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ON THE MIDDLE DIVISORS OF AN INTEGER

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Dedicated to the memory of Professor János Galambos

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Abstract. Given a positive integer n, let $\rho_1(n) = \max\{d \mid n : d \le \sqrt{n}\}$ and $\rho_2(n) = \min\{d \mid n : d \ge \sqrt{n}\}$ stand for the middle divisors of n. We obtain improvements and new estimates for sums involving these two functions.

1. Introduction

Given a positive integer n, we define the numbers $\rho_1(n)$ and $\rho_2(n)$ as

$$\rho_1(n) := \max\{d \mid n : d \le \sqrt{n}\}$$
$$\rho_2(n) := \min\{d \mid n : d \ge \sqrt{n}\}$$

and call them the *middle divisors* of n. It is clear that $\rho_1(n)\rho_2(n) = n$ and also that if n is not a perfect square, then $\rho_1(n) < \rho_2(n)$.

In 1976, Tenenbaum [5] proved that

$$\sum_{n \le x} \rho_2(n) = \frac{\pi^2}{12} \frac{x^2}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right) \tag{1.1}$$

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and that, given any $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon)$ such that for all $x \ge x_0$,

$$\frac{x^{3/2}}{(\log x)^{\delta+\varepsilon}} < \sum_{n\le x} \rho_1(n) \ll \frac{x^{3/2}}{(\log x)^{\delta}(\log\log x)^{1/2}},$$

where

$$\delta = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.086071. \tag{1.2}$$

More recently, Ford [1] showed that

$$\sum_{n\le x} \rho_1(n) \asymp \frac{x^{3/2}}{(\log x)^{\delta}(\log\log x)^{3/2}}. \tag{1.3}$$

Here, we provide a refinement and a generalisation of (1.1) as well as a generalisation of (1.3), and we then use these results to obtain estimates for $\sum_{n\le x} \rho_2(n)/\rho_1(n)^r$, for every fixed real $r > -1$, and for $\sum_{n\le x} \rho_1(n)/\rho_2(n)$, thereby improving an earlier estimate by Roesler [4] in the case of the second sum.

2. Main theorems

Theorem 1. *Let $a > 0$ be a real number. Then, for each positive integer k,*

$$\sum_{n\le x} \rho_2(n)^a = c_0\frac{x^{a+1}}{\log x} + c_1\frac{x^{a+1}}{\log^2 x} + \cdots + c_{k-1}\frac{x^{a+1}}{\log^k x} + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right)$$

where, for $\ell = 0, 1, \ldots, k-1$,

$$c_\ell = c_\ell(a) = \frac{\ell!}{(a+1)^{\ell+1}} \sum_{j=0}^{\ell} \frac{(a+1)^j(-1)^j\zeta^{(j)}(a+1)}{j!}$$

with ζ standing for the Riemann zeta function.

Theorem 2. *Let $a > 0$ be a real number and let δ be as in (1.2). Then,*

$$\sum_{n\le x} \rho_1(n)^a \asymp \frac{x^{\frac{a+2}{2}}}{(\log x)^{\delta}(\log\log x)^{3/2}}. \tag{2.1}$$

Theorem 3. *Given any integer $k \ge 1$ and any real number $r > -1$, we have*

$$\sum_{n\le x} \frac{\rho_2(n)}{\rho_1(n)^r} = e_0\frac{x^2}{\log x} + e_1\frac{x^2}{\log^2 x} + \cdots + e_{k-1}\frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right)$$

where $e_0 = \dfrac{\zeta(r+2)}{2}$ *and for each* $1 \le \ell \le k-1$,

$$e_\ell = \left(\frac{r+2}{2}\right) c_\ell + \sum_{\nu=0}^{\ell-1} \frac{r c_\nu}{2} \prod_{m=\nu}^{\ell-1} \left(\frac{m+1}{2}\right),$$

with, for each $\nu = 0, 1, \dots, \ell$,

$$c_\nu = \frac{\nu!}{(r+2)^{\nu+1}} \sum_{j=0}^{\nu} \frac{(r+2)^j (-1)^j \zeta^{(j)}(r+2)}{j!}.$$

Remark. Interestingly, as a consequence of Theorem 3,

$$T_r(x) := \sum_{n \le x} \frac{\rho_2(n)}{\rho_1(n)^r} \sim \frac{\zeta(r+2)}{2} \frac{x^2}{\log x} \qquad \text{as } x \to \infty,$$

implying that all sums $T_r(x)$ are of the same order, independently of the chosen number $r > -1$. For instance, although it may at first appear counterintuitive, we do have that $\sum_{n \le x} \rho_2(n)\sqrt{\rho_1(n)} \asymp \sum_{n \le x} \dfrac{\rho_2(n)}{\sqrt{\rho_1(n)}}$.

Theorem 4. *With* δ *as in* (1.2), *we have*

$$\sum_{n \le x} \frac{\rho_1(n)}{\rho_2(n)} \asymp \frac{x}{(\log x)^\delta (\log\log x)^{3/2}}.$$

3. Preliminary results

Let $\pi(x)$ stand for the number of primes not exceeding x and let

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

We will be using the prime number theorem with an error term which is sufficient for our purposes, namely the original one found by de la Vallée Poussin [6] in 1899.

Proposition 1. (Prime number theorem.) *There exists a positive constant* C *such that*

$$\pi(x) - \mathrm{Li}(x) = O\left(x \exp\left\{-C\sqrt{\log x}\right\}\right).$$

Lemma 1. *Assume that* $n \le x$ *with* $\rho_2(n) > x^{2/3}$. *Then,* $\rho_2(n)$ *is a prime.*

Proof. Since $\rho_2(n) > x^{2/3}$, we have that $\rho_1(n) < x^{1/3}$. Set $m = \rho_2(n)$. It is clear that both $\rho_1(m)$ and $\rho_2(m)$ are divisors of n. Hence, in order to prove that $\rho_2(n)$ is prime, it is sufficient to prove that $\rho_2(m) = m$. Now, since $\rho_2(m) \geq \sqrt{m} = \sqrt{\rho_2(n)} > x^{1/3} > \rho_1(n)$, it follows hat $\rho_1(n) < \rho_2(m) \leq \rho_2(n)$, which implies, by the definition of $\rho_1(n)$ and $\rho_2(n)$ that $\rho_2(m) = \rho_2(n) = m$, thus proving our claim. ■

The following result is not new. We include it here for the sake of completeness.

Lemma 2. *Given any fixed real number $a > 0$,*

$$S(x) = S_a(x) := \sum_{p\leq x} p^a = \int_2^x \frac{t^a}{\log t}dt + O\left(\frac{x^{a+1}}{e^{C\sqrt{\log x}}}\right). \tag{3.1}$$

Proof. Using partial summation with $A(x) = \sum_{n\leq x} a(n) = \pi(x)$ and $\varphi(t) = = t^a$, we have

$$S(x) = x^a\pi(x) - \int_2^x at^{a-1}\pi(t)\,dt. \tag{3.2}$$

Using Proposition 1, it follows from (3.2) and integration by parts that

$$\begin{aligned} S(x) &= x^a\pi(x) - a\int_2^x t^{a-1}\left(\mathrm{Li}(t) + O(te^{-C\sqrt{\log t}})\right)dt = \\ &= x^a\pi(x) - a\int_2^x t^{a-1}\mathrm{Li}(t)\,dt + O\left(\int_2^x t^a e^{-C\sqrt{\log t}}\,dt\right) = \\ &= x^a\pi(x) - a\left(\frac{t^a}{a}\mathrm{Li}(t)\Big|_2^x - \int_2^x \frac{t^a}{a}\frac{1}{\log t}dt\right) + O\left(\frac{x^{a+1}}{e^{C\sqrt{\log x}}}\right) = \\ &= x^a\pi(x) - x^a\mathrm{Li}(x) + \int_2^x \frac{t^a}{\log t}dt + O\left(\frac{x^{a+1}}{e^{C\sqrt{\log x}}}\right). \end{aligned} \tag{3.3}$$

Using Proposition 1 one more time, we have that

$$x^a\pi(x) - x^a\mathrm{Li}(x) = x^a(\pi(x) - \mathrm{Li}(x)) = O\left(\frac{x^{a+1}}{e^{C\sqrt{\log x}}}\right),$$

which substituted in (3.3) completes the proof of (3.1). ■

Lemma 3. *Let $a > 0$ be an arbitrary real number. Then,*

$$\sum_{\sqrt{x}<p\le x} p^a \left\lfloor \frac{x}{p} \right\rfloor = \int_{\sqrt{x}}^{x} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt + O\left(\frac{x^{a+1}}{e^{\frac{C}{2}\sqrt{\log x}}} \right). \tag{3.4}$$

Proof. We follow an approach used by Naslund [3] to estimate a similar sum. Let B be a positive integer. Then,

$$\begin{aligned}
\sum_{x/B<p\le x} p^a \left\lfloor \frac{x}{p} \right\rfloor &= \sum_{n\le B-1} n \sum_{x/(n+1)<p\le x/n} p^a = \\
&= \sum_{n\le B-1} n(S(x/n) - S(x/(n+1))) = \\
&= S(x) + S(x/2) + \cdots + S(x/(B-1)) - (B-1)S(x/B) = \\
&= \sum_{n\le B-1} (S(x/n) - S(x/B)) .
\end{aligned}$$

Using Lemma 2 in this last estimate, we obtain, provided that $B \ge x^{1/4}$,

$$\begin{aligned}
\sum_{x/B<p\le x} p^a \left\lfloor \frac{x}{p} \right\rfloor &= \sum_{n\le B-1} \int_{x/B}^{x/n} \frac{t^a}{\log t} dt + O\left(\sum_{n\le B-1} \frac{(x/n)^{a+1}}{e^{C\sqrt{\log(x/n)}}} \right) = \\
&= \int_{x/B}^{x} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt + O\left(\frac{x^{a+1}}{e^{C\frac{1}{2}\sqrt{\log x}}} \sum_{n=1}^{\infty} \frac{1}{n^{a+1}} \right).
\end{aligned}$$

Choosing $B = \lfloor \sqrt{x} \rfloor$ allows us to write this last equation as

$$\sum_{\sqrt{x}<p\le x} p^a \left\lfloor \frac{x}{p} \right\rfloor = \int_{\sqrt{x}}^{x} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt + O\left(\frac{x^{a+1}}{e^{\frac{C}{2}\sqrt{\log x}}} \right), \tag{3.5}$$

thereby completing the proof of (3.4). ∎

Lemma 4. *Let $a > 0$ be an arbitrary real number. Then,*

$$\sum_{p\le x} p^a \left\lfloor \frac{x}{p} \right\rfloor = \int_{2}^{x} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt + O\left(\frac{x^{a+1}}{e^{\frac{C}{2}\sqrt{\log x}}} \right). \tag{3.6}$$

Proof. Since the two quantities $\sum_{p\le\sqrt{x}} p^a \left\lfloor \frac{x}{p} \right\rfloor$ and $\int_{2}^{\sqrt{x}} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt$ are each of smaller order than the error term appearing in (3.5), we may indeed conclude from (3.5) that (3.6) holds. ∎

Lemma 5. *For all $s>1$ and for each integer $k\geq 1$,*

$$\zeta^{(k)}(s)=(-1)^k\sum_{n=1}^{\infty}\frac{\log^k n}{n^s}.$$

Proof. Differentiating k times with respect to s both sides of equation $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ yields the result. ■

Lemma 6. *Let $a>0$ be an arbitrary real number. Then, for each integer $k\geq 1$,*

$$\int_2^x \frac{t^a\lfloor x/t\rfloor}{\log t}dt=c_0\frac{x^{a+1}}{\log x}+c_1\frac{x^{a+1}}{\log^2 x}+\cdots+c_{k-1}\frac{x^{a+1}}{\log^k x}+O\left(\frac{x^{a+1}}{\log^{k+1}x}\right),$$

where

$$c_\ell=c_\ell(a)=\frac{\ell!}{(a+1)^{\ell+1}}\sum_{j=0}^{\ell}\frac{(a+1)^j(-1)^j\zeta^{(j)}(a+1)}{j!}.$$

Proof. We use the same technique that Naslund [2] used to estimate a similar integral. With the change of variable $t=x/u$, we obtain

$$\begin{aligned}\nu_a(x) &:= \int_2^x \frac{t^a\lfloor x/t\rfloor}{\log t}dt=x^{a+1}\int_1^{x/2}\frac{\lfloor u\rfloor}{u^{a+2}\log\left(\frac{x}{u}\right)}du=\\ &= \frac{x^{a+1}}{\log x}\int_1^{x/2}\frac{\lfloor u\rfloor}{u^{a+2}}\left(1-\frac{\log u}{\log x}\right)^{-1}du.\end{aligned}$$

Since $1\leq u\leq x/2$, we have $\frac{\log u}{\log x}<1$. We can therefore write that for each integer $k\geq 1$,

$$\left(1-\frac{\log u}{\log x}\right)^{-1}=1+\frac{\log u}{\log x}+\cdots+\left(\frac{\log u}{\log x}\right)^{k-1}+\left(\frac{\log u}{\log x}\right)^{k}\left(1-\frac{\log u}{\log x}\right)^{-1}.$$

From this, it follows that

$$\begin{aligned}\nu_a(x) &= \frac{x^{a+1}}{\log x}\sum_{\ell=0}^{k-1}\frac{1}{\log^\ell x}\int_1^{x/2}\frac{\lfloor u\rfloor}{u^{a+2}}\log^\ell u\,du+\\ &\quad+\frac{x^{a+1}}{\log^{k+1}x}\int_1^{x/2}\frac{\lfloor u\rfloor}{u^{a+2}}\log^{k+1}u\left(1-\frac{\log u}{\log x}\right)^{-1}du.\end{aligned}$$

Since the integral $\displaystyle\int_1^{x/2} \frac{\lfloor u \rfloor}{u^{a+2}} \log^{k+1} u \left(1 - \frac{\log u}{\log x}\right)^{-1} du$ converges, we have that

$$\begin{aligned}
\nu_a(x) &= \frac{x^{a+1}}{\log x} \sum_{\ell=0}^{k-1} \frac{1}{\log^\ell x} \int_1^{x/2} \frac{\lfloor u \rfloor}{u^{a+2}} \log^\ell u \, du + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right) = \\
&= \frac{x^{a+1}}{\log x} \sum_{\ell=0}^{k-1} \frac{1}{\log^\ell x} \left(\int_1^\infty \frac{\lfloor u \rfloor}{u^{a+2}} \log^\ell u \, du - \int_{x/2}^\infty \frac{\lfloor u \rfloor}{u^{a+2}} \log^\ell u \, du \right) + \\
&\quad + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right). \qquad (3.7)
\end{aligned}$$

On the other hand, since

$$\int_{x/2}^\infty \frac{\lfloor u \rfloor}{u^{a+2}} \log^\ell u \, du \le \int_{x/2}^\infty \frac{\log^\ell u}{u^{a+1}} du = O\left(\frac{\log^\ell x}{x^a}\right),$$

it follows from (3.7) that

$$\nu_a(x) = \frac{x^{a+1}}{\log x} \sum_{\ell=0}^{k-1} \frac{c_\ell}{\log^\ell x} + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right),$$

where $c_\ell = \displaystyle\int_1^\infty \frac{\lfloor u \rfloor}{u^{a+2}} \log^\ell u \, du.$

It remains to obtain explicit expressions for the constants c_ℓ. We have

$$c_\ell = \int_1^\infty \frac{\lfloor u \rfloor}{u^{a+2}} \log^\ell u \, du = \sum_{s=1}^\infty s \int_s^{s+1} \frac{\log^\ell u}{u^{a+2}} du.$$

Performing integration by parts k times yields

$$\int_s^{s+1} \frac{\log^\ell u}{u^{a+2}} du = \sum_{i=0}^{\ell} \frac{\ell!}{(\ell-i)!(a+1)^{i+1}} \left(\frac{\log^{\ell-i} s}{s^{a+1}} - \frac{\log^{\ell-i}(s+1)}{(s+1)^{a+1}} \right),$$

so that, using Lemma 5, we get

$$\sum_{s=1}^\infty s \int_s^{s+1} \frac{\log^\ell u}{u^{a+2}} du =$$

$$
\begin{aligned}
&= \sum_{s=1}^{\infty}\left(s\sum_{i=0}^{\ell}\frac{\ell!}{(\ell-i)!(a+1)^{i+1}}\left(\frac{\log^{\ell-i}s}{s^{a+1}}-\frac{\log^{\ell-i}(s+1)}{(s+1)^{a+1}}\right)\right)=\\
&= \sum_{i=0}^{\ell}\frac{\ell!}{(\ell-i)!(a+1)^{i+1}}\left(\sum_{s=1}^{\infty}s\left(\frac{\log^{\ell-i}s}{s^{a+1}}-\frac{\log^{\ell-i}(s+1)}{(s+1)^{a+1}}\right)\right)=\\
&= \sum_{i=0}^{\ell}\frac{\ell!}{(\ell-i)!(a+1)^{i+1}}\left(\sum_{s=1}^{\infty}\frac{\log^{\ell-i}s}{s^{a+1}}\right)=\\
&= \sum_{i=0}^{\ell}\frac{\ell!}{(\ell-i)!}\frac{(-1)^{\ell-i}\zeta^{(\ell-i)}(a+1)}{(a+1)^{i+1}}.
\end{aligned}
$$

Setting $j=\ell-i$, we conclude that $c_\ell = \dfrac{\ell!}{(a+1)^{\ell+1}}\sum_{j=0}^{\ell}\dfrac{(a+1)^j(-1)^j\zeta^{(j)}(a+1)}{j!}$, thus completing the proof of Lemma 6. ■

Let $H(x,y,z)$ stand for the number of positive integers $n\le x$ having a divisor in the interval $(y,z]$.

Theorem A. (Ford [1], Théorème 1(v)) *Let x,y,z be real numbers all strictly positive. If $x>100000$, $100\le y\le z-1$, $y\le\sqrt{x}$ and $2y\le z\le y^2$, then*

$$H(x,y,z)\asymp xu^{\delta}\left(\log\frac{2}{u}\right)^{-3/2},$$

where u is defined implicitly by $z=y^{1+u}$ and where δ is the constant defined in (1.2).

Theorem B. (Ford [1], Théorème 2) *For $y_0\le y\le\sqrt{x}$, $z\ge y+1$ and $\dfrac{x}{\log^{10}z}\le\Delta\le x$, we have*

$$H(x,y,z)-H(x-\Delta,y,z)\asymp\frac{\Delta}{x}H(x,y,z).$$

4. Proof of Theorem 1

Using Lemma 1, we easily obtain that

$$
\begin{aligned}
&\sum_{n\le x}\rho_2(n)^a = \\
&= \sum_{\substack{n\le x\\ \rho_2(n)>x^{2/3}}} \rho_2(n)^a + O\left(x^{\frac{2a+3}{3}}\right) = \sum_{x^{2/3}<p\le x} p^a \sum_{\substack{n\le x\\ \rho_2(n)=p}} 1 + O\left(x^{\frac{2a+3}{3}}\right) = \\
&= \sum_{x^{2/3}<p\le x} p^a \sum_{mp\le x} 1 + O\left(x^{\frac{2a+3}{3}}\right) = \sum_{x^{2/3}<p\le x} p^a \left\lfloor \frac{x}{p}\right\rfloor + O\left(x^{\frac{2a+3}{3}}\right) = \\
&= \sum_{p\le x} p^a \left\lfloor \frac{x}{p}\right\rfloor - \sum_{p\le x^{2/3}} p^a \left\lfloor \frac{x}{p}\right\rfloor + O\left(x^{\frac{2a+3}{3}}\right) = \\
(4.1)\quad &= \Sigma_1 - \Sigma_2 + O\left(x^{\frac{2a+3}{3}}\right),
\end{aligned}
$$

say. From Lemma 2, we obtain that

$$
(4.2)\qquad \Sigma_2 = \sum_{p\le x^{2/3}} p^a \left\lfloor \frac{x}{p}\right\rfloor \le x \sum_{p\le x^{2/3}} p^{a-1} \ll x \int_2^{x^{2/3}} \frac{t^{a-1}}{\log t}\,dt \ll \frac{x^{\frac{2a+3}{3}}}{\log x}.
$$

Hence, it follows from (4.1) and (4.2) that

$$
(4.3)\qquad \sum_{n\le x}\rho_2(n)^a = \sum_{p\le x} p^a \left\lfloor \frac{x}{p}\right\rfloor + O\left(x^{\frac{2a+3}{3}}\right).
$$

Finally, combining the results of Lemmas 4 and 6 in (4.3), the proof of Theorem 1 is complete.

5. Proof of Theorem 2

Observe that the relation (2.1) we need to prove is equivalent to

$$
(5.1)\qquad \frac{x^{\frac{a+2}{2}}}{(\log x)^{\delta}(\log\log x)^{3/2}} \ll \sum_{n\le x}\rho_1(n)^a \ll \frac{x^{\frac{a+2}{2}}}{(\log x)^{\delta}(\log\log x)^{3/2}}.
$$

We will first show the first inequality in relation (5.1). We start by observing that if $x/2 < n \le x$, then n has a divisor d_1 satisfying $\frac{\sqrt{x}}{2} < d_1 \le \sqrt{x}$ if and only if $\rho_1(n) > \frac{\sqrt{x}}{2}$. It follows from this that

$$\begin{aligned}
\sum_{n\le x} \rho_1(n)^a &\ge \sum_{\substack{x/2<n\le x\\ \rho_1(n)>\sqrt{x}/2}} \rho_1(n)^a > \left(\frac{\sqrt{x}}{2}\right)^a \sum_{\substack{x/2<n\le x\\ \rho_1(n)>\sqrt{x}/2}} 1 \ge \\
&\ge \left(\frac{\sqrt{x}}{2}\right)^a \sum_{\substack{x/2<n\le x\\ \exists\ d_1|n\\ d_1\in(\sqrt{x}/2,\sqrt{x}]}} \ge \\
&\ge \left(\frac{\sqrt{x}}{2}\right)^a \left(H\left(x,\frac{\sqrt{x}}{2},\sqrt{x}\right) - H\left(\frac{x}{2},\frac{\sqrt{x}}{2},\sqrt{x}\right)\right).
\end{aligned}$$

Using Theorem B followed by Theorem A (with $\Delta = x/2$), we find that

$$\begin{aligned}
H\left(x,\frac{\sqrt{x}}{2},\sqrt{x}\right) - H\left(\frac{x}{2},\frac{\sqrt{x}}{2},\sqrt{x}\right) &\asymp \frac{x/2}{x}\cdot H\left(x,\frac{\sqrt{x}}{2},\sqrt{x}\right) \asymp \\
&\asymp x\cdot\left(\frac{2\log 2}{\log x}\right)^\delta \cdot (\log\log x)^{-3/2} \asymp \\
&\asymp \frac{x}{(\log x)^\delta(\log\log x)^{3/2}}.
\end{aligned}$$

Combining these last two estimates, it follows that

$$\sum_{n\le x} \rho_1(n)^a \gg \frac{x^{\frac{a+2}{2}}}{(\log x)^\delta(\log\log x)^{3/2}},$$

thus establishing the first inequality in (5.1).

In order to prove the second inequality in (5.1), first observe that if $n \le x$, then it is obvious that $\frac{\sqrt{x}}{2^k} < \rho_1(n) \le \frac{\sqrt{x}}{2^{k-1}}$ for some integer $k \ge 1$, and therefore that

$$\sum_{n\le x} \rho_1(n)^a \le \sum_{k\ge 1} \left(\frac{\sqrt{x}}{2^{k-1}}\right)^a H\left(x,\frac{\sqrt{x}}{2^k},\frac{\sqrt{x}}{2^{k-1}}\right). \tag{5.2}$$

Then, using Theorem A, we find that

$$\begin{aligned}
\sum_{k\ge 1}\left(\frac{\sqrt{x}}{2^{k-1}}\right)^a H\left(x,\frac{\sqrt{x}}{2^k},\frac{\sqrt{x}}{2^{k-1}}\right) &\ll \sum_{k\ge 1}\left(\frac{\sqrt{x}}{2^{k-1}}\right)^a x\cdot\frac{1}{(\log x)^\delta}\frac{1}{(\log\log x)^{3/2}} \ll \\
&\ll \frac{x^{\frac{a+2}{2}}}{(\log x)^\delta(\log\log x)^{3/2}}. \qquad (5.3)
\end{aligned}$$

Combining estimates (5.2) and (5.3), the second inequality in (5.1) is proved.

6. Proof of Theorem 3

First observe that for each positive integer n, we have $\dfrac{\rho_2(n)}{\rho_1(n)^r} = \dfrac{\rho_2(n)^{r+1}}{n^r}$. On the other hand, it follows from Theorem 1 that for each positive integer k,

$$A(x) := \sum_{n\le x} \rho_2(n)^{r+1} = \frac{x^{r+2}}{\log x}\sum_{\ell=0}^{k-1} \frac{c_\ell}{\log^\ell x} + \vartheta(x), \tag{6.1}$$

where

$$\vartheta(x) = O\left(\frac{x^{r+2}}{\log^{k+1} x}\right) \quad \text{and} \quad c_\ell = \frac{\ell!}{(r+2)^{\ell+1}} \sum_{j=0}^{\ell} \frac{(r+2)^j(-1)^j\zeta^{(j)}(r+2)}{j!}.$$

Hence, using (6.1) and partial summation, we obtain

$$\begin{aligned}
\sum_{n\le x}\frac{\rho_2(n)}{\rho_1(n)^r} &= \sum_{n\le x}\frac{\rho_2(n)^{r+1}}{n^r} = 1 + \sum_{2\le n\le x}\frac{\rho_2(n)^{r+1}}{n^r} = \\
&= 1 + \frac{A(x)-1}{x^r} + \int_2^x \frac{r}{t^{r+1}}\left(A(t)-1\right)dt = \\
&= \frac{A(x)}{x^r} + O(1) + \int_2^x \frac{r}{t^{r+1}} A(t)dt = \\
&= \frac{x^2}{\log x}\sum_{\ell=0}^{k-1}\frac{c_\ell}{\log^\ell x} + O\left(\frac{x^2}{\log^{k+1} x}\right) + \\
&\quad + r\sum_{\ell=0}^{k-1} c_\ell \int_2^x \frac{t}{\log^{\ell+1} t}dt + r\int_2^x \frac{\vartheta(t)}{t^{r+1}}dt.
\end{aligned} \tag{6.2}$$

It is easily seen that

$$r\int_2^x \frac{\vartheta(t)}{t^{r+1}}dt \ll \int_2^x \frac{t}{\log^{k+1} t}dt \ll \frac{x^2}{\log^{k+1} x}. \tag{6.3}$$

Moreover, integrating by parts, we have

$$c_{k-1}\int_2^x \frac{t}{\log^k t}dt = \frac{c_{k-1}}{2}\frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right)$$

and for $0 \leq \ell \leq k-2$,

$$c_\ell \int_2^x \frac{t}{\log^{\ell+1} t} dt = \frac{c_\ell}{2} \frac{x^2}{\log^{\ell+1} x} + \frac{c_\ell}{2} \sum_{i=\ell}^{k-2} \frac{x^2}{\log^{i+2} x} \prod_{m=\ell}^{i} \left(\frac{m+1}{2} \right) + O\left(\frac{x^2}{\log^{k+1} x} \right).$$

Summing on ℓ from 0 to $k-1$, we then obtain that

$$r \sum_{\ell=0}^{k-1} c_\ell \int_2^x \frac{t}{\log^{\ell+1} t} dt = \frac{x^2}{\log x} \sum_{\ell=0}^{k-1} \frac{d_\ell}{\log^\ell x} + O\left(\frac{x^2}{\log^{k+1} x} \right), \tag{6.4}$$

where $d_0 = \dfrac{rc_0}{2}$, and for $1 \leq \ell \leq k-1$,

$$d_\ell = \frac{rc_\ell}{2} + \sum_{\nu=0}^{\ell-1} \frac{rc_\nu}{2} \prod_{m=\nu}^{\ell-1} \left(\frac{m+1}{2} \right). \tag{6.5}$$

This is why, combining estimates (6.2), (6.3), (6.4), (6.5), we may conclude that

$$\sum_{n \leq x} \frac{\rho_2(n)}{\rho_1(n)^r} = \frac{x^2}{\log x} \sum_{\ell=0}^{k-1} \frac{e_\ell}{\log^\ell x} + O\left(\frac{x^2}{\log^{k+1} x} \right)$$

with $e_0 = \dfrac{(r+2)c_0}{2}$ and for $1 \leq \ell \leq k-1$,

$$e_\ell = \left(\frac{r+2}{2} \right) c_\ell + \sum_{\nu=0}^{\ell-1} \frac{rc_\nu}{2} \prod_{m=\nu}^{\ell-1} \left(\frac{m+1}{2} \right).$$

7. Proof of Theorem 4

First observe that for each positive integer n, we have $\dfrac{\rho_1(n)}{\rho_2(n)} = \dfrac{\rho_1(n)^2}{n}$. Set

$$A(x) := \sum_{n \leq x} \rho_1(n)^2 \quad \text{and} \quad \alpha := \sum_{n < e^e} \rho_1(n)^2.$$

Then, using Theorem 1 along with partial summation, we obtain that

$$\begin{aligned}\sum_{n\le x}\frac{\rho_1(n)}{\rho_2(n)} &= \sum_{n\le x}\frac{\rho_1(n)^2}{n} = O(1)+\sum_{e^e\le n\le x}\frac{\rho_1(n)^2}{n} = \\ &= O(1)+\frac{A(x)-\alpha}{x}+\int_{e^e}^{x}\frac{1}{t^2}(A(t)-\alpha)dt = \\ &= \frac{A(x)}{x}+O(1)+\int_{e^e}^{x}\frac{A(t)}{t^2}dt \asymp \\ &\asymp \frac{x}{(\log x)^{\delta}(\log\log x)^{3/2}}+\int_{e^e}^{x}\frac{dt}{(\log t)^{\delta}(\log\log t)^{3/2}}.\end{aligned}$$

Since

$$\int_{e^e}^{x}\frac{dt}{(\log t)^{\delta}(\log\log t)^{3/2}} \ll \frac{x}{(\log x)^{\delta}(\log\log x)^{3/2}},$$

the proof of Theorem 4 is complete.

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