z_k : 0 \le k < N\} \subset \overline{\mathbb{D}}$, i.e. $B_N(z) = u$ has N solutions. This implies that $B'_N(z) \neq 0$ at these points. Then by the relation above we have that $K_N(z, z^*) \neq 0$ for $z \in \mathcal{Z}_{N,u}$. Consequently,

$$\sum_{k=0}^{N-1} \frac{\phi_k(z)\overline{\phi_k(\zeta^*)}}{K_N(z, z^*)} = \delta_{z,\zeta} \qquad (z, \zeta \in \ Z^{\mathbf{a}}_{N,u}). \tag{2.3}$$

Then introducing the matrices

$$\begin{aligned} A &= \Big[a_{ik}\Big]_{i,k=0}^{N-1}, \quad a_{ik} = \phi_k(z_i)/K_N(z_i, z_i^*) \\ B &= \Big[b_{jk}\Big]_{j,k=0}^{N-1}, \quad b_{kj} = \phi_k(z_j^*) = \phi_k^{\divideontimes}(z_j) \end{aligned}$$

(2.3) can be written in the form

$$AB^* = E \quad \Longleftrightarrow \quad A = (B^*)^{-1} \quad \Longleftrightarrow \quad B^*A = E.$$

Here B^* stands for the adjoint of B, and E denotes the identity matrix of $\mathbb{C}^N$.

An equivalent form is

$$\delta_{ij} = \sum_{k=0}^{N-1} \bar{b}_{kj} a_{ki} = \sum_{k=0}^{N-1} \overline{\phi_j^{\divideontimes}(z_k)}\phi_i(z_k)/K_N(z_k, z_k^*) \qquad (0 \le i, j < N).$$

As a conclusion we have the following theorem which is the generalization of the result on discrete orthogonality of MT–systems.

Theorem 2.1. *Let u be a parameter for which the condition $\mathcal{Z}_u \cap \mathcal{K} = \emptyset$ holds. Then the $\phi_n, \phi_n^{\divideontimes}$ $(0 \le n < N)$ systems are biorthogonal*

$$\Big[\phi_n, \phi_m^{\divideontimes}\Big]_{\mathbf{a},u} := \sum_{z \in \mathcal{Z}^{\mathbf{a}}_{N,u}} \phi_n(z)\overline{\phi_m^{\divideontimes}(z)}/K^{\mathbf{a}}_N(z, z^*) = \delta_{mn} \qquad (0 \le m, n < N).$$

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